QUASILINEAR THEORY OF RELAXATION OF AN ELECTRON BEAM IN A MAGNETOACTIVE PLASMA

V. D. SHAPIRO and V. I. SHEVCHENKO

Physico-technical Institute, Ukrainian Academy of Sciences

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We investigate the dynamics of quasilinear relaxation of an electron beam in a plasma in the presence of a strong magnetic field ($\omega_H \gg \omega_0$). We show that the plateau $\partial f/\partial v_z = 0$ on the longitudinal velocity distribution function, which occurs under Cerenkov excitation, is unstable against cyclotron excitation of oscillations by the anomalous Doppler effect. This instability leads to an increase of the transverse energy in the beam and to an inclination of the longitudinal-velocity plateau, with $\partial f/\partial v_z < 0$. A solution is obtained for the quasilinear system of equations describing the dynamics of this relaxation stage.

1. Several recent papers[1-6] are devoted to investigations of quasilinear relaxation of an electron beam, made under the assumption that the spectrum of the excited oscillations is one-dimensional. As is well known[7,8], the assumption that the spectrum of the Langmuir oscillations is one-dimensional is satisfied if the magnetic field in the plasma-beam system is sufficiently strong, such that $\omega_H \gg \omega_0$. In this case the maximum of the growth increment corresponds to Cerenkov excitation ($k_z v_z^{res} = \omega_k$) of oscillations propagating along the magnetic field. The diffusion of the beam particles in the field of the waves excited by them leads to the formation of the plateau $\partial f/\partial v_z = 0$ on the longitudinal-velocity distribution function, and to a limitation on the growth of the Cerenkov branch of the oscillations.

We shall show below, however, that the plateau $\partial f/\partial v_z = 0$ on the distribution function is unstable against cyclotron excitation of oscillations by the anomalous Doppler effect ($k_z v_z^{res} = \omega_k + \omega_H$). For this reason, the analysis in[1-6] describes only the initial stage of the process of quasilinear relaxation of the beam in a strong magnetic field with $\omega_H \gg \omega_0$. During the succeeding and slower stage, oscillations are excited by the anomalous Doppler effect, and lead to diffusion of the resonant particles along the lines

$$v_\perp^2 + v_z^2 - 2\int \frac{\omega_k}{k_z} dv_z = \text{const.}$$

On these lines we have

$$\frac{dv_\perp}{dv_z} = -\frac{v_z - \omega_k/k_z}{v_\perp} < 0$$

and the diffusion is thus accompanied by an increase of the transverse energy in the beam and by a decrease of the longitudinal energy, i.e., an inclination of the plateau. The plasma relaxes to a state in which $\partial f^\infty/\partial v_z < 0$ for the resonant particles (see Fig. 1). An examination of this stage of relaxation is contained in the present paper[1)]. The results obtained here are in accord

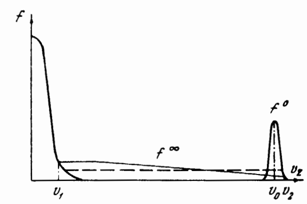

FIG. 1. Quasilinear relaxation of the longitudinal-velocity distribution function in the interaction of a beam with a plasma.

with experimental investigations performed by Frank[9] and by Kharchenko et al.[10] It was shown in these papers that the process of collisionless relaxation in the plasma-beam system does not terminate when a plateau is established on the longitudinal-velocity distribution function ($\partial f/\partial v_z = 0$). The oscillations continue to build up in the system, and an inclination of the plateau takes place ($\partial f/\partial v_z < 0$). It should be noted, however, that the mechanism of "plateau" transformation considered by us is not the only one. Thus, Mikhaĭlovskiĭ and Jungwirth[11] considered the possibility of a quasilinear transformation of the "plateau" in a spatially-inhomogeneous beam, connected with the development of a beam-drift instability. To explain the mechanism with which the experimentally-observed inclination of the plateau is connected, it is necessary to perform additional measurements. Thus, if the inclination of the plateau is connected with cyclotron excitation of oscillations, it should be accompanied by an appreciable increase of the transverse energy in the beam. In addition, in the case of cyclotron excitation decreases the growth increment of the oscillations with increasing ratio ω_H/ω_0, therefore an increase of this ratio should be accompanied by an increase of the characteristic time within which the inclination of the plateau takes place.

2. The quasilinear equations for the distribution functions of resonant particles $f(t, v_\perp, v_z)$ and the spectral density of the oscillation energy $|E_k(t)|^2$ in a longitudinal magnetic field $H_0 \parallel Oz$ have the following form[12]:

$$\frac{\partial f}{\partial t} = \frac{e^2}{2\pi m^2}\int k_\perp dk_\perp \sum_{n=-\infty}^{\infty}\left(\frac{n\omega_H}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{\partial}{\partial v_z}k_z\right) \times\left[\frac{|E_k|^2}{k^2}\frac{J_n^2(\lambda)}{|v_z - d\omega_k/dk_z|}\left(\frac{n\omega_H}{v_\perp}\frac{\partial f}{\partial v_\perp} - k_z\frac{\partial f}{\partial v_z}\right)\right]; \qquad (1)$$

[1)]During the performance of this work, the authors learned that analogous considerations concerning the existence of an instability mechanism connected with cyclotron excitation of oscillations, and leading to an inclination of the plateau, were advanced by B. B. Kadomtsev and O. P. Pogutse in connection with a discussion of collisionless heating in the Tokomak device.

$$\frac{\partial |E_{\mathbf{k}}|^2}{\partial t} = -\frac{4\pi^2 e^2}{m k_z^2}\frac{\omega_{\mathbf{k}}^3}{|k_z|}\Pi(\omega_{\mathbf{k}})\sum_{n=-\infty}^{\infty}\int dv_\perp J_n^2(\lambda)$$
$$\times\left(\frac{n\omega_H}{v_\perp}\frac{\partial f}{\partial v_\perp} - k_z\frac{\partial f}{\partial v_z}\right)|E_{\mathbf{k}}|^2;$$
$$k_z v_z = \omega_{\mathbf{k}} + n\omega_H,\quad \lambda = \frac{k_\perp v_\perp}{\omega_H},\quad \omega_H = \frac{eH_0}{mc},$$
$$\Pi(\omega_{\mathbf{k}}) = \frac{(\omega_{\mathbf{k}}^2 - \omega_H^2)(\omega_{\mathbf{k}}^2 - \omega_0^2 - \omega_H^2)}{\omega_0^2\omega_H^2(\omega_0^2 + \omega_H^2 - 2\omega_{\mathbf{k}}^2)},\quad \omega_0^2 = \frac{4\pi e^2 n_0}{m}. \tag{2}$$

The frequency $\omega_{\mathbf{k}}$, as usual, is determined from the equation

$$1 = \frac{\omega_0^2}{\omega_{\mathbf{k}}^2}\frac{k_z^2}{k^2} + \frac{\omega_0^2}{\omega_{\mathbf{k}}^2 - \omega_H^2}\frac{k_\perp^2}{k^2}. \tag{3}$$

It follows from (3) that the plasma oscillations exist only when $0 < \omega_{\mathbf{k}} < \min(\omega_0, \omega_H)$, or else at $\max(\omega_0, \omega_H) < \omega_{\mathbf{k}} < (\omega_0^2 + \omega_H^2)^{1/2}$. For such $\omega_{\mathbf{k}}$, the function $\Pi(\omega_{\mathbf{k}})$ is positive.

Changing over in the right sides of (1) and (2) to the variables v_z and

$$w(n, k_\perp) = v_\perp^2 + v_z^2 - 2\int\frac{\omega_{\mathbf{k}}}{k_z}dv_z$$

(the dependence on n enters in w via the resonance condition $k_z v_z = \omega_{\mathbf{k}} + n\omega_H$) we write down these equations in the following manner:

$$\frac{\partial f}{\partial t} = \frac{e^2}{2\pi m^2}\int k_\perp dk_\perp\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial v_z}\left[|E_{\mathbf{k}}|^2\frac{k_z^2}{k^2}\frac{J_n^2(\lambda)}{|v_z - d\omega_{\mathbf{k}}/dk_z|}\frac{\partial f}{\partial v_z}\right], \tag{4}$$
$$\frac{\partial |E_{\mathbf{k}}|^2}{\partial t} = \frac{4\pi^2 e^2}{m k_z}\frac{\omega_{\mathbf{k}}^3}{|k_z|}\Pi(\omega_{\mathbf{k}})\sum_{n=-\infty}^{\infty}\int dv_\perp J_n^2(\lambda)\frac{\partial f}{\partial v_z}|E_{\mathbf{k}}|^2. \tag{5}$$

It follows from (4) that the diffusion of the resonant particles occurs only along the lines w = const. In the general case, this diffusion is not one-dimensional, owing to the crossing of the lines corresponding to different n and $k_\perp$. The diffusion remains one-dimensional only in the case of excitation of oscillations at definite values of n and $k_\perp$, as is the case in a strong magnetic field, when the maximum of the increment corresponds to Cerenkov excitation (n = 0) of oscillations with $k_\perp = 0$. In this case the diffusion leads to establishment of a plateau along the line $v_\perp$ = const. The increment of the Cerenkov excitation (n = 0) vanishes in this case for all $k_\perp$, but, as can be shown, the increment remains positive when $n > 0$.

Indeed, let us consider the stability of the plateau along the lines

$$w^\alpha = v_\perp^2 + v_z^2 - 2\int v_{\text{ph}}^\alpha dv_z = \text{const}$$

($v_{\text{ph}} = \omega_{\mathbf{k}}/k_z$) against excitation of any definite branch of oscillations leading to the diffusion of the particles along the lines

$$w^\beta = v_\perp^2 + v_z^2 - 2\int v_{\text{ph}}^\beta dv_z = \text{const}.$$

Determining the growth increment for this branch from (5) and substituting the distribution function in the form $f = f(w^\alpha)$, we obtain

$$\gamma^\beta = \frac{4\pi^2 e^2}{m k_z}\frac{\omega_{\mathbf{k}}^3}{|k_z|}\Pi(\omega_{\mathbf{k}})\int dv_\perp J_n^2(\lambda)\frac{\partial}{\partial v_z}f$$
$$\times\left[w^\beta + 2\int(v_{\text{ph}}^\beta - v_{\text{ph}}^\alpha)dv_z\right] = \frac{8\pi^2 e^2}{m k_z}\frac{\omega_{\mathbf{k}}^3}{|k_z|}\Pi(\omega_{\mathbf{k}})$$
$$\times[v_{\text{ph}}^\beta(v_z) - v_{\text{ph}}^\alpha(v_z)]\int dv_\perp J_n^2(\lambda)\frac{\partial f}{\partial v_\perp^2}. \tag{6}$$

In this paper we confine ourselves to electron distributions that decrease monotonically with the increasing transverse velocity, $\partial f/\partial v_\perp^2 < 0$ [2)]. Then, inasmuch as $\Pi > 0$, we get from (6) that in the velocity distribution $f(w^\alpha)$ the increment for the branch β is positive if the condition

$$v_{\text{ph}}^\beta(v_z) < v_{\text{ph}}^\alpha(v_z) \tag{7}$$

is satisfied for v_z lying in the region of the plateau. In the case of Cerenkov excitation $v_{\text{ph}}^\alpha = v_z$. Substituting $v_{\text{ph}}^\beta = v_z - n\omega_H/k_z$, we find that in the state with the longitudinal-velocity plateau $\partial f/\partial v_z = 0$ all the resonances with $n > 0$, corresponding to excitation of oscillations by the anomalous Doppler effect, make a positive contribution to the increment (5) (see also Fig. 2). This result, which was first obtained by Andronov and Trakhtengerts[15], does not always signify, however, instability of the plateau, since the stabilizing action on the spectral of the resonances with $n < 0$ may turn out to be appreciable. However, in our case of a strong magnetic field, when the frequencies of the excited oscillations $\omega_{\mathbf{k}} \approx \omega_0 k_z/k \ll \omega_H$, resonances with $n < 0$ correspond to interaction of an exponentially small number of particles with the oscillations. It is therefore possible to neglect in (5) the terms with $n < 0$, and the "plateau" $\partial f/\partial v_z = 0$ thus turns out to be unstable.

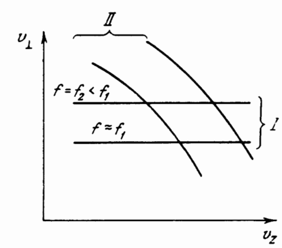

FIG. 2. Diffusion lines of resonant particles: I—diffusion lines in the case of Cerenkov excitation (n = 0), II—diffusion lines in the case of cyclotron excitation by the anomalous Doppler effect (n > 0). The height of the plateau on the lines I decreases with increasing $v_\perp$. For such a velocity distribution $\partial f/\partial v_z > 0$ along the lines II, and this can lead, in accordance with (5), to instability of the plateau.

3. In the succeeding sections of this paper we shall examine, with the aid of the quasilinear equations (1) and (2), the time variation of the distribution function of the beam and of the spectral density of the excited oscillations. We shall assume here that the stage of formation of the plateau $\partial f/\partial v_z = 0$, which has been investigated in detail earlier, has already been completed. In this stage, the beam excites one dimensional ($\mathbf{k} \parallel \mathbf{H}_0$) oscillations with frequencies $\omega_{\mathbf{k}} \approx \omega_0$ and wave vectors $k_z = \omega_0/v_z$. The energy of the excited oscillations is

$$\mathcal{E}^C = \sum_{\mathbf{k}}\frac{|E_{\mathbf{k}}|^2}{4\pi} \sim n_1 m v_0\,\Delta v$$

(n_1—density in the beam, v_0—initial beam velocity, Δv—width of plateau). The characteristic time of this stage is

2)By the same token, we exclude from consideration the instabilities connected with the presence of an average transverse velocity of the beam particles. The linear theory of such instabilities was considered in [13,14].

$$t^C \sim \frac{1}{\omega_0}\frac{n_0}{n_1}\frac{(\Delta v)^2}{v^2}\ln\frac{\mathscr{E}^C_{max}}{\mathscr{E}_0{}^C}$$

($\mathscr{E}^C_{max}$ and $\mathscr{E}^C_0$ are the final and initial energies of the Cerenkov branch of the oscillations), and in a sufficiently strong magnetic field the cyclotron excitation of the oscillations with $k_z = n\omega_H/v_z$ cannot develop within that time. However, cyclotron buildup continues also in the state with a plateau on the distribution function, and should lead to a slow transformation of the plateau.

In the investigation of the quasilinear equations we shall assume that the Larmor radius of the beam particles is small, $\lambda \ll 1$. In this case only the resonance at $n = 1$ is significant. In addition, we note that in a strong magnetic field, $\omega_H \gg \omega_0$, it is sufficient to take into account the cyclotron excitation of oscillations with frequency $\omega_k = \omega_0 k_z/k$. When $\partial f/\partial v_z = 0$, the growth increment for these oscillations is equal to

$$\gamma^D = \frac{\pi^2 e^2}{m}\frac{k_\perp{}^2}{k^3}\frac{\omega_0}{\omega_H}\int dv_\perp f(t, v_\perp{}^2). \tag{8}$$

The increment for the oscillations of frequency

$$\omega_{\mathbf{k}} = \omega_H\left(1 + \frac{\omega_0{}^2}{2\omega_H{}^2}\frac{k_\perp{}^2}{k^2}\right)$$

is small compared with (8) in the ratio ω_0/ω_H.

The cyclotron buildup of the oscillations leads to diffusion of the resonant particles along the constant-energy lines $v_\perp^2 + v_z^2 = \text{const}$ ($v_{ph} \ll v_z$ for the cyclotron branch in a strong magnetic field). The particles diffuse to smaller values of v_z, and the plateau of the longitudinal velocities becomes inclined. As a result of the diffusion, the increment for the cyclotron branch of the oscillations decreases. However, owing to the mechanism of the Cerenkov absorption, the plateau $\partial f/\partial v_z = 0$ is restored, and the energy of the Cerenkov branch of the oscillations previously excited in the plasma then decreases. By the same token, conditions are again created for cyclotron buildup of the oscillations, etc. This process continues until the Cerenkov branch of the oscillations attenuates to zero. A stationary spectrum of cyclotron oscillations is excited in the plasma, and the distribution function of the resonant particles will be constant along the lines $v_\perp^2 + v_z^2 = \text{const}$.[3)]

The quasilinear system of equations describing such a process can be obtained from (1) and (2), and has the following form:

$$\frac{\partial f}{\partial t} = \frac{e^2}{2\pi m^2}\frac{\partial}{\partial v_z}\left[\frac{1}{v_z}\int dk_\perp k_\perp |E_{\mathbf{k}}{}^C|^2 \frac{\partial f}{\partial v_z}\right] + \frac{e^2}{8\pi m^2}\left(\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{\partial}{\partial v_z}\frac{1}{v_z}\right)$$
$$\times\left[\int dk_\perp \frac{k_\perp{}^3}{k^2}|E_{\mathbf{k}}{}^D|^2 \frac{1}{v_z}\left(v_\perp\frac{\partial f}{\partial v_\perp} - \frac{v_\perp{}^2}{v_z}\frac{\partial f}{\partial v_z}\right)\right], \tag{9}$$

$$\frac{\partial |E_{\mathbf{k}}{}^C|^2}{\partial t} = \frac{4\pi^2 e^2}{mk^3}\omega_0 k_z \int dv_\perp \frac{\partial f}{\partial v_z}|E_{\mathbf{k}}{}^C|^2, \tag{10}$$

$$\frac{\partial |E_{\mathbf{k}}{}^D|^2}{\partial t} = -\frac{\pi^2 e^2}{m}\frac{\omega_0}{\omega_H}\frac{k_\perp{}^2}{k^3}\int dv_\perp\left(v_\perp\frac{\partial f}{\partial v_\perp} - \frac{v_\perp{}^2}{v_z}\frac{\partial f}{\partial v_z}\right)|E_{\mathbf{k}}{}^D|^2. \tag{11}$$

In the derivation of (9)—(11) we confined ourselves to the higher-order terms in the parameter ω_0/ω_H; $|E_k^C|^2$ in these equations is the spectral density of the oscillation energy at $k_z = \omega_0/v_z$ (Cerenkov branch of the oscillations), and $|E_k^D|^2$ is the same quantity for $k_z = \omega_H/v_z$ (cyclotron branch of the oscillations, excited by the anomalous Doppler effect).

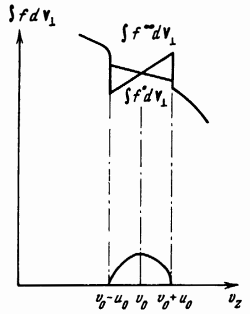

FIG. 3. Distribution function and spectral energy density of the oscillations for the solution described by (15). The spectrum boundaries remain stationary when the instability develops ($\partial f/\partial v_z|_{v_z = v_0 \pm u_0} \to -\infty$.

It follows from (11) that the growth increment of the cyclotron oscillations is maximal when $k_\perp^2 = 2k_z^2$, and thus $|E_k^D|^2$ should have a sharp maximum at this value of $k_\perp$[4)]. Using this circumstance, we can simplify Eqs. (9)—(11), introducing into consideration the spectral energy density of the oscillations, integrated over the transverse component of the wave vector, $W(v_z) = \int dk_\perp k_\perp |E_k|^2$:

$$\frac{\partial f}{\partial t} = \frac{e^2}{2\pi m^2}\frac{\partial}{\partial v_z}\left[\frac{W^C}{v_z}\frac{\partial f}{\partial v_z}\right] + \frac{e^2}{12\pi m^2}\left(\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{\partial}{\partial v_z}\frac{1}{v_z}\right)$$
$$\times\left[\frac{W^D}{v_z}\left(v_\perp\frac{\partial f}{\partial v_\perp} - \frac{v_\perp{}^2}{v_z}\frac{\partial f}{\partial v_z}\right)\right], \tag{12}$$

$$\frac{\partial W^C}{\partial t} = \frac{4\pi^2 e^2}{mk_z{}^2}\omega_0\int dv_\perp \frac{\partial f}{\partial v_z}W^C, \qquad k_z = \frac{\omega_0}{v_z}, \tag{13}$$

$$\frac{\partial W^D}{\partial t} = -\frac{2\pi^2 e^2}{3^{3/2} mk_z}\frac{\omega_0}{\omega_H}\int dv_\perp\left(v_\perp\frac{\partial f}{\partial v_\perp} - \frac{v_\perp{}^2}{v_z}\frac{\partial f}{\partial v_z}\right)W^D,$$
$$k_z = \frac{\omega_H}{v_z}. \tag{14}$$

4. We consider first the solution of the system (12)—(14) for the model transverse-velocity distribution shown in Fig. 3, wherein a narrow spectrum of oscillations, with an increment independent of k_z, is excited in the plasma. In this case we seek the solution of (12)—(14) in the form

$$f = A(v_\perp{}^2) + B(t, v_\perp{}^2)(v_z - v_0),$$
$$W^C = C(t)\frac{u_0{}^2 - (v_z - v_0)^2}{2},$$
$$W^D = D(t)\frac{u_0{}^2 - (v_z - v_0)^2}{2}. \tag{15}$$

If the conditions $Bu_0 \ll A$ and $v_0 u_0 \ll \overline{v_\perp^2}$ is satisfied, we have for the quantities $B(t, v_\perp^2)$, $C(t)$, and $D(t)$ the following system of equations:

$$\frac{\partial B}{\partial t} = -\alpha BC - \frac{\alpha}{6}\frac{v_\perp{}^2}{v_0{}^2}(B - B^\infty)D,$$

$$\frac{\partial C}{\partial t} = \beta^C\int_0^\infty Bv_\perp dv_\perp C,$$

$$\frac{\partial D}{\partial t} = \beta^D\int_0^\infty (B - B^\infty)v_\perp{}^3 dv_\perp D. \tag{16}$$

3)The quasilinear relaxation of the electron distribution in the opposite limiting case $\omega_0 \gg \omega_H$, $k v_\perp \gg \omega_H$ was considered in [16]. In that case, even if the spectrum of the oscillations excited in a direction perpendicular to the magnetic field or at an angle to the field is one-dimensional, the oscillation energy does not suffice to establish a plateau at all values of the resonant-particle velocity. As a result, the oscillations attenuate to zero, and only a certain approach to a state with a plateau (quasi-plateau) occurs in the course of the resonant-particle diffusion.

4)Actually such a spectrum of $|E_k^D|^2$ is quasistationary. If we take into account in (11) the next higher order of smallness in the parameter $\sim\omega_0/\omega_H$, then we find that the energy in the spectrum should be pumped over to smaller v_{ph}, i.e., to larger $k_\perp$, with an increment $\sim\gamma^D\omega_0/\omega_H$.

Here

$$\alpha = \frac{e^2}{2\pi m^2 v_0}, \quad \beta^C = \frac{8\pi^3 e^2}{m\omega_0} v_0^2,$$
$$\beta^D = \frac{4\pi^3 e^2}{3^{1/2} m\omega_H^2}\omega_0 = \frac{1}{6\cdot 3^{1/2}}\frac{\omega_0^2}{\omega_H^2}\frac{1}{v_0^2}\beta^C,$$

$B^\infty = 2v_0 dA/dv_\perp^2 < 0$ is the value of B as $t \to \infty$, when the distribution function is constant along the lines $v_\perp^2 + v_z^2 = \text{const}$. The system (16) has an integral

$$\frac{\alpha}{\beta^C}\left[C(t) + 3^{1/2}\frac{\omega_H^2}{\omega_0^2}D(t)\right] + \int_0^\infty dv_\perp v_\perp (B(t, v_\perp^2) - B^0(v_\perp^2)) = 0, \quad (17)$$

corresponding to the usual energy integral of quasilinear equations. In (17) we put $B^0 = B(t = 0)$, and in its derivation we neglected the contribution of the initial noise, proportional to C(0) and D(0).

During the initial stage of the instability, when the excitation of the Cerenkov branch takes place, it is possible to neglect the terms ~ D(t) in (16). During this stage, within a time

$$t^C \sim \frac{1}{\gamma^C}\ln\frac{C^{max}}{C(0)} \quad \left(\gamma^C = \beta^C \int_0^\infty B^0 v_\perp\, dv_\perp\right)$$

the function C(t) reaches a maximum equal to

$$C^{max} = \frac{\beta^C}{\alpha}\int_0^\infty B^0 v_\perp\, dv_\perp,$$

while B(t) decreases to zero. Subsequently, excitation of the cyclotron branch with a characteristic time

$$t^D \sim \frac{1}{\gamma^D}\ln\frac{D^{max}}{D(0)} \sim t^C \frac{\omega_H^2}{\omega_0^2}\frac{v_0^2\int_0^\infty B^0 v_\perp\, dv_\perp}{\int_0^\infty |B^\infty| v_\perp^3\, dv_\perp}$$

$$\left(\gamma^D = -\beta^D \int_0^\infty B^\infty v_\perp^3\, dv_\perp\right)$$

takes place. With this, in accordance with the first equation of (16), the derivative $\partial f/\partial v_z = B(t)$ becomes negative, and the quantity C(t), which determines the spectral energy density of the Cerenkov oscillations, begins to decrease with time. It is important, however, that in accordance with (17) the maximum spectral energy density of the cyclotron oscillations is smaller by a factor ω_0^2/ω_H^2 then that of the Cerenkov oscillations [5)],

$$D^{max} = \frac{\beta^C}{3^{1/2}\alpha}\frac{\omega_0^2}{\omega_H^2}\int_0 dv_\perp v_\perp (B^0 - B^\infty).$$

Therefore in this stage of the instability, when both branches of the oscillations exist simultaneously, by virtue of the smallness of the ratio $D/C \ll 1$, |B| remains small and can be neglected compared with $|B^\infty|$ (the condition of applicability of such an approximation will be clarified later). Then the last equation of (16) can be easily integrated:

$$D(t) = D(0)\exp(\gamma^D t), \quad (18)$$

and substitution of C(t) from (17) into the first equation of (16) yields the following equation for $I = \int_0^\infty B(t, v_\perp^2) v_\perp dv_\perp$:

$$\frac{dI}{dz} + \delta I\left(\frac{1}{z} - 1\right) - \delta\frac{I^2}{zI^0} + I^0 = 0 \quad (19)$$

In this equation

$$I^0 = \int_0^\infty B^0 v_\perp\, dv_\perp, \quad z = \frac{D}{I^0}\frac{3^{1/2}\alpha}{\beta^C}\frac{\omega_H^2}{\omega_0^2}, \quad \delta = \frac{\beta^C I^0}{\gamma^D} = \frac{\gamma^C}{\gamma^D} \gg 1.$$

When $1 - v \gg 1/\sqrt{\delta}$, we can neglect in (19) the derivative dI/dz and the nonlinear term $\sim I^2$. Then

$$I = -I^0 z / \delta(1 - z). \quad (20)$$

This solution cannot be used when $z \to 1$. Putting in (19) $z = 1 - \zeta$, $\zeta \ll 1$, we obtain the equation

$$\frac{dI}{d\zeta} + \delta I\left(\frac{I}{I^0} - \zeta\right) = I^0, \quad (21)$$

the solution of which is (see [17])

$$I = I^0\left[\zeta - \frac{\exp(-\delta\zeta^2/2)}{\sqrt{2\delta}\,\Phi(\sqrt{\delta/2}\,\zeta)}\right], \quad (22)$$

where

$$\Phi(x) = \int_x^\infty e^{-t^2}dt.$$

when $\zeta \gg 1/\sqrt{\delta}$, using the asymptotic form

$$\exp\left\{\frac{\delta}{2}\zeta^2\right\}\Phi\left(\sqrt{\frac{\delta}{2}}\zeta\right) \approx \frac{1}{\sqrt{2\delta}\,\zeta}\left(1 - \frac{1}{\delta\zeta^2}\right),$$

we obtain (20) from (22). On the other hand, if $\zeta < 0$ and $|\zeta| \gg 1/\delta$, then we get from (22)

$$I = I^0\left[\zeta - \frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{\delta}{2}\zeta^2\right)\right]. \quad (23)$$

At sufficiently large $|\zeta|$, when $|\zeta| \sim I^\infty/I^0$, the condition $|B| \ll |B^\infty|$ is violated. However, for appreciably smaller $|\zeta|$, when $|\zeta| \gtrsim 1/\sqrt{\delta}$, we can confine ourselves in (23) to only the first term which is linear in ζ. This, as can be readily seen, corresponds to the fact that only the contribution from the cyclotron oscillations is significant in the equation for B. Namely, using (17), we can show that C(t) is exponentially small under these conditions:

$$C = -\frac{\beta^C}{\alpha}(I - I^0\zeta) = \frac{\beta^C}{\alpha}\frac{I^0}{\sqrt{2\pi\delta}}\exp\left\{-\frac{\delta}{2}\zeta^2\right\}. \quad (24)$$

When $\delta\zeta > 1$ we obtain for $B(t, v_\perp^2)$ with the aid of (23)

$$B = I^0\zeta v_\perp^2 B^\infty \Big/ \int_0^\infty B^\infty v_\perp^3\, dv_\perp. \quad (25)$$

In the case when only the cyclotron branch of the oscillations is significant in (16), the solution of this system of equations can be readily obtained also without assuming that |B| is small compared with $|B^\infty|$. Its form is

$$B = B^\infty[1 - \exp(-\beta^D v_\perp^2 \tau)], \quad (26)$$

5)The total energy $\sim\int dk_z W^D$ of the cyclotron oscillations is smaller than the energy of the Cerenkov oscillations by a factor ω_0/ω_H, inasmuch as the width of the spectrum for the cyclotron oscillations is larger by a factor ω_H/ω_0, namely, $\Delta K_z^D \sim \Delta k_z^C \omega_H/\omega_0$. The smallness of the energy of the cyclotron oscillations is due to the fact that in the case of cyclotron excitation in a strong magnetic field, the lines of diffusion of the resonant particles are close to the constant-energy lines (the difference is in first order of smallness with respect to the parameter $\sim\omega_0/\omega_H$).

$$D=\frac{\beta^C}{3^{1/2}\alpha}\frac{\omega_0^2}{\omega_H^2}\Big[I^0-\int_0^\infty dv_\perp v_\perp B^\infty(1-\exp(-\beta^D v_\perp^2\tau))\Big]. \quad (27)$$

The connection between t and τ is determined from the equation

$$\frac{dt}{d\tau}=\frac{1}{I^0-I}=\Big[I^0-\int_0^\infty dv_\perp v_\perp B^\infty(1-\exp(-\beta^D v_\perp^2\tau))\Big]^{-1}. \quad (28)$$

At small values of τ, when, in accord with (27),

$$\tau=-\Big[\frac{3^{1/2}\alpha}{\beta^C}\frac{\omega_H^2}{\omega_0^2}D-I^0\Big]\Big/\beta^D\int_0^\infty dv_\perp v_\perp^3B^\infty=I^0\zeta/\beta^D\int_0^\infty dv_\perp v_\perp^3B^\infty,$$

formula (26) for $B(t, v_\perp^2)$ goes over into (25). At large values of τ we have $B\to B^\infty$, corresponding to a plateau along the lines $v_\perp^2+v_z^2=\text{const}$, and $D\to D^{max}$.

5. The solution obtained in the preceding section describes the dynamics of the quasilinear relaxation of the velocity distribution, shown in Fig. 3. With such a distribution, there is excited in the plasma a sufficiently narrow oscillation spectrum, all the harmonics of which with respect to k_z have the same increment. Let us proceed now to investigate the system (12)—(14) in the case when the plateau $\partial f/\partial v_z=0$ is established in a broad velocity interval $v_{min}<v_z<v_{max}$, $v_{min}\approx v_1$, $v_{max}\approx v_2$ (see Fig. 1). The maximum of the growth increment of the cyclotron oscillations with respect to $k_\perp$ is determined in the states with the plateau by the formula

$$\gamma_0^D=\frac{2\pi^2e^2}{3^{3/2}m}\frac{\omega_0}{\omega_H^2}v_z\int dv_\perp f(t,v_\perp^2) \quad (29)$$

and consequently, it is primarily the cyclotron oscillations at $v_z\approx v_0$ which build up and reach saturation. Subsequently the boundary of the region in which the energy of the cyclotron oscillations is maximal shifts towards smaller v_z.[6)] It can be expected that, just as in the case of excitation of the Cerenkov branch of the oscillations, considered by Ivanov and Rudakov[6], the solution of (12)—(14) will be a wave whose front moves in accordance with $v_z=u(t)$. The distribution function $f(t, v_\perp^2, v_z)$, which is the solution of Eqs. (12)—(14), will be sought in the form

$$f=f_0(t,v_\perp^2)+f_1(t,v_\perp^2,v_z).$$

Ahead of the wave front, i.e., as $W^D\to 0$, the distribution function is close to the longitudinal-velocity plateau. Indeed, for f_1 we have in this velocity region, in accordance with (12),

$$\frac{\partial f_1}{\partial v_z}=\frac{2\pi m^2}{e^2}\frac{v_z(v_z-v_1)}{W^C(v_z)}\frac{\partial f_0}{\partial t} \quad (30)$$

The value of W^C ahead of the front, W^C_{max}, will be determined from the energy integral of the quasilinear equations (12)—(14):

$$W^C(v_z)+3^{1/2}\frac{\omega_H^2}{\omega_0^2}W^D(v_z)=\frac{8\pi^3mv_z^3}{\omega_0}\int_{v_1}^{v_z}dv_z\int dv_\perp(f(t,\mathbf{v})-f(0,\mathbf{v})). \quad (31)$$

Hence

$$W^C_{max}(v_z)=\frac{8\pi^3mv_z^3(v_z-v_1)}{\omega_0}\int dv_\perp f_0(t,v_\perp^2) \quad (31')$$

[6)]We assume that when $v_z>v_2\approx v_0$ the distribution function decreases sufficiently rapidly with increasing velocity, and the cyclotron oscillations do not build up at these values of v_z.

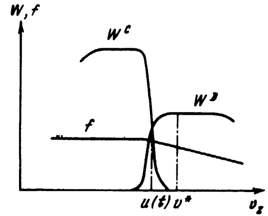

FIG. 4. Distribution function of the resonant particles and spectral density of the oscillation energy vs. v_z.

(we are considering the values of v_z such that $f(t=0)=0$). Substituting W^C_{max} in (30), we obtain

$$f_1\approx\frac{1}{\gamma^C}\frac{\partial f_0}{\partial t}<\frac{\gamma^D}{\gamma^C}f_0\ll f_0.$$

On the wave front, the distribution function changes from a value $f_0(t, v_\perp^2)$ to a certain value $f(t, v_\perp^2+v_z^2)$, $W^C(v_z)$ attenuates to zero, and $W^D(v_z)$ increases from an initial value W^D_0 to a value

$$W^D_{max}(v_z)=\frac{\omega_0^2}{3^{1/2}\omega_H^2}W^C_{max}(v_z)$$

(see Fig. 4). Following[6], we seek the solution on the wave front in the form

$$f_1(t,v_\perp^2,v_z)=f_1(t,v_\perp^2,v_z-u(t))\equiv f_1(t,v_\perp^2,\eta),$$
$$W(t,v_z)=W(t,v_z-u(t))\equiv W(t,\eta).$$

The width of the wave front $\Delta\eta$ is small ($\Delta\eta\sim u\ln^{-1}(W^D_{max}/W^D_0)$, see below), and we therefore assume that the following conditions

$$\Big|\frac{\partial f_1}{\partial t}\Big|\ll\Big|\dot u\frac{\partial f_1}{\partial\eta}\Big|,\quad\Big|\frac{\partial W}{\partial t}\Big|\ll\Big|\dot u\frac{\partial W}{\partial\eta}\Big| \quad (32)$$

are satisfied on the wave front (the dependence on η determines the rapid variation of f_1 and W on the wave front, while the dependence on t determines the slow variation of these quantities as the front advances).

Using conditions (32), we rewrite Eqs. (13), (14), and (31) on the wave front as follows:

$$-\dot u\frac{\partial W^C}{\partial\eta}=\frac{4\pi^2e^2}{m\omega_0}u^2W^C\int dv_\perp\frac{\partial f_1}{\partial\eta}, \quad (33)$$

$$-\dot u\frac{\partial W^D}{\partial\eta}=\frac{4\pi^2e^2}{3^{3/2}m\omega_H^2}\omega_0uW^D\int dv_\perp\Big(f+\frac{v_\perp^2}{u}\frac{\partial f_1}{\partial\eta}\Big), \quad (34)$$

$$\frac{\partial W^C}{\partial\eta}+3^{1/2}\frac{\omega_H^2}{\omega_0^2}\frac{\partial W^D}{\partial\eta}=\frac{8\pi^3mu^3}{\omega_0}\int dv_\perp f. \quad (35)$$

We consider first the solution of these equations at values of η such that in terms of the variables (34) and (35) we can neglect the terms proportional to $\partial f_1/\partial\eta$ and f_1. Then these equations can be simply integrated and lead to the following relations for W^D and W^C:

$$W^D(t,\eta)=W^D_{max}(u)Z(\eta),\quad W^C(t,\eta)=W^C_{max}(u)(1-Z(\eta)) \quad (36)$$

($\eta<0$). We put here

$$Z(\eta)=\exp\Big[-\frac{4\pi^2e^2}{3^{3/2}m}\frac{\omega_0}{\omega_H^2}\frac{u}{\dot u}\int dv_\perp f_0\eta\Big]$$

and W^C_{max} and W^D_{max} were defined above.

Integrating (33), we get

$$\int dv_\perp f_1=-\frac{m}{4\pi^2e^2}\omega_0\frac{\dot u}{u^2}\ln[1-Z(\eta)] \quad (37)$$

and

$$\int dv_\perp\frac{\partial f_1}{\partial\eta}=-\frac{\omega_0^2}{3^{1/2}\omega_H^2}\frac{1}{u}\int dv_\perp f_0\frac{Z(\eta)}{1-Z(\eta)}$$

It follows from the last equation that the condition

$$\int \frac{v_\perp^2}{u}\frac{\partial f_1}{\partial \eta} dv_\perp \sim \frac{\overline{v_\perp^2}}{u}\int \frac{\partial f_1}{\partial \eta} dv_\perp \ll \int f_0\, dv_\perp$$

is violated only if

$$1 - Z(\eta) \sim \frac{\omega_0^2}{\omega_H^2}\frac{\overline{v_\perp^2}}{u^2} \ll 1,$$

when $W^D \approx W^D_{max}(u)$. Then, as before

$$\int f_1\, dv_\perp \ll \int f_0\, dv_\perp.$$

Thus, up to values $W^D \approx W^D_{max}$, the buildup of the cyclotron oscillations proceeds with the increment given by formula (29). The displacement of the wave front boundary u(t), can then be determined from the following approximate equation:

$$u(t)\int_0^t dt' \int dv_\perp f_0(t', v_\perp^2) = \frac{3^{1/2}m}{4\pi^2 e^2}\frac{\omega_H^2}{\omega_0}\ln\frac{W^D_{max}}{W_0^D}. \tag{38}$$

For the front width $\Delta\eta$ we have from (36) and (38)

$$\Delta\eta \approx \left|\frac{3^{1/2}m}{4\pi^2 e^2}\frac{\omega_H^2}{\omega_0}\frac{\dot u}{u}\Big/\int dv_\perp f_0\right| \approx u/\ln\frac{W^D_{max}}{W_0^D}.$$

The small width of the front makes the derivative $\partial f_1/\partial\eta$ on the wave front large if f_1 is small. The equation for the determination of the function f_1 on the wave front can then be simplified, retaining only terms containing the higher-derivative with respect to η. As a result, from the Eq. (12) written in terms of the variables η and t,

$$\frac{\partial f_0}{\partial t} - \dot u\frac{\partial f_1}{\partial \eta} = \frac{e^2}{2\pi m^2}\frac{\partial}{\partial\eta}\left(\frac{W^C(t,\eta)}{u}\frac{\partial f_1}{\partial\eta}\right)$$
$$+\frac{e^2}{12\pi m^2}\left(\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{1}{u}\frac{\partial}{\partial\eta}\right)\frac{W^D(t,\eta)}{u}\left(v_\perp\frac{\partial f_0}{\partial v_\perp} - \frac{v_\perp^2}{u}\frac{\partial f_1}{\partial\eta}\right),$$

we get, integrating once with respect to η:

$$\frac{\partial f_1}{\partial\eta} = \frac{v_\perp}{6u}\frac{\partial f_0}{\partial v_\perp}\frac{W^D(t,\eta)}{W^C(t,\eta) + (v_\perp^2/6u^2)\, W^D(t,\eta)}. \tag{39}$$

In deriving this equation we neglect the terms that are small compared with $\ln^{-1}(W^D_{max}/W^D_0) \ll 1$ and $u\Delta\eta/\overline{v_\perp^2} \ll 1$. Inasmuch as $\overline{v_\perp^2} \sim v_2(v_2 - u)$ (see (49′) below), the last condition denotes smallness of the front width compared with the distance traversed by the wave. When $W^C \gg (\overline{v_\perp^2}/u^2)W^D$, we get formulas (37) from (39). In the opposite limiting case, $W^C \ll (\overline{v_\perp^2}/u^2)W^D$, we get from (39)

$$\frac{\partial f_1}{\partial\eta} = \frac{u}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\left(1 - \frac{6W^C}{W^D}\frac{u^2}{v_\perp^2}\right).$$

Substituting $\partial f_1/\partial\eta$ in (34) and (35) and integrating, we find that for these values of η we have $W^D \to W^D_{max}$, and W^C tends to zero quite rapidly, with a characteristic value

$$\Delta\eta_1 = \left|\frac{m\omega_0}{24\pi^2 e^2}\frac{\dot u}{u^3}\Big/\int_0^\infty \frac{dv_\perp}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right| \approx \Delta\eta\frac{\omega_0^2}{\omega_H^2}\frac{\overline{v_\perp^2}}{u^2} \ll \Delta\eta$$

(the rapid attenuation of W^C in this region of η is due to the large value of the derivative $\partial f_1/\partial\eta$).

According to (39), the total change of f on the front of the wave is small:

$$\Delta f \lesssim \frac{u\Delta\eta}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \ll f_0.$$

Therefore the distribution function behind the wave front, i.e., as $W^C \to 0$, is equal to $f_0(t, v_\perp^2 + v_z^2 - u^2(t))$.

6. We now proceed to derive an equation for $f_0(t, v_\perp^2)$. To this end, we integrate (12) from v_1 to v^*. Inasmuch as we have $W^C = 0$ when $v_z = v_1$ and $v_z = v^*$ (see Fig. 4), we obtain as a result of the integration

$$\int_{v_1}^{v^*}\frac{\partial f_0}{\partial t} dv_z = \frac{e^2}{12\pi m^2}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{v_\perp}{u}\int d\eta\, W^D(t,\eta)\left(\frac{\partial f_0}{\partial v_\perp} - \frac{v_\perp}{u}\frac{\partial f_1}{\partial\eta}\right)\right.$$
$$\left.- W^D(t, v_z)\frac{v_\perp^2}{v_z^2}\left(\frac{1}{v_\perp}\frac{\partial f}{\partial v_\perp} - \frac{1}{v_z}\frac{\partial f}{\partial v_z}\right)\Big|_{v_z = v^*}\right\}. \tag{40}$$

In the right side of this equation, the integration was carried out over the region of the wave front. Substituting under the integral sign $\partial f_1/\partial\eta$ from (39), we transform the integral in the following manner:

$$\frac{e^2}{12\pi m^2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{v_\perp}{u}\int d\eta\, W^D(t,\eta)\left(\frac{\partial f_0}{\partial v_\perp} - \frac{v_\perp}{u}\frac{\partial f_1}{\partial\eta}\right)$$
$$= \frac{e^2}{12\pi m^2 u}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial f_0}{\partial v_\perp}\int d\eta\frac{W^D W^C}{W^C + (v_\perp^2/6u^2)W^D}$$
$$= \frac{e^2}{12\pi m^2 u}\int_{-\infty}^{0} d\eta\, W^D(t,\eta)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial f_0}{\partial v_\perp} = -\frac{\dot u(u - v_1)}{2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial f_0}{\partial v_\perp}. \tag{41}$$

In the integration we have neglected the narrow region of η, in which $W^C \lesssim (v_\perp^2/u^2)W^D$, which makes a small contribution to the integral, and used formula (36) for $W^D(t, \eta)$. We transform the second term in the right side of (40) by substituting the distribution function at $v_z = v^*$ in the form

$$f = f_0(t, v_\perp^2 + v_z^2 - u^2(t)) + f_*(t, \mathbf{v}), \qquad f_* \ll f_0. \tag{42}$$

Going over to the variables v_z and $w = v_\perp^2 + v_z^2$, we write down this term as follows:

$$\frac{e^2}{12\pi m^2}\frac{v_\perp^2}{v_z^2}W^D(t, v_z)\left(\frac{1}{v_\perp}\frac{\partial f}{\partial v_\perp} - \frac{1}{v_z}\frac{\partial f}{\partial v_z}\right)\Big|_{v_z = v^*}$$
$$= -\frac{e^2}{12\pi m^2}\frac{W^D(t, v_z)}{v_z^3}v_\perp^2\frac{\partial f_*}{\partial v_z}\Big|_{v_z = v^*}, \tag{43}$$

where the derivative with respect to v_z in the right side of this relation is calculated at w = const.

The equation for the determination of the distribution function behind the wave front, where only the cyclotron oscillations are significant, has in terms of the variables v_z and w the form

$$\frac{\partial f}{\partial t} = \frac{e^2}{12\pi m^2}\frac{\partial}{\partial v_z}W^D\frac{v_\perp^2}{v_z^3}\frac{\partial f}{\partial v_z}.$$

Substituting f from (42) in this equation and integrating along the line w = const, we get

$$\int_{v^*}^{v_m}\frac{\partial f_0}{\partial t}[t, w - u^2(t)]\, dv_z = -\frac{e^2}{12\pi m^2}W^D(t, v_z)\frac{v_\perp^2}{v_z^3}\frac{\partial f_*}{\partial v_z}\Big|_{v_z = v^*}. \tag{44}$$

In this equation

$$v_m = \begin{cases}\sqrt{w}, & \text{if} \quad w < v_2^2\\ v_2, & \text{if} \quad w > v_2^2\end{cases}.$$

With this

$$W^D(v_z)v_\perp^2|_{v_z = v_m} = 0.$$

Using (41)–(44), we rewrite (40) in the form

$$\int_{v_1}^{u} \frac{\partial f_0}{\partial t}(t, v_\perp^2)\,dv_z + \int_{u}^{v_m} \frac{\partial f_0}{\partial t}(t, w - u^2(t))\,dv_z$$
$$= -\frac{\dot{u}u(u-v_1)}{2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \frac{\partial f_0}{\partial v_\perp}. \qquad (45)$$

The contour of integration with respect to v_z in this equation is shown in Fig. 5.

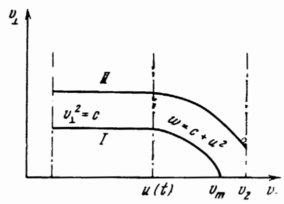

FIG. 5. Contour of integration with respect to v_z in Eq. 45: I—$v_\perp^2 < v_z^2 - u^2$, II—$v_\perp^2 > v_z^2 - u^2$

Neglecting small quantities of order $\Delta\eta/u$, we replace v^* by u within the integration limits. Integrating with respect to v_z, we obtain finally the following equation for $f_0(t, v_\perp^2)$:

$$[v_m(v_\perp, u) - v_1]\frac{\partial f_0}{\partial t} - \frac{\dot{u}u}{v_\perp}[v_m(v_\perp, u) - u]$$
$$\times \frac{\partial f_0}{\partial v_\perp} + \frac{\dot{u}u(u-v_1)}{2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \frac{\partial f_0}{\partial v_\perp} = 0. \qquad (46)$$

We have put here

$$v_m(v_\perp, u) = (v_\perp^2 + u^2)^{1/2}$$
$$\text{if} \quad v_\perp^2 < v_2^2 - u^2,$$
$$v_m(v_\perp, u) = v_2 \ \text{if} \ v_\perp^2 > v_2^2 - u^2.$$

Equation (46), as well as the initial equation (12), was obtained under the condition $\lambda = k_\perp^2 \overline{v_\perp^2}/\omega_H^2 \ll 1$. Inasmuch as $k_\perp^2 \sim k_z^2 \sim \omega_H^2/v_z$ and $\overline{v_\perp^2} \sim v_2(v_2 - u)$ for cyclotron oscillations, as will be shown below, it follows that the condition for applicability of (46) is

$$1/\ln\frac{W_{max}^D}{W_0^D} \ll \frac{v_2 - u}{v_2} \ll 1$$

(the distance traversed by the wave is large compared with the width of its front, but is small compared with the width of the plateau). For (46) to be valid at all values of t, this condition must be satisfied by the lower boundary u_{min} of the spectrum of the cyclotron oscillations: $v_2 - u_{min} \ll v_2$.

In the present paper we consider the case when u_{min} is determined by the Cerenkov absorption of the cyclotron branch of oscillations on the plasma electrons. When this absorption is taken into account, the formula for the increment of the cyclotron oscillations with $k_z = \omega_H/v_z$ becomes

$$\gamma^D = \gamma_0^D(v_z) + \frac{2\pi^2 e^2}{3^{1/2} m\omega_0} v_p^2 \int dv_\perp \frac{\partial f}{\partial v_p}\Big|_{v_p = \omega_0 v_z/\sqrt{3}\omega_H}. \qquad (47)$$

The first term in this formula determines the cyclotron buildup of the oscillations by the beam particles with velocity v_z. In the state with the plateau on the distribution function, this term is given by formula (29). The second term in (47) determines the Cerenkov absorption of the same oscillations on particles with much smaller velocities, $v_p = \omega_0 v_z/\sqrt{3}\,\omega_H \ll v_z$. When $v_p > v_1$, this absorption is insignificant, since the distribution function at such values of v_p is close to the plateau. When $v_p < v_1$, the absorption becomes large and can change the sign of the increment. Indeed, assuming that when $v_p < v_1$ the distribution over the longitudinal velocities is Maxwellian, we obtain from (47) and (29) the following formula for the increment:

$$\gamma^D = \frac{\pi}{6\sqrt{3}}\frac{n_1}{n_0}\frac{\omega_0^3}{\omega_H^2}\frac{v_z}{v_2} - \frac{1}{2}\sqrt{\frac{\pi}{6}}\,\omega_0\left(\frac{mv_p^2}{T}\right)^{3/2}\exp\left\{-\frac{mv_p^2}{2T}\right\}.$$

Determining v_1 from the relation (see [4])

$$\exp\left\{-\frac{mv_1^2}{2T}\right\} \approx \frac{n_1}{n_0}\sqrt{\frac{2\pi T}{mv_1^2}},$$

we can readily show that when $v_p < v_1$ we have $\gamma^D < 0$. Thus, the lower limit of the spectrum of the cyclotron oscillations is

$$u_{min} = 3^{1/2}\omega_H v_1 / \omega_0 \gg v_1.$$

In the case considered by us, the inclination of the plateau takes place only if $u_{min} < v_z < v_2$. When $v_z < u_{min}$, the distribution with respect to the longitudinal velocities remains close to the plateau, but an appreciable smearing of the distribution with respect to the transverse velocities takes place; this smearing is described by Eq. (46). Confining ourselves in this equation to terms of first order of smallness in the parameter $(v_2 - u)/v_2 \ll 1$, we obtain

$$\frac{\partial f_0}{\partial u} + \frac{u}{2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \frac{\partial f_0}{\partial v_\perp} = \frac{v_m - u}{v_2 - v_1}\left(\frac{v_2}{v_\perp}\frac{\partial f_0}{\partial v_\perp} - \frac{\partial f_0}{du}\right),$$
$$v_m = \begin{cases} u + v_\perp^2/2v_2 & \text{if} \quad v_\perp^2 < 2v_2(v_2 - u) \\ v_2 & \text{if} \quad v_\perp^2 > 2v_2(v_2 - u) \end{cases}.$$

In the highest order in the parameter $(v_2 - u)/v_2 \ll 1$, when the terms transferred to the right side of (48) can be neglected, the solution of this equation entails no difficulty. Thus, if the initial distribution with respect to the transverse velocities is Maxwellian, the distribution $f_0(T; V_\perp^2)$ remains Maxwellian in this approximation

$$f_0(t, v_\perp^2) = \frac{n_1}{v_2 - v_1}\frac{m}{2\pi T(u)}\exp\left(-\frac{mv_\perp^2}{2T(u)}\right), \qquad (49)$$

and the change of the transverse temperature on approaching the front of the wave is determined from the equation

$$dT/du \approx -mu, \text{ i.e., } T(u) = T_0 + m(v_2^2 - u^2)/2 \approx mv_2(v_2 - u) \qquad (49')$$

(The initial value of T is assumed to be sufficiently small: $T_0 \ll mv_2(v_2 - u)$).

When small terms $\sim (v_2 - u)/v_2$ are taken into account in (48), the height of the plateau with respect to the longitudinal velocities, $\sim \int f_0 dv_\perp$, changes. This change is a result of the diffusion of the resonant particles on the wave front in the region $v_z < u$. We then get from (48)

$$\frac{d}{du}\int f_0\,dv_\perp = \frac{\pi}{v_2 - v_1}\left[\frac{1}{v_2}\int_0^{2v_2(v_2-u)} dv_\perp v_\perp^3\left(\frac{v_2}{v_\perp}\frac{\partial f_0}{\partial v_\perp} - \frac{\partial f_0}{\partial u}\right)\right.$$
$$\left. + 2(v_2 - u)\int_{2v_2(v_2-u)}^{\infty} dv_\perp v_\perp\left(\frac{v_2}{v_\perp}\frac{\partial f_0}{\partial v_\perp} - \frac{\partial f_0}{\partial u}\right)\right].$$

Substituting in the right side of this equation f_0 from (49) and integrating with respect to u and $v_\perp$, we get

$$\Delta\int f_0\,dv_\perp \approx \frac{n_1}{e}\frac{v_2 - u}{(v_2 - v_1)^2}. \qquad (50)$$

Owing to the increase in the height of the plateau during the considered stage of relaxation, an increase takes place in the longitudinal energy of the resonant particles when $v_z < u_{min}$. In first order in the parameter $(v_2 - u)/v_2 \ll 1$, the increase of this energy is equal to

$$\Delta\mathscr{E}_{\parallel}^{(1)} = \frac{m}{2}\Delta\int f_0\, dv_\perp \int_{v_1}^{u_{min}} v_z^2\, dv_z = \frac{n_1 m}{6e}\frac{(v_2 - u_{min})(v_2^2 + v_2 v_1 + v_1^2)}{v_2 - v_1} \approx \frac{n_1 m v_0 (v_0 - u_{min})}{6e} \quad (51)$$

(in obtaining the last relation we replaced approximately v_2 by v_0 and neglected v_1 compared with v_0).

For the increase of the transverse energy of the resonant particles upon relaxation, we have from (49) and (49′), with the same accuracy,

$$\Delta\mathscr{E}_{\perp}^{(1)} = n_1 T \approx n_1 m v_0 (v_0 - u_{min}). \quad (52)$$

The distribution function of the resonant particles in the region behind the front of the wave $v_z > u(t)$ is, in accord with (42) and (49),

$$f \approx \frac{n_1}{v_2 - v_1}\frac{m}{2\pi T(u)}\exp\left[-\frac{m}{2T(u)}(v_\perp^2 + v_z^2 - u^2)\right].$$

The inclination of the plateau in this velocity region is accompanied by loss of energy by the resonant particles:

$$\Delta\mathscr{E}^{(2)} \approx \frac{n_1 m}{2(v_2 - v_1)}\int_0^\infty dv_\perp v_\perp \int_{u_{min}}^{v_2} dv_z (v_\perp^2 + v_z^2)\left[-\frac{\exp\{-mv_\perp^2/2T_0\}}{T_0/m} + \frac{\exp\{-m(v_\perp^2 + v_z^2 - u_{min}^2)/2T\}}{T/m}\right] \approx -\frac{n_1 m v_0 (v_0 - u_{min})}{2e}. \quad (53)$$

Finally, the change of the energy of the Cerenkov branch of the oscillations in the relaxation stage under consideration is

$$\Delta\mathscr{E}^{C} \approx n_1 m v_0 (v_0 - u_{min})\left(\frac{1}{3e} - 1\right). \quad (54)$$

The first term in this formula is the increase of the energy of the Cerenkov oscillation branch when $v_z < u_{min}$, connected with the increase of the height of the plateau and determined from (31′). The second term is the energy of the Cerenkov oscillations in the state with the plateau when $v_z > u_{min}$. In the stage under consideration, the Cerenkov oscillations in this region of v_z, i.e., behind the wave front, attenuate to zero.

The energy of the cyclotron branch of the oscillations, as shown above, is smaller than $\Delta\mathscr{E}^C$ by a factor ω_0/ω_H. Thus, (51)—(54) yield the energy conservation law in the higher order in the parameter ω_0/ω_H: the energy lost by the oscillations and by the resonant particles with velocities $v_z > u_{min}$ is equal to the energy acquired by the resonant particles at $v_z < u_{min}$.

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