

POSITRONIUM SINGULARITIES IN QUANTUM ELECTRODYNAMICS AND
PERTURBATION THEORY

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The charge singularities in quantum electrodynamics which appear as a result of positronium formation are investigated. Terms exhibiting such singularities in the proper mass of the electron and the vacuum polarization are computed explicitly. It is shown that in perturbation theory dispersion relations should be written without taking into account poles related to composite particles.

1. INTRODUCTION

As is well known, the occurrence of composite particles in nonrelativistic quantum mechanics leads to singularities with respect to the coupling constant in the corresponding Green's function, thus making it impossible to use perturbation theory^[1]. It is a characteristic feature that the singularities with respect to the coupling constant do not appear at all energies, but only in a region immediately adjacent to the pole produced by the composite particle. Outside this region the Green's function remains an analytic function of the interaction constant.

In quantum field theory the two-particle Green's function can occur as a building block of some Feynman diagram. The question arises as to how the Green's function singularities due to the presence of composite particles reflect themselves in the properties of the diagram. In particular, is it still possible to use perturbation theory for such diagrams, taking into account the fact that the two-particle Green's function cannot be expanded in a perturbation theory series for some region of its arguments?

Concretely, we consider quantum electrodynamics. The two-particle Green's function will refer to the electron (e^-) and the positron (e^+), and the singularities in which we are interested are due to the formation of positronium. At the same time we shall have to consider the two-particle e^-e^- (or e^+e^+) Green's function which does not have pole singularities on the physical sheet of the energy, but still cannot be expanded in a perturbation theory series.

Thus we shall be interested in finding out how the singularity with respect to $\alpha = e^2/4\pi$ in the e^-e^+ or e^-e^- Green's function manifests itself in the diagrams containing these functions as blocks, and whether it is possible to compute such diagrams in the usual manner according to perturbation theory.

In brief, the results of the paper are the following. All diagrams which contain as blocks the e^-e^+ or e^-e^- Green's functions are themselves nonanalytic functions of α in a certain region of the external momenta where perturbation theory is not applicable. This region of external momenta contains, however, only momenta situated in a neighborhood (at distances $\sim \alpha^2 m^2$) of the singularities of the Feynman diagram under consideration, singularities defined by the well-

known Landau equations^[2] and coming from the Landau diagrams with "positronium" (for e^-e^+) or "antipositronium" (for e^-e^-) lines of mass $\sim 2m$. For the simplest diagrams the singularities in α appear only when real positronium can be formed in the intermediate state, and are closely related to threshold singularities which appear when a new reaction channel opens up. In bound state problems (e.g., the levels of a hydrogen atom), no positronium singularities appear.

As an illustration, we compute the parts of the proper mass of the electron, and of the vacuum polarization, which are non-analytic in α due to positronium. In conclusion we discuss briefly the relation between formalisms which utilize off-shell and on-shell quantities in perturbation theory. It turns out that in the presence of composite particles a formalism operating with quantities defined off the mass shell has the advantage that it allows the use of perturbation theory. In formalisms using only quantities defined on the mass shell (dispersion relations and unitarity) this is not possible.

2. SINGULARITIES IN α IN QUANTUM ELECTRODYNAMICS

The singularities of the relativistic two-particle Green's function of the e^-e^+ (G) or e^-e^- (G_1) with respect to α , and related to positronium or antipositronium, are analogous to the singularities of the non-relativistic Green's function in a Coulomb field. The relativistic Green's function satisfies the equation

$$G = G_0 + \alpha G_0 I G, \tag{1}$$

where G_0 is the product of the one-particle Green's functions of the e^- and e^+ , and I represents the interaction. For analyzing the positronium singularity it suffices to consider the one-photon exchange only, since the other terms in I only produce insignificant displacements of the positronium levels. Then I does not depend on α . Following Cutkosky and Leon^[3] and Weinberg^[4], we introduce the eigenvectors Φ_N and the eigenvalues of the homogeneous equation

$$G_0 I \Phi_N = \eta_N \Phi_N, \quad N = 1, 2, \dots, \tag{2}$$

expand G in terms of Φ_N and obtain (cf. ^[3,4])

$$G(p, k, k') = G_0(p, k, k') + \sum_{N=1}^{\infty} \frac{\Phi_N(p, k) \bar{\Phi}_N(p, k')}{\langle \Phi_N | I | \Phi_N \rangle} \frac{\alpha \eta_N^2}{1 - \alpha \eta_N}. \tag{3}$$

Here p is the total momentum, and k and k' are the relative momenta of the e^- and e^+ . In Eq. (3) Φ_N and $\eta_N(p^2)$ do not depend on α . The total dependence on α is exhibited in explicit form. The positronium levels are determined from the equation $\alpha = \eta_N^{-1}$. A comparison with the well-known formula for the levels of hydrogen-like systems yields $\eta_N = m/N(4m^2 - p^2)^{1/2}$. It follows from (3) that the whole nonanalyticity in α is concentrated in the factor $1/(1 - \alpha\eta_N) \sim 1/(4m^2 - p^2 - \alpha^2 m^2/N^2)$. In other words, the nonanalyticity in α is completely related to the positronium pole, and to the dependence of the positronium mass on α . The residue at this pole is analytic in α .

The Green's function G cannot be expanded in a perturbation theory series for $|\alpha\eta_N| > 1$, i.e.

$$|4m^2 - p^2| < \alpha^2 m^2/N^2, \quad N = 1, 2, \dots \quad (4)$$

This is, of course, a well-known fact, which follows directly from the nonrelativistic theory.

The results also remain valid for the two-electron Green's function G_1 , if one replaces α by $-\alpha$. The pole appears on the second sheet (antipositronium), but the region of non-expansion in powers of α remains the same.

We now turn to an arbitrarily complicated Feynman diagram in quantum electrodynamics, containing either G or G_1 as a whole (cf. e.g., Fig. 1, which represents the electron self-energy, or Fig. 2, which represents vacuum polarization). We substitute (3) into the diagram, and consider the contribution of the N -th term.

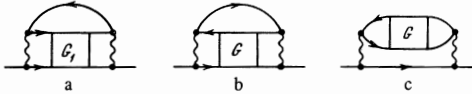


FIG. 1. The electron proper mass

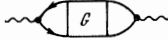


FIG. 2. Vacuum polarization

We carry out all the integrations over internal momenta, with the exception of the last one, which is over the positronium momentum p . In the complex p_0 plane the singularities of the integrand will consist of the positronium pole at $p^2 = M_N^2 \equiv 4m^2 - \alpha^2 m^2/N^2$, cuts for $p^2 \geq 4m^2$, and the singularity produced by the remainder of the diagram, $p_0 = s(q_i)$, the position of which depends on the external momenta q_i of the diagram (cf. Fig. 3).

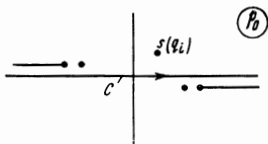


FIG. 3. The singularities in the complex p_0 plane; C is the integration path.

So long as the point $s(q_i)$ is not close to the positronium singularity, we can deform the integration contour in the p_0 -plane and avoid the positronium pole by a distance $\sim m$. Then the denominator $p^2 - M_N^2$ can definitely be expanded in powers of α and the whole diagram can be represented as a series in powers of α . It becomes impossible to expand with respect to α when $s(q_i)$ comes into the vicinity of the positronium

pole (i.e., into the region (4)), from the other side of the integration path. If $s(q_i)$ coincides exactly with the positronium pole, the integral ceases to exist, and the Feynman diagram acquires a singularity as a function of its external momenta for $q_i = q_i^{(0)}$. The position of this singularity is determined by the well-known equations and diagrams given by Landau^[2], and is such that in the Landau diagram there must exist a line of mass M_N , corresponding to the positronium pole. Assuming the derivatives of $s(q_i)$ with respect to q_i to be of order of unity, we find that the region where the Feynman diagram cannot be expanded in powers of α is

$$|q_i - q_i^{(0)}| \lesssim \alpha^2 m/N^2. \quad (5)$$

We consider as examples the self-mass of the electron and the vacuum polarization. The simplest Landau diagrams for them are illustrated in Fig. 4 (the double line denotes the positronium). The corresponding singularities are at $q^2 = (m + M_N)^2$ and $q^2 = M_N^2$, respectively. The domain of inapplicability of perturbation theory is situated near these values, at a distance $\sim \alpha^2 m^2/N^2$. It is obvious that all singularities in α are directly related to thresholds in the energy for real production of positronium in the intermediate state. More complicated Landau diagrams correspond to the production of positronium plus several photons in the intermediate state and are of higher order in α .

FIG. 4. The simplest Landau diagrams for the electron proper mass and vacuum polarization, involving positronium lines.



Diagrams with three or four external lines will have singularities produced by positronium of exactly the same nature. Owing to the large mass of positronium, there are no anomalous singularities, and only the simplest Landau diagrams, corresponding to production of positronium and some other particles in the intermediate state, remain. The domain where perturbation theory is not applicable will refer to energies near the production threshold.

It is important that positronium singularities do not manifest themselves at all in bound state problems for an electron in an external field (hydrogen atom), since here the electron energy $E < m$, and for positronium production E should be of the order $3m$. This result can also be seen directly from the corresponding Bethe-Salpeter equation, by rotating the path of integration with respect to the relative energy by 90° and thus transforming to an Euclidean equation^[5]. The kernel of the Euclidean equation can be considered to contain only Euclidean integrations, and one can see that in the denominator $p^2 - M_N^2$ corresponding to positronium, $p^2 < 0$, and the denominator can be expanded in powers of α .

3. THE SINGULARITIES IN α OF THE VACUUM POLARIZATION

In order to illustrate the general considerations given above we compute the terms containing singularities in α in the electron self-energy and in the vacuum

polarization. We start with the vacuum polarization tensor $\Pi^{\alpha\beta}(q) = \Pi(q^2)(g^{\alpha\beta} - q^\alpha q^\beta/q^2)$.

The totality of diagrams in Fig. 2 corresponds to the contribution

$$\Pi^{\alpha\beta}(q) = \frac{e^2}{(2\pi)^4 i} \int d^4k d^4k' \gamma_{\mu\nu}^\alpha G_{\nu\nu', \mu\mu'}(q, k, k') \gamma_{\nu'\mu'}^\beta, \quad (6)$$

where the spinor subscripts of G refer to the first ($\nu\nu'$) and second ($\mu\mu'$) particles. In the region of interest $q^2 \approx 4m^2$. It is clear that the principal part of the singularity is given by the nonrelativistic region $|k| \ll m$. In the region of large k there appears at most another α (ultraviolet divergences being removed by renormalization). Therefore in the first approximation one may assume that the e^- and e^+ move slowly, and then

$$G_{\nu\nu', \mu\mu'} = \frac{1}{2}(1 + \gamma_0)_{\nu\nu'} \frac{1}{2}(1 - \gamma_0)_{\mu\mu'} 4m^2 \tilde{G}, \quad (7)$$

where $\tilde{G}(q, k, k')$ is the four-dimensional Green's function corresponding to scalar particles which interact with an attractive Coulomb potential. Substituting (7) into (6), we obtain

$$\Pi(q^2) = 8\pi a g(\epsilon, 0, 0). \quad (8)$$

The function $g(\epsilon, \mathbf{x}, \mathbf{y})$ is the nonrelativistic three-dimensional Green's function in an attractive Coulomb field for a particle of mass $\frac{1}{2}m$ and energy $\epsilon(q^2) = (q^2 - 4m^2)/4m$. Making use of the known formulas for this function^[6] we find that the part singular in α in $g(\epsilon, 0, 0)$ is

$$g_s(\epsilon, 0, 0) = -\frac{am^2}{4\pi} \psi(1 - iv), \quad v = am/\sqrt{q^2 - 4m^2}. \quad (9)$$

Here $\psi(z) \equiv d \ln \Gamma(z)/dz$. It follows that the part of Π which is singular in α has the form:

$$\Pi_s(q^2) = -2\alpha^2 m^2 \psi(1 - iv). \quad (10)$$

It can be seen that the proper mass of the photon has poles at points corresponding to positronium ($i\nu = N, N = 1, 2, \dots$). The photon Green's function is given by the usual expression

$$D(q^2) = (\Pi(q^2) - q^2)^{-1} \quad (11)$$

and will have poles at slightly displaced points, corresponding to the displacement of the positronium level due to the contact potential produced by the photon pole. Perturbation theory becomes invalid for $|\nu| > 1$, where one would have to add to the standard perturbative expression of Π in n -th order of α the correction

$$\Delta\Pi^{(n)} = -2\alpha^2 m^2 \left(\psi(1 - iv) + C + \sum_{k=2}^{n-1} \zeta(k) (iv)^{k-1} \right). \quad (12)$$

The correction has the order α^2 and is large in the immediate neighborhood of the positronium poles and at the scattering threshold ($\nu \rightarrow \infty$), where it behaves like $\alpha^2 \nu^{n-2}$.

4. THE SINGULARITIES IN α OF THE ELECTRON PROPER MASS

We now pass on to the electron proper mass $\Sigma(\hat{q})$ (Fig. 1). The part of the mass operator Σ_1 related to the Green's function e^-e^- (Fig. 1a) can be written in the form

$$\begin{aligned} (\Sigma_1)_{\mu\mu'} &= \frac{e^4}{(2\pi)^8} \int d^4p d^4k d^4k' \gamma_{\mu\nu}^\alpha G_{1\nu\nu', \mu'\xi}(p, k, k') \gamma_{\nu'\mu'}^\beta \\ &\times (\gamma_\beta(m - \hat{p} + \hat{q})^{-1} \gamma_\alpha)_{\xi\mu''} / (q - \frac{1}{2}p - k)^2 (q - \frac{1}{2}p - k')^2. \end{aligned} \quad (13)$$

We again take into consideration the fact that the essential contribution is made by small k : $|k| \ll m$. In addition the condition for the appearance of a singularity requires $q^2 \approx 9m^2$, $p^2 \approx 4m^2$ and $(p - q)^2 \approx m^2$. In the frame where $q = 0$ it follows that $p_0 \approx 2m$ and consequently $|p| \ll m$. This allows us to set $(\hat{q} - \frac{1}{2}p - k)^2 = (q - \frac{1}{2}p - k')^2 = 4m^2$, and $m + p - q = m(1 - \gamma_0)$. For G_1 one can write down an expression of the type (7), but involving two projection operators $\frac{1}{2}(1 + \gamma_0)$ and the function G_1 corresponding to repulsion. We obtain (in the frame where $q = 0$)

$$\Sigma_1(\hat{q}) = (1 - \gamma_0) \frac{3\alpha^2 i}{8\pi^2 m} \int d^4p \frac{g_1(\epsilon(p^2), 0, 0)}{m^2 - (p - q)^2}, \quad (14)$$

where $g_1(\epsilon, \mathbf{x}, \mathbf{y})$ is the repulsive Coulomb Green's function. Substituting the expression for the singular part of $g_1(\epsilon, 0, 0)$ we find the part of Σ_1 singular in α in the form

$$\Sigma_{1s}(\hat{q}) = (3m - \hat{q}) \frac{\alpha^3 i}{32\pi^3 m^2} \int d^4p \frac{\psi(1 + iv)}{m^2 - (p - q)^2}, \quad (15)$$

where $\nu(p^2) = \alpha m / (p^2 - 4m^2)^{1/2}$. The integral in (15) diverges, but the part which is singular in α is finite. It can be found by using a series representation for the ψ -function.

The contribution from all diagrams in Fig. 1b is the same, and is given by (15), but with α replaced by $-\alpha$. The final result for the part Σ_S of the proper mass of the electron which is singular in α is

$$\Sigma_s(\hat{q}) = (\hat{q} - 3m) \frac{\alpha^6}{8\pi \sqrt{3}} \sigma(q^2), \quad (16)$$

where

$$\sigma(q^2) = \sum_{N=1}^{\infty} \frac{1}{N^4} \xi_N^{-1} \sqrt{1 + \xi_N^2} \left(\ln(\xi_N + \sqrt{1 + \xi_N^2}) + i \frac{\pi}{6} \right) \quad (17)$$

and $\xi_N = (\frac{3}{2})^{1/2} \alpha m / N (q^2 - 9m^2)^{1/2}$. Perturbation theory becomes invalid for $|\xi_N| > 1$. However, the corrections to perturbation theory are unusually small: $\sim \alpha^6$.

In a completely analogous manner one could find the part singular in α of other quantities, e.g., for Compton scattering. It is similar to Σ_S , extremely small ($\sim \alpha^7$) and scarcely presents any practical interest.

5. DISPERSION RELATIONS AND PERTURBATION THEORY

We consider the interesting problem of a dispersion relation in the presence of composite particles. For simplicity we consider the photon Green's function $D(q^2)$ defined according to (11). In the vicinity of $q^2 \approx 4m^2$ the part $D_S(q^2)$ singular in α can be written in first approximation in the form

$$D_s(q^2) \approx -(4m^2)^{-2} \Pi_s(q^2) = \frac{\alpha^2}{8m^2} \psi(1 - iv). \quad (18)$$

Far from the point $q^2 = 4m^2$, D_S can be expanded in a convergent series in powers of α . One might think that the whole function $D(q^2)$ can be expanded in this region into a (perhaps only asymptotically) convergent perturbation theory series:

$$D(q^2) = \sum_{n=0}^{\infty} D^{(n)}(q^2), \quad |q^2 - 4m^2| > \alpha^2 m^2. \quad (19)$$

$D^{(n)}(q^2)$ can be represented as a sum of Feynman diagrams, each being an analytic function of q^2 with singularities dictated by the Landau rules. As can be seen from (18), for $q^2 = 4m^2$ $D^{(n)}(q^2)$ has an integrable pole or square-root singularity. This singularity is not related to the production of composite particles, but is a consequence of the vanishing photon mass. If, as is usually done, one starts from a theory with photon mass $\lambda \neq 0$, the nonintegrable singularity of $D^{(n)}(q^2)$ at $q^2 = 4m^2$ disappears. In order to see this, we replace the Coulomb Green's function g in (8) by the Green's function for a Hulthén potential $V_0 e^{-\lambda r} (1 - e^{-\lambda r})^{-1}$, for $\lambda \rightarrow 0$ and $V_0 = \alpha\lambda$. This does not exactly correspond to the introduction of a photon mass λ , but is more like the introduction of an assembly of photons with masses which are multiples of λ . Nevertheless it is natural to assume that the results do not depend on method of going to the limit $\lambda \rightarrow 0$. Then the function $\psi(1 - i\nu)$ in (10) will be replaced by the expression

$$\sum_{N=1}^{\infty} \left(\frac{2}{N} - \frac{2\lambda N - i\sqrt{q^2 - 4m^2}}{\lambda N^2 - iN\sqrt{q^2 - 4m^2} - \alpha m} \right) - \psi \left(1 - \frac{i}{\lambda} \sqrt{q^2 - 4m^2} \right), \quad (20)$$

which goes over into $\psi(1 - i\nu)$ for $\lambda \rightarrow 0$. Expanding (20) in a series in α will preserve the expressions $\lambda N - i(q^2 - 4m^2)^{1/2}$ in the denominator, and for $\lambda \neq 0$ these never vanish. Therefore for $\lambda \neq 0$, $D^{(n)}(q^2)$ is analytic in q^2 with a cut along $q^2 \geq 0$, with a branch point at $q^2 = 4m^2$ which is integrable at that point. We emphasize that $D^{(n)}$ has no poles corresponding to positronium, since it is simply a sum of a finite number of Feynman diagrams. Assuming that $D^{(n)}(q^2)$ decreases as $|q^2| \rightarrow \infty$, we can write a dispersion relation

$$D^{(n)}(q^2) = \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} D^{(n)}(s) ds}{s - q^2}. \quad (21)$$

The imaginary part $\text{Im} D^{(n)}$ can be obtained from the Feynman diagram by means of the Landau-Cutkosky rules, i.e., using generalized unitarity in n -th order of perturbation theory.

Using (19) we sum the $D^{(n)}(q^2)$ and obtain

$$D(q^2) = -\frac{1}{q^2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\text{Im} D^{(n)}(s) ds}{s - q^2} \quad (22)$$

for $|q^2 - 4m^2| > \alpha^2 m^2$.

At a first glance, Eq. (22) looks paradoxical. The function $D(q^2)$ is represented as a dispersion integral which does not contain any positronium poles. But we

know that in fact such poles should be present: they are explicitly exhibited in Eq. (18). Therefore the correct dispersion relation for $D(q^2)$ must look as follows:

$$D(q^2) = -\frac{1}{q^2} + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} D(s) ds}{s - q^2} + \sum_{N=1}^{\infty} \frac{c_N}{M_N^2 - q^2}, \quad (23)$$

where M_N are the positronium masses. The apparent contradiction between (22) and (23) can, of course, be explained by the impossibility of interchanging the order of summation and integration in (22). The sum over n in (22) converges, but $\sum_n \text{Im} D^{(n)}(s)$ diverges in the region $|s - 4m^2| \sim \alpha^2 m^2$. The difference between the sum of integrals in (22) and the integral in (23) is exactly equal to the sum of pole terms in (23). This can be seen directly by taking an individual term in the sum (20).

We come to the conclusion that in the presence of composite particles the Feynman diagram method which operates with off-mass-shell quantities, and the axiomatic dispersion approach in conjunction with unitarity, are not equivalent insofar as perturbation theory is concerned. The method of Feynman diagrams continues to work far from the singularities produced by virtual composite particles. The dispersion method requires the knowledge of the discontinuity across the cut throughout the whole energy range, including the domain where perturbation theory is manifestly inapplicable. Therefore this method is incompatible with perturbation theory in the presence of composite particles.

In concrete computations of Feynman diagrams one can, of course, use dispersion relations in the form (22). Practically this means that in dispersion relations combined with perturbation theory there is no need to take into account the contribution from poles produced by composite particles.

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