MAGNETOHYDRODYNAMIC CUMULATION NEAR A ZERO FIELD LINE

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Boundary conditions are determined, under which the previously obtained self-similar solutions of nonlinear equations of magnetohydrodynamics are realized for the vicinity of a zero magnetic-field line. The corresponding flows, which lead to the compression of the plasma pinch and to an unlimited increase of the current density, occur at a definite law governing the variation of the vector potential of the external field.

EXACT particular solutions for the equations of magnetohydrodynamics, for nonstationary two-dimensional flow in the vicinity of a null line of a hyperbolic magnetic field, were obtained by Chapman and Kendall[1] for an incompressible liquid and subsequently by Imshennik and the author[2] for a compressible liquid. These solutions are characterized by a linear dependence of the velocity components and by a quadratic dependence of the vector potential of the field on the spatial coordinates. The density of the medium in the case of a compressible liquid depends only on the time. The dependence of the quantities on the time is such that one of the components of the magnetic field, the current density in the medium, and also certain other quantities, increase without limit. In the case of a compressible medium, the singularity is reached within a finite time interval.

The obtained solution was interpreted in[11], as an instability of the initial equilibrium state of the plasma in the vicinity of the null line of the magnetic field. In contrast, arguments were presented in[2] in favor of assuming that the flow in question has the character of cumulation and is due to the action of forces that are external with respect to the fixed value of the plasma. This conclusion results, in particular, from a calculation of the energy balance and of the Poynting vector, which show that the energy of each material volume of the plasma is increased by influx of energy from the outside. The hypothesis was also advanced that the obtained solution is realized when the external currents producing the magnetic field in the plasma are varied, and corresponds to the process of dynamic dissipation of the magnetic field, which was considered in[3,4].

However, the question of the exact boundary conditions under which the obtained solutions are realized remained unexplained in both[1] and[2]. We shall show below that the self-similar solutions obtained in[1,2] can be set in correspondence with exact boundary conditions that have a sufficiently simple physical meaning. These conditions are a particular case of the conditions considered in[3,4], and correspond to a change of the potential of the external currents producing the hyperbolic magnetic field in accordance with a fully defined law.

Let us stop first to discuss the properties of the self-similar solutions under consideration.

The initial state of a homogeneous plasma in the vicinity of a null magnetic-field line is characterized by a constant density and by a hyperbolic vector potential (by virtue of the two-dimensional character of the problem, only the z-component of the vector potential is significant):

$$\rho = \rho_0, \quad A = A_0(x^2 - y^2) + L_0. \tag{1}$$

We shall use the dimensionless variables assumed in[2], wherein the units of density, vector potential, and time are chosen to be ρ_0, A_0, and

$$t_0 = (\pi\rho_0)^{1/2} / |A_0|. \tag{2}$$

The length unit is arbitrary in this case, so that we assume it to equal the radius r_0 of the initial cylindrical plasma column.

In dimensionless variables, the self-similar solutions of[1] and[2] can be written in the following form:

$$A = \frac{x^2}{\xi^2} - \frac{y^2}{\eta^2} + L, \quad \mathbf{B} = \left\{\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0\right\},$$
$$\mathbf{v} = \left\{\frac{\dot\xi}{\xi}x, \frac{\dot\eta}{\eta}y, 0\right\}, \quad \rho = \frac{1}{\xi\eta}. \tag{3}$$

Here $\mathbf{B}$ and $\mathbf{v}$ are the vectors of the magnetic field intensity and of the velocity; $\xi = \xi(t)$, $\eta = \eta(t)$ and $L = L(t)$ are certain functions of the time with initial values $\xi(0) = 1$, $\eta(0) = 1$, and $L(0) = L_0$.

For an ideally conducting plasma, the quantity L, which does not depend on the coordinates, is likewise independent of the time, i.e., $L = L_0$. This follows, for example, from the absence from such a plasma, in a coordinate system tied to the plasma, of an electric field $\mathbf{E} = -c^{-1}\partial A/\partial t$, and from the fact that when $x = 0$ and $y = 0$ the plasma is at rest. However, the solution (3) can be generalized also to the case of a limited plasma conductivity. For an incompressible liquid, this was demonstrated by Uberoi[5] (see also[6]).

In the general case, when the conductivity is limited it is necessary to use the equation

$$\partial\mathbf{B} / \partial t = \operatorname{rot}[\mathbf{vB}] + \nu_m\Delta\mathbf{B}, \tag{4}$$

where $\nu_m = c^2/4\pi\sigma$ is the magnetic "viscosity," and the conductivity σ, generally speaking, is a function of the time. For the two-dimensional problem Eq. (4) reduces to the following equation for the vector potential $A = A(x, y, t)$:

$$\partial A / \partial t + \mathbf{v}\nabla A - \nu_m\Delta A = 0. \tag{5}$$

It is clear therefore that if the potential A from the solution (3) is a solution at $\nu_m = 0$, then the potential

$$A' = A + 2\int_0^t \nu_m\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right)dt \tag{6}$$

will be a solution of the problem for $\nu_m \neq 0$. This denotes simply that in the case of limited conductivity there appears in the plasma an additional electric field

$$E_\sigma = -\frac{1}{c}\frac{\partial}{\partial t}(A' - A) = -\frac{2\nu_m}{c}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right) = \frac{j}{\sigma}. \tag{7}$$

We took into account here the fact that, by virtue of the solution (3), the current density j is

$$j = -\frac{c}{4\pi}\Delta A = -\frac{c}{2\pi}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right). \tag{8}$$

Thus, in a plasma with limited conductivity the quantity L in the solution (3) is a function of the time and is equal to

$$L = L_0 + 2\int_0^t \nu_m\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right)dt. \tag{9}$$

The functions $\xi(t)$ and $\eta(t)$ were defined for incompressible and compressible media in[1] and[2], respectively. For an incompressible liquid[1] we have

$$\xi(t) = 1/\eta(t) = e^{s(t)}, \tag{10}$$

and $s(t)$ is determined by the equation

$$\dot{s}^2 = 2 - (2 - \dot{s}_0^2)/\operatorname{ch} 2s, \tag{11}$$

where $\dot{s}_0 = \dot{s}(0)$ is the initial value of the derivative ds/dt and $s(0) = 0$. In[1] are presented numerical solutions of Eq. (11) for values of $\dot{s}_0$ equal to 0.01, 0.05, 0.2, 0.5, 1, 1.414, 3.5, and 5. Asymptotically, at large values of t, the function $s(t)$ increases linearly with time:

$$s = t\sqrt{2} + a, \tag{12}$$

where a is a constant that depends on the initial value of $\dot{s}_0$ and is determined by a numerical solution.

For a compressible medium, the quantities ξ and η are defined by the equations[2]

$$\ddot{\xi} = -\eta\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right), \quad \ddot{\eta} = \xi\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right). \tag{13}$$

Numerical solutions of these equations for a number of values of $\dot{\xi}_0$ and $\dot{\eta}_0$ are given in[2]. An important difference between these solutions and the solutions for an incompressible liquid is the fact that one of the quantities, ξ or η, namely that corresponding to the algebraically smaller value of the initial velocity $\dot{\xi}_0$ or $\dot{\eta}_0$, vanishes after a finite time interval t_c. With this, the second of the quantities ξ and η remains finite at $t = t_c$. By suitable choice of the spatial coordinates (interchange of the axes $x \to y'$ and $y \to x'$) it is possible to make the function $\eta(t)$ vanish at $t = t_c$. Then the instant t_c and the value $\xi_c = \xi(t_c)$ depend on the initial values of $\dot{\xi}_0$ and $\dot{\eta}_0$ and are determined by a numerical solution of (13).

As follows from (3), the vanishing of $\eta(t)$ corresponds to a singularity in the potential, the field intensity, the velocity, and the density. The principal terms in the functions $\xi(t)$ and $\eta(t)$ near the singularity $t = t_c$ are

$$\xi = \xi_c + \ldots, \qquad \eta = (9/2\xi_c)^{1/3}(t_c - t)^{2/3} + \ldots \tag{14}$$

A characteristic property of the solution (3) is that any function of the quantities x/ξ and y/η remains constant at points moving together with the plasma:

$$\frac{d}{dt}\Phi\left(\frac{x}{\xi}, \frac{y}{\eta}\right) = \frac{\partial\Phi}{\partial t} + v_x\frac{\partial\Phi}{\partial x} + v_y\frac{\partial\Phi}{\partial y} = 0. \tag{15}$$

In other words, any function $\Phi(x/\xi, y/\eta)$ defines a Lagrangian surface. This property is possessed by the potential A in the case of ideal conductivity and the difference $A - L(t)$ in the case of finite conductivity, and also by the quantity of the plasma contained in an arbitrary elliptical cylinder with boundary $x^2/\xi^2 + y^2/\eta^2 = \text{const}$. If at the initial instant $(t = 0,\ \xi(0) = 1,\ \eta(0) = 1)$ the plasma has occupied the volume of a circular cylinder of unit radius with boundary $x^2 = y^2 = 1$, then at any subsequent instant of time the plasma will be contained in an elliptical cylinder with boundary

$$x^2/\xi^2 + y^2/\eta^2 = 1. \tag{16}$$

Let us consider now the conditions that must be satisfied on the boundary of a plasma occupying this volume, and moving in accordance with the solution (3). The problem is to separate in the solution (3) the field due to the external sources, and to clarify the properties of this field. To this end we calculate from the known current density (8) inside the cylinder the potentials A_j^e and A_j^i and their derivatives should be continuous on the boundary (16) and satisfy the equations

$$\Delta A_j^e = 0, \quad \Delta A_j^i = 2\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right). \tag{17}$$

Proceeding in analogy with[1], we can show that the potentials satisfying these conditions are

$$A_j^i = \frac{\eta - \xi}{\xi\eta}\left(\frac{x^2}{\xi} + \frac{y^2}{\eta}\right) + C_j(t),$$

$$A_j^e = \frac{\eta^2 - \xi^2}{\xi\eta}\left[\frac{x^2}{\xi^2 + \lambda'} + \frac{y^2}{\eta^2 + \lambda'} + \frac{1}{2}L(t, \lambda')\right] + C_j(t), \tag{18}$$

where

$$\lambda' = \lambda + [(\xi^2 + \lambda)(\eta^2 + \lambda)]^{1/2},$$

$$L(t, \lambda') = \ln\frac{2\lambda' + \xi^2 + \eta^2}{(\xi + \eta)^2}, \tag{19}$$

and $\lambda = \lambda(x, y, t)$ is defined by

$$\frac{x^2}{\xi^2 + \lambda} + \frac{y^2}{\eta^2 + \lambda} = 1. \tag{20}$$

The function $C_j(t)$, which appears in (18) as a constant of integration, can be determined by considering the asymptotic behavior of the potential A_j^e at large distances from the plasma column. When $x^2 + y^2 \gg \xi^2 + \eta^2$ we obtain from (19) and (20) that $\lambda \to x^2 + y^2$ and $\lambda' \to 2\lambda$, and

$$A_j^e = \frac{\eta^2 - \xi^2}{2\xi\eta}\left[1 + \ln\frac{4(x^2 + y^2)}{(\xi + \eta)^2}\right] + C_j(t). \tag{21}$$

On the other hand, at distances from the column that are large compared with its transverse dimensions, the potential should equal

$$A_j^e = \frac{1}{c}\int\frac{j\,dv}{R} = -\frac{\eta^2 - \xi^2}{2\xi\eta}\oint\frac{dl}{R_0} \tag{22}$$

where the current density j is taken from (8), $\pi\xi\eta$ is the area of the column cross section, R_0 is the distance from the point of observation to the point on the axis of the column, and the last integral is taken along a counter made up of the plasma column and the conductors closing its circuit. It is assumed here that the length of the column and the distance to the closing conductors are larger than all the distances of interest to us. Under these assumptions, just as for a linear conductor, we have

$$\oint\frac{dl}{R_0} = -\ln(x^2 + y^2) + L_j, \tag{23}$$

where the constant L_j is determined by the total geometry of the current circuit, and obviously does not depend on the time.

From a comparison of (21) and (22) and (23) it follows that the function $C_j(t)$ should be equal to

$$C_j(t) = \frac{\eta^2 - \xi^2}{\xi\eta} \ln \frac{\xi + \eta}{2} - \frac{\eta^2 - \xi^2}{2\xi\eta}(1 + L_j). \tag{24}$$

From this and from (18) we get

$$A_j{}^i = \frac{\eta - \xi}{\xi\eta}\left(\frac{x^2}{\xi} + \frac{y^2}{\eta}\right) + \frac{\eta^2 - \xi^2}{\xi\eta}\left(\ln\frac{\xi + \eta}{2} - \frac{1 + L_j}{2}\right). \tag{25}$$

Defining now the potential difference A and A_j^i from (3) and (25), we can readily obtain the potential $A_e = A - A_j^i$, produced by currents that are external relative to the plasma cylinder:

$$A_e = \frac{1}{\xi\eta}(x^2 - y^2) + \frac{\xi^2 - \eta^2}{\xi\eta}\left(\ln\frac{\xi + \eta}{2} - \frac{1 + L_j}{2}\right) + L. \tag{26}$$

Thus, the motion of a plasma described by the solution (3) occurs in an external field with potential (26). This potential depends in a strictly defined manner on the time via the functions $\xi(t)$ and $\eta(t)$, and the dependence on the spatial coordinates corresponds to the vicinity of the null line of the magnetic field. Such a field A_e can be realized by various means, for example by using two conductors with equal and parallel currents, located symmetrically with respect to the plasma column; the necessary time dependence can be ensured here by suitably varying the current in the conductors and the distance between them (see[4]).

Under laboratory conditions it is easiest to realize a potential (26) with the aid of a magnetic quadrupole (four conductors symmetrically arranged with respect to the plasma column, with equal currents of alternating directions), and a homogeneous external electric field. In this case the current I in the quadrupole conductors and the external electric field E should vary in accordance with

$$I = \frac{I_0}{\xi\eta}, \quad E = -\frac{A_0}{ct_0}\frac{\partial A'}{\partial t}, \tag{27}$$

where

$$A'(t) = \frac{\xi^2 - \eta^2}{\xi\eta}\left(\ln\frac{\xi + \eta}{2} - \frac{1 + L_j}{2}\right) + L \tag{28}$$

and in the general case of limited conductivity $L = L(t)$ is a function of the time (9). The constant A_0 is connected with the value of the initial current I_0 in the conductors of the quadrupole by the relation $A_0 = 4I_0e^{-1}(r_0/R_0)^2$, where r_0 is the radius of the plasma column and R_0 is the distance of the conductors from the axis of the column (it is assumed that $R_0 \gg r_0$).

Expressions (26)—(28) are valid for both a compressible and an incompressible medium using the corresponding relations $\xi(t)$ and $\eta(t)$. We note that in the case of an incompressible liquid the current in the quadrupole should be maintained constant (see (10), (27)) and the considered self-similar motion is due entirely to the external electric field, which varies in time like

$$E = -\frac{4A_0}{ct_0}\left[\operatorname{ch} 2s\left(\ln \operatorname{ch} s - \frac{1 + L_j}{2}\right) + \operatorname{sh}^2 s\right]\dot{s} + \frac{4A_0\nu_m}{cr_0^2}\operatorname{sh} 2s. \tag{29}$$

For a compressible medium, the current in the quadrupole and the external field near the singularity $t = t_c$ should vary in accordance with

$$I = I_0(9/2)^{1/3}[\xi_c^2(t_c - t)]^{-2/3},$$

$$E = -\frac{A_0}{ct_0}\left[3\left(\frac{2}{9}\right)^{4/3}\xi_c^{2/3}(t_c - t)^{-5/3}\left(\ln\frac{\xi_c}{2} - \frac{1 + L_j}{2}\right) - \frac{2t_0}{r_0^2}\nu_m\left(\frac{9}{2}\xi_c\right)^{-2/3}(t_c - t)^{-4/3}\right]. \tag{30}$$

As is clear from (26)—(30), the singularity as $\eta \to 0$ in the considered self-similar solution can be attained only as a result of an unlimited increase of the potential of the external field, i.e., at infinitely large external currents and electric fields. Thus, the discussed plasma motions actually have the character of cumulation due to external forces.

It follows from the properties of the obtained solutions that the relative current density j/n (n—plasma density) can be made as large as desired if the external fields are sufficiently strong. This realizes the conditions for dynamic dissipation of the magnetic field[3,4], a dissipation which does not depend on the plasma conductivity and takes place, in particular, also in a collisionless plasma. In this process, the plasma loses its magnetohydrodynamic property of "freezing" the external magnetic field, and leads in our particular case, at sufficiently large values of j/n, to an acceleration of the plasma particles by the external electric field.

We note that the magnetic field outside and inside the plasma column, which was calculated above, is continuous and does not produce surface tension on the plasma boundary. Therefore, for a bounded plasma column in a compressible medium it is necessary to stipulate, strictly speaking, continuity of the gas pressure on the plasma surface. For example, in the adiabatic process, by virtue of the solution (3), the pressure in the column is $p \propto \rho^\gamma = (\xi\eta)^{-\gamma}$, and consequently the external pressure (for example the pressure of the neutral gas) should vary in similar fashion. Within certain limits, however, this difficult-to-realize condition does not play a major role in the case of a cold plasma, when the effect of the gas plasma pressure can be neglected.[1)]

Besides the boundary conditions considered above, the discussed self-similar solutions imply also a strictly defined, namely linear (see (3)), distribution of the initial plasma velocity in space. This raises the question of establishment of the self-similar regime. It is natural to assume that the real motion of a plasma which is at rest at t = 0 will differ little from self-similar motion corresponding to small initial velocities $\dot{\xi}_0$ and $\dot{\eta}_0 \ll 1$.

In the general case of arbitrary $\dot{\xi}_0$ and $\dot{\eta}_0$, as indicated in[2], the discussed self-similar solution should be regarded as the principal term of a general solution corresponding to the developing plasma flow in the vicinity of the axes x = 0 and y = 0. In this case the boundary-value problem considered above becomes meaningless, and the ensuing limitations on the external potential vanish. In such a general case, the self-similar solution (3) is apparently established in the nearest vicinity of the null line for arbitrary

1)The initial pressure of the plasma is an independent parameter, and can in principle be chosen to be sufficiently small.

variations of the external potential, as considered in[3,4].

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