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# *MOTION OF COULOMB POLES OVER THE J-PLANE IN QUANTUM ELECTRODYNAMICS*

L. N. LIPATOV

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences

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The asymptotic behavior for large energies and fixed unphysical momentum transfers: $Q^2 - (m + \mu)^2 \sim \alpha m^2$, of the $\mu^+\mu^- \to e^+e^-$ annihilation amplitudes is calculated by means of direct summation of the essential contributions of Feynman diagrams. The asymptotic expression is used to obtain explicit expressions for the partial waves with complex total angular momenta $J \sim (\alpha)^{1/2} \ll 1$ in the $Q^2$-channel, thus allowing a determination of the motion of the Coulomb poles situated farthest to the right in the complex J-plane.

## 1. INTRODUCTION

It is known[1] that the behavior of the amplitudes in quantum field theory at large energies $S^{1/2}$ and fixed momentum transfer $(-Q^2)^{1/2}$ is related to the analytic properties of the partial wave $f_J(Q^2)$ in the J-plane of the crossed channel, where $(Q^2)^{1/2}$ plays the role of the total c.m.s. energy. It is usually assumed[2] that the only singularities in the J-plane are Regge poles and the accompanying Mandelstam branch points.

On the other hand several authors[3-7] have found a doubly logarithmic behavior of some processes in quantum electrodynamics. This problem is equivalent to the computation of the partial waves $f_J(Q^2)$ for

$$Q^2 < 0, \quad J \sim \sqrt{\alpha} \ll 1, \tag{1}$$

where $\alpha$ is the fine-structure constant.

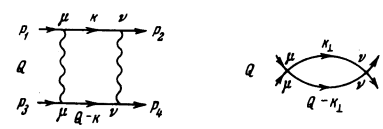
It turned out[6,7] that in the region (1) the partial waves in the $Q^2$-channel have only stationary singularities: poles in the $\mu^-e^-$ system and a square-root branch point in the $\mu^-e^+$ system. It is clear a priori that for $Q^2$ sufficiently close to $(m + \mu)^2$ (m is the electron mass, $\mu$ is the muon mass), i.e., in the unphysical region of the S-channel, this result will not be valid. The reason for this is that there are bound states in the $\mu^-e^+$ system, which unavoidably leads to the appearance of Regge poles in the J-plane, which for $Q^2$ sufficiently close to $(m + \mu)^2$ will move arbitrarily far to the right.

The motion of the Coulomb poles has been investigated in several papers (cf. e.g. [8]), however it was usually assumed that there are no other singularities in the J-planes of the partial waves. Here we compute $f_J(Q^2)$ in the region

$$|(m+\mu)^2 - Q^2| \sim \alpha m^2 \ll m^2, \qquad J \sim \sqrt{\alpha} \ll 1. \tag{2}$$

For such values of $Q^2$ the leading Coulomb pole just reaches the region $J \sim (\alpha)^{1/2}$ and the partial waves start depending on $Q^2$.

The problem of computing $f_J(Q^2)$ in the region (2) is interesting in two respects. Firstly, one gains considerable insight on the general structure of the simply logarithmic terms in the physical region $Q^2 < 0$. Secondly, the knowledge of the analytic properties of the partial waves in the J-plane could be useful in computing corrections to positronium energy levels.



a b

FIG. 1.

## 2. COMPUTATION OF THE ASYMPTOTIC BEHAVIOR OF THE $\mu^+\mu^- \to e^+e^-$ ANNIHILATION AMPLITUDE IN THE UNPHYSICAL REGION $|Q^2 - (m + \mu)^2| \sim \alpha m^2$

We shall for definiteness consider the annihilation of a pair $\mu^-\mu^+$ with momenta $p_1$ and $-p_2$, respectively, into a $e^+e^-$ pair, with momenta $-p_3$, $p_4$ at large energies. It is assumed that the muon mass $\mu$ is of the order of the electron mass m. We introduce the quantity S—the square of the c.m.s. energy:

$$S = (p_1 - p_2)^2 \gg m^2 \sim \mu^2, \tag{3}$$

and also the momentum transfers

$$Q = p_1 - p_4 \tag{3a}$$

for forward scattering, and

$$Q = p_1 + p_3 \tag{3b}$$

for backward scattering.

The immediate reason for the fact that in the region (2) the previously found expressions for the partial waves[6,7] become incorrect is the circumstance that in perturbation theory in addition to the doubly logarithmic terms

$$\sim \left(\frac{\alpha}{\pi} \ln^2 \frac{S}{m^2}\right)^n$$

there also appear large singly logarithmic terms

$$\sim \left[\alpha \sqrt{m\mu} \ln \frac{S}{m^2} \Big/ \sqrt{(m+\mu)^2 - Q^2}\right]^n,$$

which, if the conditions (2) are satisfied are just of the order of the doubly logarithmic terms. Therefore there arises the problem of summing large contributions of both types.

As a characteristic example we consider the simplest diagram of Fig. 1, a for backward annihilation. We expand the integration momentum in terms of the vectors

$p_1$ and $p_2$ by means of the use of Sudakov parameters:[3]

$$k = -\alpha p_2 + \beta p_1 + k_\perp, \quad -k_\perp^2 > 0. \tag{4}$$

As was shown in [6], the contribution of this diagram to the asymptotic behavior is expressed in terms of a two-dimensional integral corresponding to the "contracted" Polkinghorne diagram[9] (cf. Fig. 1, b):

$$F_2 = \frac{e^2}{S}\left(-\frac{\alpha}{(2\pi)^2}\right)\int_{m^2/S}^{1}\frac{d\beta}{\beta}\int_{-k_\perp^2 \ll \beta S} d^2k_\perp \times \frac{\gamma_\nu^\perp(\hat{k}_\perp + \mu)\gamma_\mu^\perp \cdot \gamma_\nu^\perp(\hat{Q}_\perp - \hat{k}_\perp + m)\gamma_\mu^\perp}{[k_\perp^2 - \mu^2][(Q+k)_\perp^2 - m^2]}. \tag{5}$$

The analytic continuation of the integral with respect to $k_\perp$ into the region (2) leads to the usual doubly logarithmic contribution from the region $-k_\perp^2 \gg m^2$, but in addition, for $Q^2 \to (m+\mu)^2$ the singularities of the integrand come close to the integration contours with respect to $k_\perp^1$ and $k_\perp^2$, which leads to the appearance of a large contribution from the integration region $-k_\perp^2 \sim m^2$. This is the simply logarithmic contribution we were looking for.

The spin structure of the expression (5) can be simplified by using the anticommutation properties of the matrices $\gamma_\nu^\perp$:

$$\gamma_\nu^\perp(\hat{k}_\perp + \mu)\gamma_\mu^\perp \cdot \gamma_\nu^\perp(\hat{Q}_\perp - \hat{k}_\perp + m)\gamma_\mu^\perp = 2(k_\perp Q - k_\perp^2)\gamma_\mu^\perp \cdot \gamma^{\mu\perp} + \mu m \gamma_\nu^\perp\gamma_\mu^\perp \cdot \gamma_\nu^\perp\gamma_\mu^\perp \tag{6}$$

(the dots between the matrices separate the electronic from the muonic $\gamma$-matrices).

Introducing the new integration variables

$$\xi = \sqrt{\frac{2\alpha}{\pi}}\ln\frac{-k_\perp^2}{m^2\beta}, \quad \eta = \sqrt{\frac{2\alpha}{\pi}}\ln\frac{1}{\beta}, \quad \rho = \sqrt{\frac{2\alpha}{\pi}}\ln\frac{S}{m^2}, \tag{7}$$

the expression (5) can be reduced to the form

$$F_2 = F_0\left(-\frac{1}{4}\right)\int_0 d\xi \int_0 d\eta \left[\theta(\xi - \eta) - 4\sqrt{\frac{\pi}{2\alpha}}\gamma(Q^2)\delta(\xi - \eta)\right], \tag{8}$$

where $F_0 = \gamma_\mu^\perp \cdot \gamma_\mu^\perp / S$ is the Born amplitude and $\gamma(Q^2)$ is the spinor "trajectory" of the Coulomb pole:

$$\gamma(Q^2) = -\frac{\alpha\sqrt{m\mu}\mathcal{P}}{\sqrt{(m+\mu)^2 - Q^2}} \qquad \mathcal{P} = \frac{1}{2}\left[1 + \frac{1}{2}\gamma_\nu^\perp \cdot \gamma_\nu^\perp\right]. \tag{9}$$

In this expression only the most singular terms as $Q^2 \to (m+\mu)^2$ have been written, leading to large simply logarithmic terms.

The asymptotic behavior of the more complicated ladder diagram (Fig. 2, a) is also determined by the "contracted" Polkinghorne diagram (Fig. 2, b); it was shown in [6] that

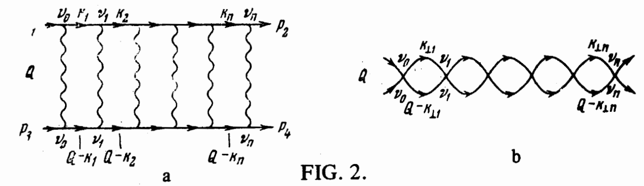
$$F_{2n} = \frac{e^2}{S}\left(-\frac{\alpha}{(2\pi)^2}\right)^n \int \prod_{i=1}^{n}\frac{d\beta_i}{\beta_i}\int\prod_{i=1}^{n} d^2k_{\perp i} \times \frac{\gamma_{\nu_n}^\perp(\hat{k}_{\perp n} + \mu)\gamma_{\nu_{n-1}}^\perp \cdots \gamma_{\nu_0}^\perp \cdot \gamma_{\nu_n}^\perp(\hat{Q} - \hat{k}_{\perp n} + m)\gamma_{\nu_{n-1}}^\perp \cdots \gamma_{\nu_0}^\perp}{[k_{\perp n}^2 - \mu^2][(Q-k_n)_\perp^2 - m^2]\cdots[k_{\perp 1}^2 - \mu^2][(Q-k_1)_\perp^2 - m^2]}, \tag{10}$$

where the integration is over the region

$$m^2/S \ll \beta_n \ll \ldots \ll \beta_2 \ll \beta_1 \ll 1, \tag{10a}$$

$$1 \gg -k_{\perp n}^2/S\beta_n \gg \ldots \gg -k_{\perp 1}^2/S\beta_1 \gg m^2/S. \tag{10b}$$

The condition (10b) appears after taking the residues of the photon propagators with respect to all $\alpha_i$'s as a condition of separating from one another the integrations with respect to the $k_{\perp i}$. If (10a) is satisfied, this condition is equivalent to the requirement that $k_i^2 \approx k_{\perp i}^2$. In the case where all $-k_{\perp i}^2 \sim m^2$, this condition is automatically



a FIG. 2. b

satisfied owing to the inequality (10a). In the other extreme situation, when all $-k_{\perp i}^2 \gg m^2$ the change of variables $-k_{\perp i}^2/S\beta_i \equiv \alpha_i$ reduces the inequality to the usual restriction[6] on the region of doubly logarithmic behavior:

$$\frac{m^2}{S} \ll \alpha_1 \ll \alpha_2 \ll \ldots \ll \alpha_n \ll 1. \tag{11}$$

As before, two regions: $-k_{\perp i}^2 \gg m^2$ and $-k_{\perp i}^2 \sim m^2$ give large contributions to the integration with respect to every $k_{\perp i}$. Introducing the new variables (7) one can reduce the expression (10) to the form

$$F_{2n} = F_0\left(-\frac{1}{4}\right)^n \int \prod_{i=1}^{n} d\xi_i\, d\eta_i \times \left[\theta(\xi_i - \eta_i) - 4\sqrt{\frac{\pi}{2\alpha}}\gamma(Q^2)\delta(\xi_i - \eta_i)\right], \tag{12}$$

where the integration is over the region

$$0 < \eta_1 < \eta_2 < \ldots < \eta_n < \rho, \tag{12a}$$

$$0 < \xi_1 < \xi_2 < \ldots < \xi_n < \rho. \tag{12b}$$

It is easy to verify that in the general case the class of diagrams giving a large contribution of the type illustrated above coincides with the class of diagrams giving a doubly logarithmic contribution.[6,7] A distinction appears only in the integration over the ladder variables $\xi_i$ and $\eta_i$, where in addition to the region $-k_{\perp i}^2 \gg m^2$ the region $-k_{\perp i}^2 \sim m^2$ also gives a large contribution. This fact can be taken into account by means of introducing a term $\sim\gamma\delta(\xi - \eta)$ into the kernel of the integral equation[7] for F, the amplitude for backward $\mu^+\mu^-$-annihilation (cf. Eq. (12)):

$$F = F_0 \exp(-\rho^2) J(\rho, \rho),$$

$$J(\xi, \eta) = 1 - \frac{1}{4}\int_0^\xi d\xi_1 \int_0^\eta d\eta_1 \left[\theta(\xi_1 - \eta_1) - 4\sqrt{\frac{\pi}{2\alpha}}\gamma(Q^2)\delta(\xi_1 - \eta_1)\right] e^{\eta_1(\xi - \xi_1)} J(\xi_1, \eta_1). \tag{13}$$

In a similar manner one modifies the equation for forward annihilation

$$M = F_0 J(\rho, \rho),$$

$$J(\xi, \eta) = 1 + \frac{1}{4}\int_0^\xi d\xi_1 \int_0 d\eta_1 \left[\theta(\xi_1 - \eta_1) + 4\sqrt{\frac{\pi}{2\alpha}}\beta(Q^2)\delta(\xi_1 - \eta_1)\right] J(\xi_1, \eta_1), \tag{14}$$

where

$$\beta(Q^2) = \frac{\alpha\sqrt{m\mu}\,\mathcal{P}_1}{\sqrt{(m+\mu)^2 - Q^2}}, \quad \mathcal{P}_1 = \frac{1}{2}\left[1 - \frac{1}{2}\gamma_\nu^\perp \cdot \gamma_\nu^\perp\right]. \tag{15}$$

The integral equations (13) and (14) can be easily solved by rewriting them in differential form,[1)] and looking for

1) The author is indebted to V. G. Gorshkov for calling to his attention the possibility of writing the integral equations for $J(\xi, \eta)$ in the simple form (13) and (14). The integral equations which the author had originally obtained were fairly involved, although they led to the same differential equations.

a solution by the method exposed in [6, 7]:

$$F=F_0\exp\left(-\frac{\rho^2}{2}\right)\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}dl\frac{e^{\rho l}}{D_{-1/4}(l)/D_{-5/4}(l)-\sqrt{\pi/2\alpha}\,\gamma(Q^2)},\quad(16)$$

$$M=F_0\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}dl\frac{e^{\rho l}}{1/2(l+\sqrt{l^2-1})-\sqrt{\pi/2\alpha}\,\beta(Q^2)},\quad(17)$$

where $D_p(l)$ are parabolic cylinder functions.[10]

The factor $\exp(-\rho^2/2)$ in Eq. (16) appears due to the fact that, following Abrikosov,[4] we cut off the infrared-divergent integrals by introducing a finite virtuality of the external particles $p_i^2-m^2=\Delta m^2\sim m^2$. In the sequel we omit this factor corresponding to a transition to the effective amplitude with the emission of an arbitrary number of quanta with $-k_\perp^2\ll m^2$,[7] or by attributing a mass to the photon, $\lambda\sim m$.

## 3. THE PARTIAL WAVES IN THE $Q^2$-CHANNEL FOR COMPLEX J

A convenient formalism for the computation of the helicity amplitudes $F^{J\pm}_{\lambda_1\lambda_3;\lambda_2\lambda_4}(Q^2)$, $M^{J\pm}_{\lambda_1\lambda_4;\lambda_2\lambda_3}$ with total angular momentum J and parity $\pm(-1)^J$ was proposed by Gell-Mann et al.[11] Using the formulas given in that paper it is easy to find expressions for the partial waves of the amplitude $F_0^J$ for $\mu^+\mu^-$-annihilation in the backward direction in the Born approximation:

$$F^{J+}_{0\,\lambda_2\lambda_4;\lambda_1\lambda_3}=0,\quad F^{J-}_{0\,1/2\,1/2;1/2\,1/2}=\frac{e^2}{2p^2}\delta_{J0},$$

$$J^{-1/2}F^{J-}_{0\,1/2\,-1/2;1/2\,1/2}=F^{J-}_{0\,1/2\,-1/2;1/2\,-1/2}=-\frac{e^2}{2p^2}\frac{1}{J},\quad(18)$$

where p is the momentum in the c.m.s. of the $Q^2$-channel. In deriving (18) use has been made of the smallness of J and p in the region (2).

It can be seen from Eqs. (18) that only one among the amplitudes, namely the physical $Q^2$-channel amplitude $F^{J-}_{\frac12\frac12;\frac12\frac12}$ is not analytic in the complex J plane. This is a well-known[11] general property of lowest order perturbation theory. If one takes into account radiative corrections this amplitude becomes an analytic function.[8]

For the computation of the helicity amplitudes for the case of the more complicated spin structure (16) it is useful to note that the operator $\mathcal{P}$ introduced in Eq. (9) when multiplied by the Born term behaves like a projection:

$$\mathcal{P}^2=\mathcal{P},\quad(19)$$

which is an immediate consequence of the commutation properties of the $\gamma_\mu^\perp$-matrices. Therefore it suffices to determine the helicity amplitudes for only two spin combinations, namely $\gamma_\mu^\perp\gamma_\nu^\perp\cdot\gamma_\mu^\perp\gamma_\nu^\perp$ and $\gamma_\nu^\perp\cdot\gamma_\nu^\perp$. The computation of these helicity amplitudes shows that in the region (2) the operator $\mathcal{P}$ is the unit matrix. After this use of the formalism developed in [11] leads in a simple manner to the helicity amplitudes for backward annihilation

$$F^{J+}_{\lambda_2\lambda_4;\lambda_1\lambda_3}=0,\quad F^{J-}_{1/2\,1/2;\,1/2\,1/2}$$

$$=\frac{e^2}{2p^2}\left\{1-J\Big/\left[\sqrt{\frac{2\alpha}{\pi}}\frac{D_{-1/4}(\sqrt{\pi/2\alpha}\,J)}{D_{-5/4}(\sqrt{\pi/2\alpha}\,J)}-\gamma(Q^2)\right]\right\},$$

$$J^{-1/2}F^{J-}_{1/2\,-1/2;1/2\,1/2}=F^{J-}_{1/2\,-1/2;1/2\,-1/2}=-\frac{e^2}{2p^2}\Big/\left[\sqrt{\frac{2\alpha}{\pi}}\frac{D_{-1/4}(\sqrt{\pi/2\alpha}\,J)}{D_{-5/4}(\sqrt{\pi/2\alpha}\,J)}-\gamma(Q^2)\right].\quad(20)$$

Comparing the expressions for $F^{J-}_{\frac12\frac12;\frac12\frac12}$ and $F^{J-}_{0\frac12\frac12;\frac12\frac12}$ we note that the $\delta_{J0}$ singularity disappears if one takes into account radiative corrections:

$$F^{J-}_{0\,1/2\,1/2;1/2\,1/2}=\lim_{\alpha\to0}F^{J-}_{1/2\,1/2;1/2\,1/2}.\quad(21)$$

In addition, for J = 0 the physical amplitude $F^{J-}_{\frac12\frac12;\frac12\frac12}$ coincides with the Born amplitude $F^{J-}_{0\frac12\frac12;\frac12\frac12}$, i.e., the residues of all Regge poles vanish for $J\to0$.

The helicity amplitudes (20) have the correct analytic properties in the $Q^2$-plane, the discontinuity across the cut in the right-hand side being determined by the two-particle unitarity condition in the $Q^2$-channel:

$$\operatorname{Im}F^J_{\lambda_2\lambda_4;\lambda_1\lambda_3}=\sum_{L_1,L_3}\frac{2m\mu p}{8\pi\sqrt{t}}F^J_{\lambda_2\lambda_4;L_1L_3}F^{J+}_{L_1L_3;\lambda_1\lambda_3}.\quad(22)$$

Conversely, the relations (22) could be used for an alternative derivation of Eqs. (20).

Similar computations for the forward annihilation amplitude lead to the following expressions:

$$M^{J-}_{\lambda_2\lambda_3;\lambda_1\lambda_4}=0,\quad M^{J+}_{1/2\,1/2;\,1/2\,1/2}$$

$$=-\frac{e^2}{2p^2}\left[1-\frac{J}{1/2(J+\sqrt{J^2-2\alpha/\pi})-\beta(Q^2)}\right].\quad(23)$$

In this case one can also check the validity of two-particle unitarity of the type (22).

In conclusion of this section we list formulas relating the amplitudes $F^{J\pm}$ and $M^{J\pm}$ introduced above with the amplitudes $f^{j\pm}_{\lambda\lambda'}$ describing transitions between states with spin projections $\lambda\lambda'=0$ or 1 on the momentum direction in the c.m.s., which were used in [6, 7]:

$$f_{00}{}^{j\pm}=4pF^{J\pm}_{1/2\,1/2;1/2\,1/2},\quad J_{01}{}^{j\pm}=-4pF^{J\pm}_{1/2\,-1/2;1/2\,1/2},\quad f_{11}{}^{j\pm}=4pF^{J\pm}_{1/2\,-1/2;\,1/2\,-1/2}.\quad(24)$$

In order to verify the relations (24) it suffices to find the limit of $f^{J\pm}_{\lambda\lambda'}$ for $Q^2\to(m+\mu)^2$.

We also note a misprint in Eq. (27) of [6]. The sign in front of the term $2j/(j+(j^2-\gamma^2)^{1/2})$ should be changed. In addition, in Eq. (24) $f^\pm_{\lambda\lambda'}$ should be replaced by $f^{j\pm}_{\lambda\lambda'}$.

## 4. THE MOTION OF COULOMB POLES IN THE J-PLANE

In nonrelativistic quantum mechanics the Coulomb pole farthest to the right of the partial wave amplitude $f_l(E)$ moves to the point $l=-1$ for large energies E. Owing to the shift discovered by Azimov,[12] the singularity in $l$ moves into the point $J=l+s$.

On the other hand the asymptotic formulas (16) and (17) allow us to compute correctly only the state of even signature, since the presence of the signature factor in the Sommerfeld–Watson integral describing the contribution to the asymptotic behavior of the negative signature state leads to the effective loss of one logarithm of S. This means that in order to find the odd-signature partial waves it is necessary to know the asymptotic behavior of the amplitudes F and M to a higher degree of accuracy. Therefore Eqs. (20) and (23) describe only the motion of one Coulomb pole corresponding to bound states, or "virtual bound states"[13] of the $e^{\pm}\mu^{\pm}$ system with quantum numbers:

$$J = 2n, \quad n = 1, 2, \ldots; \quad l = J - 1, \quad n_r = 0 \tag{25}$$

($n_r$ is the radial quantum number).

We first consider the motion of the Coulomb pole in the J-plane of the $\mu^- e^+$ system for the $Q^2$-channel. Its trajectory can be obtained from the vanishing of the denominator in the partial waves (23):

$$J = \beta(Q^2) + \alpha / 2\pi\beta(Q^2), \tag{26}$$

where

$$\beta(Q^2) = \alpha\sqrt{m\mu} / \sqrt{(m + \mu^2) - Q^2}. \tag{26a}$$

In the physical region of the S-channel, i.e., for $Q^2 < 0$ the Regge pole is situated on the second sheet of the J-plane, to the right of the branch-point $J = (2\alpha/\pi)^{1/2}$, and therefore does not influence the asymptotic behavior of the process of forward $\mu^+\mu^-$ annihilation into an $e^+e^-$ pair.[6] As $Q^2$ increases the Coulomb pole moves to the left, reaches the point $J = (2\alpha/\pi)^{1/2}$ exits via the cut onto the first sheet of the J-plane, and as $Q^2$ increases further it reaches the points J = 2, 4, ..., leading to bound states with the quantum numbers (25).

In the $Q^2$-channel of the $\mu^- e^-$ system it can be seen from Eq. (23) that there are an infinity of Regge poles. Their trajectories can be found by solving the following equation

$$\sqrt{\frac{2\alpha}{\pi}}D'_{-1/4}\left(\sqrt{\frac{\pi}{2\alpha}}J\right)\Big/ D_{-5/4}\left(\sqrt{\frac{\pi}{2\alpha}}J\right) = \gamma(Q^2), \tag{27}$$

where

$$\gamma(Q^2) = -\alpha\sqrt{m\mu} / \sqrt{(m + \mu)^2 - Q^2}. \tag{27a}$$

In the physical region of the S-channel, i.e., for $Q^2 < 0$, the quantity $\gamma(Q^2)$ is of the order $\alpha \ll 1$, i.e., the position of the Regge poles is determined by the zeroes of $D_{-1/4}(l)$. These zeroes are situated between the bisectors of the second and third coordinate quadrants and asymptotically approach these bisectors (cf. Fig. 3). Making use of the asymptotic behavior of the functions $D_p(l)$[10] one can determine the positions of the zeroes:

$$|l_n| = [2(^1/_2\varphi_n + \pi/4 + 2\pi n)/\sin 2\varphi_n]^{1/2}$$
$$\text{for } |n| \gg 1, \quad n = -1, -2 \ldots,$$

$$\cos 2\varphi_n = |l_n|^{-2}\left(\ln|l_n| - 2\ln\frac{\sqrt{2\pi}}{\Gamma(^1/_4)}\right), \quad l_n \equiv |l_n| e^{i\varphi_n}, \tag{28}$$

As $Q^2$ increases the function $\gamma(Q^2)$ increases in absolute value, going to $-\infty$ as $Q^2 \to (m + \mu)^2$. In this case the position of the Regge poles is obviously determined by the zeroes of $D_{-5/4}(l)$, which are situated in the sector $|\varphi| < {}^3/_4\pi$ and tend asymptotically to the bisectors of the second and third quadrants (cf. Fig. 3)

$$|l_n| = [2(-{}^3/_2\varphi_n + 5\pi/4 + 2\pi n)/\sin 2\varphi_n]^{1/2} \text{ for } |n| \gg 1.$$
$$n = -1, -2, \ldots,$$

$$\cos 2\varphi_n = |l_n|^{-2}\left(-3\ln|l_n| - 2\ln\frac{\sqrt{2\pi}}{\Gamma(^5/_4)}\right). \tag{29}$$

The trajectories of the various Regge poles,

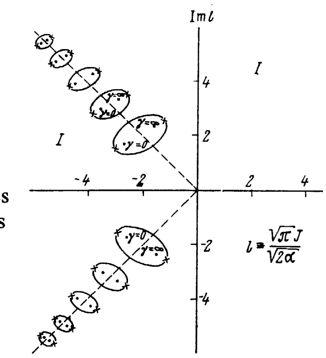



FIG. 3. The J-plane for the partial wave $F^J(Q^2)$. The closed curves separate the regions within which the Regge poles $J = J_i(\gamma)$ move around for all complex values of $\gamma$. The dots denote the positions of these poles for $\gamma = 0$ and $\gamma = \infty$. The crosses denote points where these poles "collide".

$J = J_i(Q^2)$ are different branches of a single non-schlicht analytic function $J = J(Q^2)$. This function has an infinite sequence of square-root branch points, corresponding to the collision of neighboring Regge poles:

$$\frac{d}{dJ}\sqrt{\frac{2\alpha}{\pi}}\frac{D'_{-1/4}(\sqrt{\pi/2\alpha}\,J)}{D_{-5/4}(\sqrt{\pi/2\alpha}\,J)} = 0. \tag{30}$$

In Fig. 3 the "collision" points of Regge poles are denoted by crosses. Utilizing the asymptotic behavior of the functions $D_p(l)$ it is easy to find the positions $\gamma^{a,b}$ of the branch points:

$$\sqrt{\frac{\pi}{2\alpha}}|\gamma_n^a| = \left[2\left(-\frac{7}{2}\varphi_n^a + \frac{\pi}{4} + 2\pi n\right)\Big/\sin 2\varphi_n^a\right]^{1/2}, \quad n \gg 1,$$

$$\cos 2\varphi_n^a = \left[-7\ln\left[|\gamma_n^a|\sqrt{\frac{\pi}{2\alpha}}\right] - 2\ln\frac{\sqrt{2\pi}}{\Gamma(^5/_4)}\right]\Big/\frac{\pi}{2\alpha}|\gamma_n^a|^2,$$
$$n = -1, -2, \ldots,$$

$$\sqrt{\frac{\pi}{2\alpha}}|\gamma_n^b| = \frac{1}{4}\left[\frac{-^1/_2\sin 2\varphi_n^b}{^5/_2(-\varphi_n^b + \pi) + \pi/4 + 2\pi n}\right]^{1/2},$$

$$\cos 2\varphi_n^b = \left[-5\ln\left(4\sqrt{\frac{\pi}{2\alpha}}|\gamma_n^b|\right) - 2\ln\left(\frac{1}{2}\frac{\sqrt{2\pi}}{\Gamma(^1/_4)}\right)\right]\left(4\sqrt{\frac{\pi}{2\alpha}}|\gamma_n^b|\right)^2. \tag{31}$$

One can verify that the Riemann surface of the function $J = J(\gamma)$ consists of a first sheet with excised sector $^3/_4\pi < \varphi < {}^5/_4\pi$, covered by an infinite number of branch cuts and an infinity of second sheets, each of them being connected to the first sheet through one branch cut (cf. Fig. 4). As it moves over the first sheet of the $\gamma$-plane, the trajectory $J = J(\gamma)$ moves along the region I of Fig. 3. To any second sheet of the $\gamma$ plane there corresponds its own pole in the J-plane, which, as it moves over the second sheet, encircles a certain region (in Fig. 3 the positions of the poles for $\gamma = 0$ and for $\gamma = \infty$, (28) and (29), respectively, is indicated inside these re-

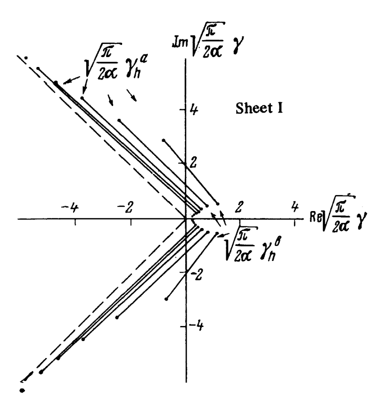



FIG. 4. The first sheet of the Riemann surface of the function $J = J(\gamma)$. The position of the square-root singularities $\gamma_n^{a,b}$, is given by Eq. (31).

gions). Thus there is a one-to-one correspondence between the points of the Riemann surface of $\gamma$ and the points of the J-plane.

We now investigate what happens with the singularities of the partial waves (20) under analytic continuation from the region $Q^2 < (m + \mu)^2$ of the first sheet of the $Q^2$-plane into the region $Q^2 < (m + \mu)^2$ of the second sheet. Under this continuation $\gamma(Q^2)$ varies from negative to positive values, and for large positive $\gamma > 0$ Eq. (27) acquires a new solution, in addition to (29):

$$J = \gamma(Q^2) + O(\alpha / \gamma), \qquad (32)$$

i.e., for $Q^2$ sufficiently close to $(m + \mu)^2$, J can reach the points J = 2, 4 leading to "virtual" bound states[13] with quantum numbers (25), for electron-$\mu^-$ meson scattering. However, as can be seen from Fig. 4, this analytic continuation leads us onto the first sheet of the $\gamma$-plane for the trajectory $J = J(\gamma)$, necessarily coming from one of the second sheets, to each of which corresponds its own pole in the J-plane. Thus, depending on the path in the $Q^2$-plane, each given Regge pole $J_i(Q^2)$, in moving from the region $Q^2 < (m + \mu)^2$ of the physical sheet of the $Q^2$-plane into the region $Q^2 < (m + \mu)^2$ of the second sheet, can be continuously moved to the physical points J = 2, 4, ..., leading to "virtual bound states"[13] in the $e^-\mu^-$ system with the quantum numbers (25). A similar situation, when to one bound state or resonance there correspond several Regge pole might also occur in the theory of strong interactions.

In conclusion we discuss the question whether it is possible to compute corrections to the positronium energy levels with the help of formulas of the type (26) and (27). It is known[14] that the principal corrections to the nonrelativistic expression for the bound states of the $\mu^-e^+$ system are of the order $m\alpha^4$, whereas (26) formally yields a correction $\sim m\alpha^3$. This, of course, is not contradictory, since Eqs. (26) and (27) are valid only for $J \sim (\alpha)^{1/2}$. Therefore, if one adds to the denominators (23) an expression of the type $\alpha f(J)$, where f(J) is a function analytic at J = 0 satisfying the only condition

$$f(J) = 1/2\pi J, \quad J = 2, 4, \ldots, \qquad (33)$$

then, without modifying $F^J$ for $J \sim (\alpha)^{1/2}$ we can achieve the required order of the corrections to the energy levels. At the same time the introduction of $\alpha f(J)$ is equivalent to a summation of singly logarithmic terms $\sim \alpha^n \ln (S/m^2)$. Therefore the requirement that the corrections to the nonrelativistic expression of the energy of the bound states be of order $\alpha^4 m$ imposes a severe restriction of the type (33) on the structure of the singly logarithmic asymptotic behavior.

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