ON SOLITONS AND THE EIGENVALUES OF THE SCHRÖDINGER EQUATION

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Some of the regularities in the formation of stationary solitary waves (solitons) in nonlinear dispersive media in which the wave processes can be described by the Korteweg–de Vries equation (1.1) are investigated. The method for determining the amplitudes of the solitons developed in the paper can also be used as a rather effective method for calculating the energy levels in a one-dimensional potential well with a potential satisfying condition (4.2). In the limiting case of a large number of levels this method leads to the results of the quasiclassical theory. The asymptotic solution of the Korteweg–de Vries equation for small values of the parameter β [or large similarity numbers σ, cf. (1.4)] is also considered.

1. INTRODUCTION

IN[1,2] some general regularities of the evolution of nonlinear waves in dispersive media satisfying the Korteweg–de Vries equation

$$u_t(x,t) + uu_x + \beta u_{xxx} = 0 \qquad (1.1)$$

were considered (there also a bibliography is given, including the papers where this equation was obtained for a number of dispersive media). It follows from the results of[1,2] that in the general case the initial perturbation, which we write in the form

$$u(x,0) = u_0\varphi(x/l) \qquad (1.2)$$

[where $\varphi(\xi)$ is a dimensionless function, and u_0 and l are the characteristic amplitude and width of the perturbation] decays for sufficiently large values of t into a set of stationary solitary waves (solitons) and a "nonsoliton tail" of the form of a wave packet which flows apart in the course of time (cf. the figures in[2]). The equation of the soliton has the form $u_s(x, t) = u_s(x - Vt)$, where

$$u_s(x) = a\,\mathrm{sech}^2[(x-x_0)(a/12\beta)^{1/2}], \quad V = a/3 \qquad (1.3)$$

(without loss of generality we can assume in the following that $\beta > 0$[2]). Thus the solitons move in the positive direction with velocities proportional to the amplitudes, whereas the "tails" move in the opposite direction (for more details, cf.[2]). It is also important that all solutions of the Korteweg–de Vries equation with the same initial profiles $\varphi(\xi)$ [cf. (1.2)] and the same value of the dimensionless similarity parameter σ,

$$\sigma = l(u_0/\beta)^{1/2}, \qquad (1.4)$$

are similar.[2] Thus the number of solitons formed from the initial perturbation, their velocities, etc., are, for the same function $\varphi(\xi)$, uniquely determined by the quantity σ.

The analysis of numerical experiments performed in[2], allowed one to advance the hypothesis that if the profile $\varphi(\xi)$ satisfies the conditions

$$\varphi(\xi) > 0, \quad \varphi(\xi) \to 0 \quad (|\xi| \to \infty) \qquad (1.5)$$

[the question as to the rate of decrease of $\varphi(\xi)$ for $|\xi| \to \infty$ will be discussed in Sec. 3] and if the similarity parameter σ is larger than some critical value σ_c determined in[2], then the initial perturbation (1.2) decays practically completely into solitons whose amplitudes can be determined from the conservation laws corresponding to Eq. (1.1).

Besides the "usual" laws of momentum (S_1) and energy (S_2) conservation

$$S_1 = \int_{-\infty}^{\infty} u(x,t)\,dx, \quad S_2 = {}^1/_2\int_{-\infty}^{\infty} u^2(x,t)\,dx,$$

the Korteweg–de Vries equation also yields a whole range of other conservation laws (in the work of Zabuski,[3] for example, five conserved quantities were listed). This leads to the natural assumption that the number of nontrivial conservation laws implied by (1.1) is generally unlimited. (This assumption has essentially been used in[2]. A rigorous proof for it was given by Miura, Gardner, and Kruskal.[4] There also a table of the first ten conserved quantities is given, which in general have a very complicated form.) As in[2], we shall denote the conserved quantities ("integrals of motion") by S_m. Then

$$S_m = \int_{-\infty}^{\infty} Q_m[u(x,t)]\,dx, \quad \frac{\partial Q_m}{\partial t} + \frac{\partial P_m}{\partial x} = 0, \quad m = 1, 2, \ldots \qquad (1.6)$$

In the present paper, as in[2], we shall use only the most general properties of the densities of the conserved quantities $Q_m[u]$. In the general case $Q_m[u]$ has the form of a polynomial in u, β, and the derivatives u_x, u_{xx}, ... [the derivatives u_t can be eliminated with the help of (1.1)]. If we expand the terms of $Q_m[u]$ in the order of increasing powers of β, then the term not containing β is proportional to u^m [using the fact that Q_m in (1.6) is given up to a constant factor, it is convenient to write this term in the form u^m/m]; in addition, the quantity $Q_m[u]$ contains terms β^k ($k \le m-2$) (in particular, the first two conserved quantities mentioned above, momentum and energy, do not contain β at all). The coefficients of β^k have in general the form of very complicated polynomials of u, u_x, u_{xx}, ..., whose "rough" structure can be established by dimensional arguments. Here we shall not dwell on this question. We only note that, as is easily seen, the coefficient of β

must consist of the single term $u_x^2u^{m-3}$. It can also be shown that the numerical coefficient of this term has the form $(m-1)(m-2)/2$, so that in the case when the quantity β in (1.1) is a small parameter, one can write

$$Q_m[u] = u^m/m - {}^1/_2\beta(m-1)(m-2)u_x^2u^{m-3} + O(\beta^2) \qquad (1.7)$$

(in the following this formula will be useful also for values of β not very small). As to the "currents" P_m in (1.6), we do not need and will therefore not discuss them.

Let us now assume that the initial perturbation (1.2) satisfying the conditions (1.5) decays practically completely into N solitons.[1)] Then the conservation laws for the first N invariants lead to the following system of equations for the amplitudes of the solitons a_r $(r = 1, 2, \ldots, N)$:[2]

$$\sum_{r=1}^{N}\left(\frac{a_r}{u_0}\right)^{(2m-1)/2} = \sigma\int_{-\infty}^{\infty} Q_m(\xi, 0)\,d\xi/\sqrt{12}\int_{-\infty}^{\infty} q_m(\xi)\,d\xi, \qquad (1.8)$$

$m = 1, 2, \ldots, N$; $N \geq 2$. Here u_0 is the amplitude of the original perturbation (1.2), σ is the similarity parameter (1.4), and $Q_m(\xi, 0)$ and $q(\xi)$ are the densities of the conserved quantities for the original perturbation (1.2) and the soliton (1.3), where all quantities are written in dimensionless form:

$$Q_m(\xi, 0) = Q_m[u]\,|_{u=\varphi(\xi),\ \beta=\sigma^{-2}}, \qquad (1.9)$$

$$q_m(\xi) = Q_m[u]\,|_{u=\mathrm{sech}^2\xi,\ \beta=\sigma_s^{-2}={}^1/_{12}} \qquad (1.10)$$

($\sigma_s = \sqrt{12}$ is the value of the similarity parameter corresponding to the soliton; it is the same for all solitons[2]).

In particular, formulas (1.7) and (1.9) lead to the relation

$$Q_m(\xi, 0) = \frac{\varphi^m(\xi)}{m} - \frac{1}{\sigma^2}\frac{(m-1)(m-2)}{2}\varphi_\xi^2\varphi^{m-3} + O(\sigma^{-4}), \qquad (1.11)$$

which can be regarded as an asymptotic expression for $Q_m(\xi, 0)$ for large σ (but not necessarily small β!).[2)]

As already noted, the numerical solutions of the Korteweg—de Vries equation (1.1) for a number of specific initial perturbations satisfying (1.5) decay into solitons (the "tails" are at the level of a background), where the amplitudes of these solitons satisfy (1.8). Unfortunately, the analysis of the numerical solutions in[2] is severely limited by the difficulties of solving the system (1.8).

In the present paper we propose, first, a general effective method for solving the system (1.8) which allows us to simplify considerably and extend the analysis of the numerical experiments, and to confirm the above-mentioned hypothesis advanced in[2] (cf. Sec. 2).

Second, we consider the asymptotic solution of the system (1.8) for large σ, which, as shown in Sec. 3, can be written in closed analytic form and which (as will be seen in the following) determines completely the asymptotic solution of the Koretweg—de Vries equation for large σ.

These questions are of particular interest in connection with the recent work of Gardner et al.[5], where it was shown that the amplitudes of the solitons formed from the initial perturbation are determined by the discrete "energy levels" in some potential well determined by the initial condition (1.2). In our notation the corresponding "Schrödinger equation" for these levels has the form

$$\Psi''(\xi) + (\sigma^2/6)\,[\varphi(\xi) + E]\Psi(\xi) = 0, \qquad (1.12)$$

where $\Psi(\xi)$ is the auxiliary wave function, $\varphi(\xi)$ is the dimensionless profile of the initial perturbation (1.2), and E is the energy level in the potential well $V(\xi) = -\varphi(\xi)$ (the parameter $\sigma^2/6$ plays in this case the role of $2m/\hbar^2$). Comparing the results of[5] and[6], we easily see that the amplitudes of the solitons a_r are related to the energy levels of the discrete spectrum by the relation

$$a_r/u_0 = -2E_r, \quad r = 1, 2, \ldots, \qquad (1.13)$$

where u_0 is the amplitude of the initial perturbation (1.2).[3)] Thus the question arises as to the connection between the solutions of the system of equations (1.8) and the energy levels in the potential well $V = -\varphi(\xi)$. The results obtained in Secs. 2 and 3 lead us to a rather simple and somewhat unexpected approximate method for the calculation of the energy levels in the one-dimensional potential well $V = -\varphi(\xi)$ under the conditions (1.5). This method is discussed in Sec. 4.

2. DETERMINATION OF THE AMPLITUDES OF THE SOLITONS FROM THE SYSTEM (1.8) AND ANALYSIS OF THE NUMERICAL EXPERIMENTS

We introduce the notation

$$\left(\frac{a_r}{u_0}\right)^{1/2} = y_r, \qquad (2.1)$$

$$\sigma\int_{-\infty}^{\infty} Q_m(\xi, 0)\,d\xi/\sqrt{12}\int_{-\infty}^{\infty} q_m(\xi)\,d\xi = s_{2m-1}.$$

Then the system of equations (1.8) takes the form

$$\sum_{r=1}^{N} y_r^{2m-1} = s_{2m-1}, \quad m = 1, 2, \ldots N. \qquad (2.2)$$

(The case where a single soliton is formed will be discussed at the end of this section).

We further introduce the notation

$$\sigma_1 = y_1 + y_2 + \ldots + y_N, \quad \sigma_2 = y_1y_2 + y_2y_3 + \ldots + y_{N-1}y_N, \ \ldots,$$
$$\sigma_{N-1} = y_1y_2\ldots y_{N-1} + \ldots + y_2y_3\ldots y_N, \quad \sigma_N = y_1y_2\ldots y_N. \qquad (2.3)$$

Clearly, the desired quantities $y_1, \ldots, y_N$ are the roots of the single algebraic equation

$$y^N - \sigma_1y^{N-1} + \sigma_2y^{N-2} - \ldots + (-1)^N\sigma_N = 0. \qquad (2.4)$$

It turns out that the coefficients σ_i of this equation can be expressed through the right-hand sides s_{2m-1} of (2.2) by solving a certain system of linear equations. The algorithm for constructing this linear system is explained in the Appendix [cf. formulas (A.3) to (A.5)]. In the simplest case $N = 2$ the system (2.2) has an elementary analytic solution (an analysis of this solution was carried out in[2]), and for $N > 2$ it is easily solved by standard numerical methods.

Solving (2.4), we find its roots as functions of the parameter σ [for fixed profile of the initial perturbation

[1)] More precisely, this means that the contribution of the "tails" can be neglected at least in the first N invariants S_m.

[2)] Formula (1.11) is exact for $m < 3$.

[3)] It is convenient to define u_0 such that the largest of the maxima of $\varphi(\xi)$ is equal to unity.

$\varphi(\xi)$ determining $\int_{-\infty}^{\infty} Q_m(\xi, 0)d\xi$ in (2.1)]. In general, for arbitrary σ, some of these roots can be complex or negative, which is forbidden [since in the derivation of (1.8) all square roots were understood in the arithmetic sense[2]]. In order that the initial perturbation decay into N solitons for a given σ, it is necessary [4)] that all roots of (2.4) be positive for this σ.

As an example we consider the real non-negative roots of (2.4) for $2 \le N \le 6$ in the case when the initial profile is given by the function

$$\varphi(\xi) = \exp(-\xi^2). \tag{2.5}$$

The right-hand sides of the system (2.2) are computed from (1.9) and (1.10), using the explicit expressions for the first six integrals of motion. It should be noted, however, that (as simple estimates show) the accuracy of the calculations is not made worse when the quantities $Q_m(\xi, 0)$ in (2.1) are calculated by the asymptotic formulas (1.11) (which are much simpler than the exact ones); the accuracy of the calculations increases here with increasing N. [5)] The denominators on the right-hand sides of (2.2) must be calculated from the exact expressions for $Q_m[u]$ substituted in (1.10), since the contribution of all terms in $Q_m[u]$ is of the same order of magnitude under the conditions indicated in (1.10). Here it is remarkable that the general properties of the conserved quantities imply the following simple formula:[7]

$$\int_{-\infty}^{\infty} q_m(\xi)\,d\xi = 2^m[(m-1)!]^2/(2m-1)!, \tag{2.6}$$

where $q_m(\xi)$ is defined by (1.10). The results of the calculations are shown in Fig. 1. The corresponding roots of the system (1.8) $\eta_r = a_r/u_0$ for given σ are determined by the points of intersection of a vertical straight line having an abscissa equal to σ, with the curves of this figure obtained by the method explained above.

Let us denote by Δ_n the interval of values σ where all roots of the system (2.2) for N = n are real and non-negative. It is easy to see that the interval Δ_n begins inside the interval Δ_{n-1}. Indeed, denoting the beginning of the interval Δ_n by σ_n, we obtain $y_1(\sigma_n) = 0$, $y_i(\sigma_n) > 0$ $(i > 1)$. But then $y_2(\sigma_n), \ldots, y_n(\sigma_n)$ are positive roots of the system (2.2) for $N = n-1$ and hence, σ_n lies in the interval Δ_{n-1}. The numerical experiment shows that the boundaries separating the regions of values σ with soliton numbers differing by one unit must lie at the points σ_n, i.e., at the origins of the dotted straight lines of Fig. 1. For an illustration the crosses in Fig. 1 indicate the dimensionless amplitudes of the solitons $\eta = a/u_0$ obtained from the initial perturbation. These amplitudes are found by numerical solution of the Korteweg—de Vries equation with the initial condition (2.5) and various values of σ. We see that the "experimental points" (crosses) fall on the curves of Fig. 1 in a completely satisfactory manner. This indicates that the contribution of the "non-soliton tails" to the first N

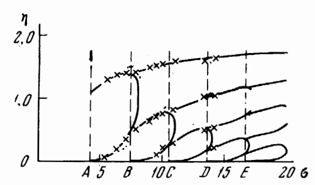

FIG. 1

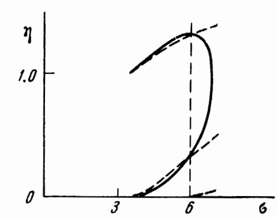

FIG. 2

invariants can be neglected in the decay of the initial perturbation (satisfying the above-mentioned conditions) into N solitons.

This result is valid for all $\sigma \ge \sigma_c$, where σ_c is the abscissa of the point A in Fig. 1 [for the analytic expression for σ_c, cf. formula (3.9) of[2]] and, as we shall see below, its accuracy increases with increasing σ.

The solid line in Fig. 2 represents the roots of the system (1.8) as functions of σ for N = 2 and the initial profile

$$\varphi(\xi) = \mathrm{ch}^{-2}\,\xi. \tag{2.7}$$

The dotted curves represent the exact values of the amplitudes of the solitons calculated by (1.13) [with condition (2.7) the Schrödinger equation (1.12) can be solved analytically[8]]. This figure illustrates the transition from N = 2 to N = 3. In the neighborhood of the point $\sigma = 6$ (to the right of it) the system (1.8) has non-negative real roots for N = 2 as well as for N = 3; however, only for N = 3 are the roots of the system close to the dotted curves representing the exact amplitudes (for N = 3, 4 the corresponding data are given in Sec. 4).

For $\sigma < \sigma_c$ no more than one soliton can evolve from the initial perturbation.[2] In this case the "tail" can evidently no longer be neglected. As to the occurrence of the soliton, the numerical results of[2] gave no unambiguous answer. However, qualitative considerations were presented in[9] (in connection with the problem of the streaming around a body in a dispersive medium) which indicate that if the initial perturbation satisfies the condition

$$\int_{-\infty}^{\infty} \varphi(\xi)\,d\xi > 0 \tag{2.8}$$

[it is not required that $\varphi(\xi)$ be positive for all ξ], then at least one soliton must occur for arbitrary (even very small) σ (in the case

$$\int_{-\infty}^{\infty} \varphi(\xi)\,d\xi = 0$$

this may not be so). Using the results of[5], we can determine the amplitude of this soliton for sufficiently small σ. This problem reduces to the determination of the energy level in a one-dimensional potential well of small depth. Applying perturbation theory to (1.12) as in[8] (cf. problem 1 of Sec. 45) and using (1.13), we ob-

4) But in general not sufficient (for more detail, cf. below).

5) This is connected with the circumstance that the larger N, the larger must σ be taken in order that the system (2.2) have a nonnegative solution [cf., for example, Fig. 1 and also formula (3.10)]. On the other hand for N = 2, 3 the system (2.2) contains $Q_m(\xi, 0)$ with $m < 3$, for which (1.11) is exact.

tain for the amplitude of the (only) soliton the following expression:

$$a \approx \sigma^2 u_0 \left\{ \int_{-\infty}^{\infty} \varphi(\xi)\, d\xi \right\}^2 \Big/ 12. \tag{2.9}$$

The condition for the applicability of perturbation theory [cf.[8], formula (45.5)] is in our case

$$\sigma^2 \ll 12 = \sigma_s^2. \tag{2.10}$$

We see that the amplitude of the solition is of second order smallness for small σ. In this case the main part of the energy remains in the "tail." As is easily verified, by comparing (2.9) with the first equation of the system (1.8), the soliton carries a momentum which is twice that of the initial perturbation, so that the momentum of the "tail" (or the area under the tail) is negative. It is remarkable that the correct order of the amplitude of the soliton is obtained from (1.8) even in this case, if one starts from the first equation of this system, keeping only one term on its left-hand side.

3. ASYMPTOTIC SOLUTION OF THE KORTEWEG–DE VRIES EQUATION FOR LARGE σ AND t[6)]

It follows from the results of the preceding section that if the initial perturbation has the form of a positive pulse which falls off sufficiently rapidly for $x \to \infty$, then the number of solitons N increases with σ, whereas the "tails" can be neglected (in the sense that they make a small contribution to the first N conserved quantities, as compared with the solitons). Therefore the asymptotic form of the solution for large t is completely determined by the amplitudes of the solitons. For a characterization of the latter for sufficiently large N one can introduce a distribution function for the amplitudes.

We introduce the dimensionless amplitude of the soliton

$$\eta = a / u_0, \tag{3.1}$$

where u_0 is the amplitude of the initial perturbation (1.2) [for definiteness it is convenient to assume that u_0 is the largest of the maxima of u(x, 0), so that the maximal value of $\varphi(\xi)$ is equal to unity). Let $f(\eta)d\eta$ denote the number of solitons having a dimensionless amplitude in the interval $(\eta, \eta + d\eta)$. For large N the left-hand sides of (1.8) take the form

$$\int_0^{\infty} \eta^{(2m-1)/2} f(\eta)\, d\eta.$$

As to the right-hand sides we can, regarding σ sufficiently large, substitute the main terms of the asymptotic expressions (1.11), i.e.,

$$\int_{-\infty}^{\infty} \varphi^m(\xi)\, d\xi / m,$$

in the numerators, and formula (2.6) in the denominators. Then the system (1.8) takes the form

$$\int_c^{\infty} \eta^{(2m-1)/2} f(\eta)\, d\eta = \frac{\sigma (2m-1)!}{\sqrt{12}\, 2^m m!\, (m-1)!} \int_{-\infty}^{\infty} \varphi^m(\xi)\, d\xi, \tag{3.2}$$

m = 1, 2, ...

Equations (3.2), which determine the sequence of moments of the function $f(\eta)$, allow one to establish the form of this function. Setting $\eta = z^2$ on the left-hand side of (3.2) and introducing the Fourier transform of the function $f(z^2)$

$$\Phi(p) = \int_{-\infty}^{\infty} f(z^2) e^{-ipz} dz, \tag{3.3}$$

we obtain

$$\int_0^{\infty} \eta^{(2m-1)/2} f(\eta)\, d\eta = (-1)^m \Phi^{(2m)}(0). \tag{3.4}$$

Determining $\Phi^{(2m)}(0)$ with the help of (3.4) and (3.2) and substituting it in the expansion

$$\Phi(p) = \sum_{m=0}^{\infty} \Phi^{(2m)}(0) p^{2m} / (2m)!, \tag{3.5}$$

we obtain the following expression for the function $\Phi(p)$:

$$\Phi(p) = \frac{\sigma}{\sqrt{48}} \int_{-\infty}^{\infty} d\xi\, J_0(\sqrt{2\varphi(\xi)}\, p), \tag{3.6}$$

where $J_0(z)$ is the Bessel function.

From (3.6), in particular, we see the importance of the requirement that the initial perturbation be nonnegative. That is, when $\varphi(\xi)$ takes negative values in some regions, then the use of the conservation laws for the calculation of the amplitudes of the solitons becomes incorrect, owing to the non-negligible contribution of the "tails." Substituting now (3.6) in (3.3) and taken the inverse Fourier transform, we can determine the form of the distribution function $f(\eta)$. Returning to the amplitudes $a = u_0\eta$, we obtain for the distribution function of the amplitudes of the solitons $F(a) = f(a/u_0)/u_0$ the following final expression:

$$F(a) = \frac{\sigma}{4\pi u_0 \sqrt{6}} \int_M \frac{d\xi}{\sqrt{\varphi(\xi) - a/2u_0}}, \tag{3.7}$$

where the region of integration M is determined by [7)]

$$0 < a < 2u_0 \varphi(\xi). \tag{3.8}$$

In particular, it follows from (3.8) and (3.7) that the distribution function for the amplitudes of the solitons vanishes when the amplitude a exceeds twice the largest maximum of the initial perturbation (1.2),

$$F(a) = 0, \quad a > 2u_0. \tag{3.9}$$

The total number of solitons is obtained by integrating (3.7) over a:

$$N = \int_0^{2u_0} F(a)\, da = \frac{\sigma}{\pi\sqrt{6}} \int_{-\infty}^{\infty} d\xi\, \varphi^{1/2}(\xi), \tag{3.10}$$

i.e., the total number of solitons in the asymptotic limit is proportional to σ. This number will be finite if the integral in (3.10) converges, which determines the rate of decrease of $\varphi(\xi)$ for $|\xi| \to \infty$ necessary for the validity of the results obtained in Sec. 2.

Applying (3.10) to the initial condition (2.5), we obtain

$$N = (3\pi)^{-1/2}\sigma + O(\sigma^{-2}). \tag{3.11}$$

This formula, which is asymptotically exact for $N \to \infty$, has completely satisfactory accuracy even for $N \sim 4$. Indeed, in this case we obtain $\sigma \approx 12.4$ from (3.11), which is in good agreement with the results shown in

6) The main results of this section are briefly mentioned in [7].

7) If we go over to the new variable $z = \varphi(\xi)$ in the integral (3.7), we obtain formula (6) of [7].

Fig. 1. The value of the increment $\Delta\sigma$ for $\Delta N = 1$ also agrees with these results. Indeed, the distances AB, etc., indicated in Fig. 1 are AB $\approx$ BC $\approx$ CD $\approx$ DE $\approx$ 3.1 $\approx (3\pi)^{1/2}$. We note, finally, that formulas (3.7) and (3.10) agree with the expressions for the densities and the total number of levels in a one-dimensional well with the potential $V(\xi) = -\varphi(\xi)$ [if we replace $\sigma^2/6$ by $2m/\hbar^2$ and set $E = -a/2u_0$, corresponding to the Schrödinger equation (1.12) (cf. e.g., the formulas of Sec. 48 of the book[8])].

This is in agreement with the results of[5], where a connection is established between the amplitudes of the solitons and the energy levels in the corresponding potential well. However, since our results were obtained from the system (1.8), which is based on the conservation laws and on the neglect of the contribution of the "tails," we thus obtain a proof of the asymptotic vanishing of the contribution of the "tails" to the integrals of motion for $N \to \infty$. If $\varphi(\xi)$ takes negative values in some region (and the potential $V = -\varphi(\xi)$ is positive], then, in the calculation of the number of levels and, therefore, solitons, one must extend the integral in (3.10) over that region of space where $\varphi(\xi) > 0$ (cf.[8], Sec. 48). In this case the contribution of the "tails" is important[8)] [cf. also the text after formula (3.6)].

Finally, we point out the following aspect of the results obtained in the present section. Since it follows from the similarity principle[2] that the limiting case $\sigma \to \infty$ can be realized also for l, u_0 finite but $\beta \to 0$ [cf. (1.4)], the results obtained above determine the asymptotic equation (1.1) for $\beta \to 0$. Let us compare it with the solution of the equation for a simple wave in gas dynamics:[10]

$$u_t + uu_x = 0 \tag{3.12}$$

(written in a system moving with the unperturbed velocity of sound c_0 with respect to the medium) and also with the asymptotic solution of the Burgers equation

$$u_t + uu_x - \mu u_{xx} = 0, \quad \mu > 0 \tag{3.13}$$

for $\mu \to 0$. [The Burgers equation can be obtained from the Navier-Stokes equation for waves with sufficiently small (but finite) amplitude in the same way as the Korteweg—de Vries equation is obtained from the equations of "hydrodynamics with dispersion," i.e., with neglect of the nonlinear terms of third order.] The last term in (3.13) describes the effect of the dissipative processes on the properties of the simple wave (up to terms of third order).[9)]

It is known that the profile of the solution satisfying (3.12) becomes many-valued with the course of time and loses its physical meaning (Fig. 3a, curve ABDEC). The profile of the permutation described by (3.13) takes a

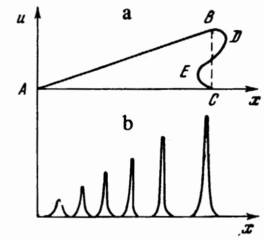

FIG. 3

"triangular form" for large t and $\mu \to 0$ (cf.[11], where the exact solution of the Burgers equation is investigated) with a shock wave in front (profile ABC in Fig. 3a). For identical initial conditions the areas ABDEC and ABC are equal, since the equations (3.12) and (3.13) imply the conservation of momentum, i.e., of the quantity $S_1 = \int_{-\infty}^{\infty} u(x, t)dx$.

The solution of the Korteweg—de Vries equation (1.1) for $\beta \to 0$ consists of a number of solitons, whose number increases as $\beta^{-1/2}$ [cf. (3.10)] and the width decreases like $\beta^{1/2}$ [cf. (1.3)] (the amplitudes of the solitons cannot exceed twice the maximum of the initial perturbation). This solution is shown in Fig. 3b. It is remarkable that for $\beta \to 0$ the integrals of motion of the Korteweg—de Vries equation take the form (according to (1.7))

$$S_m(\beta \to 0) = \frac{1}{m}\int_{-\infty}^{\infty} u^m(x, t)\,dx, \quad m = 1, 2, \ldots, \tag{3.14}$$

i.e., agree with the integrals of motion for equation (3.12) [this is easily seen by multiplying both sides of (3.12) by u^{m-1} and integrating over all x]. In other words, for identical initial conditions the profile ABDEC and the profile in Fig. 3b have not only identical areas for $\beta \to 0$, but also the same integrals of the form (3.14).

Clearly, the solution of the Korteweg—de Vries equation for $\beta \to 0$ is a generalized solution of (3.12) in the sense defined in[11] but, moreover, has a number of other features, for example, the same integrals of motion. [The solution of the Burgers equation for $\mu \to 0$ is also a generalized solution of (3.12), but has only one integral of motion—the area of the profile.]

4. ON THE CALCULATION OF THE ENERGY LEVELS IN A ONE-DIMENSIONAL POTENTIAL WELL

Summarizing the results of Secs. 2 and 3, we can formulate the following simple method for an approximate calculation of the energy levels in a one-dimensional well with the potential

$$U(x) = -u_0\varphi(x/l), \tag{4.1}$$

where

$$\varphi(\xi) \geqslant 0, \quad \int_{-\infty}^{\infty} \varphi^{1/2}(\xi)\,d\xi < \infty. \tag{4.2}$$

With the conditions (4.2) the eigenvalues of the Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\frac{d^2\Psi(x)}{dx^2} + [U(x) - E]\Psi = 0 \tag{4.3}$$

can be determined with satisfactory accuracy (cf. the

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8) In this case the assumption that the "tails" are formed mainly from those regions of the initial perturbation where $\varphi(\xi) < 0$ is evidently correct (at least for large N). We recall in this connection that, in the terminology of [5], the "tails" are determined by the "reflection coefficients" b(k).

9) The parameter μ is simply expressed through the coefficient γ for the absorption of sound ([10], sec. 77):

$$\mu = \gamma c_0^3/\omega^2 = (2\rho_0)^{-1}[4\eta/3 + \zeta + \varkappa(1/c_v - 1/c_p)],$$

where η, ζ, and κ are the coefficients of first and second viscosity and heat conductivity, respectively.

Energy levels in the potential well $u(x) = u_0 \cosh^{-2}(x/l)$

	Solutions of the system (4.4)				Exact solutions			
$6\alpha^2$	$\lvert E_1 \rvert$	$\lvert E_2 \rvert$	$\lvert E_3 \rvert$	$\lvert E_4 \rvert$	$\lvert E_1 \rvert$	$\lvert E_2 \rvert$	$\lvert E_3 \rvert$	$\lvert E_4 \rvert$
39	0,676	0.184	0,002	—	0,676	0,184	0.0004	—
44	0,692	0,214	0,008	—	0,692	0.216	0,003	—
71	0,748	0.330	0,080	—	0.748	0,332	0,078	—
78	0,758	0.352	0,100	—	0,763	0,330	0.126	—
77	0,757	0,348	0,097	0,001	0,757	0,348	0,097	0.0003
84	0.766	0.370	0,116	0,006	0,766	0,369	0.118	0,002
116	0.797	0.442	0,192	0,044	0.797	0.440	0,196	0,040
120	0,800	0,450	0.200	0,050	0,800	0.450	0.200	0,051

table) from the following system of equations:

$$\sum_{r=1}^{N} k_r^{2n-1} = s_{2n-1} \equiv \sqrt{\frac{2\mu u_0 l^2}{\hbar^2}} \frac{(2n-1)!}{2^{2n}[(n-1)!]^2} \int_{-\infty}^{\infty} Q_n(\xi)\, d\xi, \qquad (4.4)$$

$$n = 1, 2, \ldots, N, \quad N \geqslant 2,$$

where N is the total number of levels, r is the number of eigenvalues (in order of increasing moduli), and $E_r = -u_0 k_r^2$,

$$Q_n(\xi) = \frac{\varphi^n(\xi)}{n} - \frac{\hbar^2}{2\mu u_0 l^2} \frac{(n-1)(n-2)}{12} \varphi_\xi^2 \varphi^{n-3}. \qquad (4.5)$$

The location of the levels and their number is, according to (4.4) and (4.5), determined by the values of the dimensionless parameter

$$\alpha^2 = 2\mu u_0 l^2 / \hbar^2. \qquad (4.6)$$

Although [cf. (4.4)] our method is valid only for $N \geq 2$, it gives the correct order of magnitude for the level energy even for $N = 1$.

For $N = 2$ the system (4.4) reduces to a quadratic equation with the roots

$$k_{1,2} = {}^1/_8 (\alpha\gamma_1 \pm \sqrt{32\gamma_2/\gamma_1 - \alpha^2\gamma_1^2/3}).$$
$$\gamma_1 = \int_{-\infty}^{\infty} \varphi(\xi)\, d\xi, \quad \gamma_2 = {}^1/_2 \int_{-\infty}^{\infty} \varphi^2(\xi)\, d\xi \qquad (4.7)$$

It follows from the condition that the roots k_r be positive and real that the "well" has more than one energy level only when the parameter α exceeds the value

$$\alpha_c = (24\gamma_2/\gamma_1^3)^{1/2}. \qquad (4.8)$$

Analogously, for larger N, the solutions of the system (4.4) are roots of a single algebraic equation of N-th degree having the form (2.4). The coefficients σ_i of this equation can be expressed through the right-hand sides of (4.4), s_{2n-1}, by solving a certain system of linear equations (cf. the Appendix). Each number of levels N corresponds to a definite region of values of the parameter α, whose beginning is determined (with the same accuracy as the levels) by the value of α for which the first root of the system of N equations (4.4) vanishes: $k_1^{(N)} = 0$; this region ends at the value of α for which $k_1^{(N+1)} = 0$ (Fig. 1). Furthermore, since the asymptotic distribution law for the levels obtained from (4.4) is the same for large N as in the classical case, the accuracy of the calculation of the eigenvalues of the Schrödinger equation with the help of (4.4) increases with increasing N. We note also that the change of the quantity α corresponding to an increase of the number of levels N by one, depends very weakly on N and can therefore be determined by its asymptotic value

$$\Delta\alpha \approx \pi \Big/ \int_{-\infty}^{\infty} d\xi\, \varphi^{1/2}(\xi) \qquad (4.9)$$

[which follows from (3.10) after replacing $\sigma^2/6$ by α^2]. This formula is satisfied already for N = 2 or 3 with good accuracy (at least for potentials with one minimum).

An idea of the accuracy of our method can be gained from the table, where the energy levels in a potential well having the profile (2.7) are compared with the values E which are obtained from the system (4.4) for N = 3, 4. For N = 2 the results are shown in Fig. 2. For $N > 4$ the accuracy increases. Analogous results (as regards the accuracy) are obtained for other potentials with one minimum.

In conclusion we express our gratitude to R. Z. Sagdeev for his interest in this work and to V. V. Sobolev for help with the numerical solution of the Korteweg–de Vries equation.

APPENDIX

We start from the relation

$$A_{2k-1} \equiv \frac{s_1^{2k-1} - s_{2k-1}}{2k-1} = \sum_{\mu} \frac{(\mu_2 + \ldots + \mu_{n+1} - 1)!}{\mu_2! \ldots \mu_{n+1}!} \rho_2^{\mu_2} \ldots \rho_{n+1}^{\mu_{n+1}},$$
$$s_{2k-1} = \sum_{i=1}^{n} y_i^{2k-1}, \qquad (A.1)$$

where

$$\rho_i = s_1\sigma_{i-1} - \sigma_i (2 \leqslant i \leqslant n), \qquad \sigma_1 = s_1, \qquad \rho_{n+1} = s_1\sigma_n. \qquad (A.2)$$

The summation in (A.1) goes over all sets of non-negative integer numbers μ_i satisfying the condition $2\mu_2 + \ldots + (n+1)\mu_{n+1} = 2k - 1$. The relation (A.1) is most easily obtained from the first Waring formula (cf. for example,[12]) for the (n + 1)st variable by substituting the auxiliary quantities $Y_i = y_i$ $(i \leq n)$, $Y_{n+1} = -\sum_{i=1}^{n} y_i$. Setting k = 2, 3, ... in (A.1), we may express ρ_i through A_{2k-1}; then one obtains a system of linear equations for ρ_i and hence for σ_i (i = 2, ..., n).

Let us consider, for example, the case n = 6. To obtain the necessary equations for σ_i it suffices to set k = 2, ..., 6 in (A.1). After simple transformations they take the form

$$\begin{gathered} \rho_3 = A_3, \quad \rho_2 A_3 + \rho_5 = A_5, \quad \rho_2 A_5 + \rho_4 A_3 + \rho_7 = A_7, \\ \rho_2 A_7 + \rho_4 A_5 + \rho_6 A_3 = A_9 - A_3^3/3, \\ \rho_2 (A_9 - A_3^3/3 + \rho_4 A_7 + \rho_6 A_5 = A_{11} - A_3^2 A_5. \end{gathered} \qquad (A.3)$$

Substituting (A.2) in (A.3), we obtain the desired system of linear equations for $\sigma_2, \ldots, \sigma_6$. It is remarkable that for $n < 6$ one need not carry out the calculations again; one must simply take the first n − 1 equations of (A.3), substitute (A.2), and strike out all terms proportional to σ_i with $i > n$.

For arbitrary k the general form of the equations determining ρ_k can be written in the form

$$(A_{2k-3}-R_{2k-3})\rho_2+(A_{2k-5}-R_{2k-5})\rho_4+\ldots+(A_3-R_3)\rho_{2k-4}+\rho_{2k-1}=A_{2k-1}-R_{2k-1}, \tag{A.4}$$

where the quantities R_{2i+1} are obtained from the recurrency formula

$$R_{2i+1}=\sum_{\mu}\frac{(\mu_3+\mu_5+\ldots+\mu_{2i-5}-1)!}{\mu_3!\,\mu_5!\ldots\mu_{2i-5}!}(A_3-R_3)^{\mu_3}\ldots(A_{2i-5}-R_{2i-5})^{\mu_{2i-5}}, \tag{A.5}$$

where $R_{2i+1}=0$ for $i\leq 3$, and the summation goes over all sets of non-negative integer numbers μ satisfying the condition

$$3\mu_3+5\mu_5+\ldots+(2i-5)\mu_{2i-5}=2i+1.$$

After substituting (A.2) in (A.4) one must strike out all σ_i with $i>n$.

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