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PROBLEMS OF THE DYNAMICAL APPARATUS OF QUANTUM FIELD THEORY IN THE AXIOMATIC APPROACH

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Several problems related to the dynamical formalism of the axiomatic approach to quantum field theory are considered. A Wick theorem is derived, connecting the quasinormal and ordinary powers of Heisenberg fields. The concept of generalized functional (variational) derivatives is introduced, a concept which in renormalizable theories has allowed to linearize the equations proposed by Medvedev for the current-like operators Λ_ν. A number of theorems are proved for these operators. An example of a theory with a finite number of nontrivial operators Λ_ν is given and explicit representations are obtained for the latter.

1. INTRODUCTION

In order to construct a quantum field theory in any form it is first of all necessary to introduce a set of local Heisenberg operators. In the usual Hamiltonian formalism the role of such operators is obviously played by the Heisenberg field operators, but this approach leads, as is well known, to serious difficulties. Another method of introducing point-dependent operators—and one which is natural in a theory based on a scattering matrix—consists in deriving the local operators by variations of the scattering matrix with respect to appropriate functional arguments, which can be interpreted as classical external fields [1] 1). In this case the whole theory has to be formulated in terms of currents and current-like operators [2].

We note that such a theory could serve as the most convenient dynamical starting point for all computations within the framework of the so-called current algebra [3]. In themselves, such postulates as the equal-time current commutation relations or PCAC could be naturally interpreted in the axiomatic approach as additional dynamical requirements, of the same type as requiring definite degrees of growth, or of an even stronger nature [4].

However, in order to get closer to the investigation of such problems in the axiomatic method it is useful to introduce the concept of Heisenberg field [2]

$$A(x) = T_W(\varphi(x)S)S^+ = \varphi(x) - \int dy\, D^{adv}(x-y)j(y), \quad (1.1)$$

where $\varphi(x)$ is the asymptotic out-field, and also to learn how to define powers of such a field. Then the dynamical principle of the theory could be formulated in the form of the requirement

$$j(x) \equiv i\frac{\delta S}{\delta\varphi(x)}S^+ = f[A(x)], \quad (1.2)$$

where $f[A(x)]$ is a local function of the operators $A(x)$ of the form (1.1). In this sense the axiomatic method somehow approaches a perfected formulation of the Lagrangian method [5], in which the principal problem is that of representing nonlinear Heisenberg fields as local functions of the operators $A(x)$.

In the axiomatic approach the dynamics has to be described by means of a set of current-like operators $\Lambda_\nu(x_1, \ldots, x_\nu)$. These operators, as well as any other Heisenberg operators have to be represented as infinite Wick polynomials of the fields $\varphi(x)$. (Finite Wick polynomials obviously lead to a trivial S-matrix.) At the same time we expect that a renormalizable theory will involve only a finite number of nonvanishing operators Λ_ν. One can reach such a conclusion considering, for instance, the unitarity condition [6], or using the analogy with perturbation theory [7]. It would be considerably simpler to establish this directly if one had a dynamical law of the type (1.2) for the current, from which one can further derive appropriate expressions for the higher order current-like operators in the form of quasilocal functions of the operators $A(x)$. In particular, in the framework of the Lagrangian formalism such a procedure was realized with concrete examples in ref. [8].

In the framework of the axiomatic approach the most natural product of Heisenberg operators is in many respects the so-called quasinormal product [2,9]

$$N_Q(A(x_1)\ldots A(x_n)) \equiv T_W[:\varphi(x_1)\ldots\varphi(x_n):S]S^+. \quad (1.3)$$

It is just this product [5] which turned out to be the most convenient for defining Heisenberg operators which are nonlinear in the fields. However, such products alone do not suffice, since, for example, the commutators $[A(x), A(y)]$ or $[j(x), j(y)]$ consist of ordinary products of Heisenberg operators. The same is also true for T_D-products of currents, products which play an important role in the theory.

It would therefore be very desirable to have at one's disposal something like a Wick theorem, relating ordinary and quasinormal products of fields. In the first place we shall be interested in establishing such a relation for products with coinciding arguments such as can actually occur in (1.2). But it is a well-known fact that products (of any kind) of fields at the same point are not mathematically well-defined objects. It is

1) We shall often use, without additional explanations, the notations and concepts from [1,2].

therefore necessary each time to make a convention as to what one means by the product of two fields at the same point.

We shall see below that if one takes into account these remarks there exists a class of theories in which the relation between quasinormal and ordinary products of fields at one point has the form of a Wick theorem sui generis. The existence of such a relation is closely related to the character of the current-like operators Λ_ν and in particular to the presence of derivatives of delta functions in these operators. Therefore, simultaneously with the derivation of the Wick theorem we have to clarify the problem of the number and structure of the current-like operators Λ_ν which determine the dynamics of the theory.

2. RELATIONS BETWEEN N_Q AND THE ORDINARY SQUARES AND CUBES OF THE FIELDS A(x)

Since obviously the first ordinary and quasinormal powers of the operator A(x) coincide, the first problem arises if one compares the corresponding squares. First we establish the relation between $N_Q(A(x)A(y))$ and the symmetrized product

$$\{A(x),\ A(y)\} \equiv A(x)A(y) + A(y)A(x). \tag{2.1}$$

We start from the generalized Yang-Feldman equation [2;9]

$$N_Q(A(x)A(y)) = :\varphi(x)\varphi(y): - i\varphi(x)\int du D^{adv}(y-u)S^{(1)}(u) - i\varphi(y)\int dz D^{adv}(x-z)S^{(1)}(z) + i^2\int dz du D^{adv}(x-z)D^{adv}(y-u)S^{(2)}(u,z), \tag{2.2}$$

for the N_Q-product of two fields (the radiative operators $S^{(1)}$ and $S^{(2)}$ are defined in [1,2]). The idea of the subsequent transformation consists in expressing (cf. [1,2]) the radiative operators in terms of currents, their usual products and functional derivatives of currents. The terms involving the currents and their products can be combined into the product of fields (2.1) and the terms involving functional derivatives remain. This leads to the relation

$$N_Q(A(x)A(y)) = {}^1/_2\{A(x),A(y)\} - {}^1/_2 D^{(1)}(x-y) + \frac{i}{2}\int dz\,du\left\{D^{ret}(y-u)\frac{\delta j(z)}{\delta\varphi(u)}D^{ret}(z-x) + D^{adv}(y-u)\frac{\delta j(u)}{\delta\varphi(z)}D^{adv}(z-x)\right\}. \tag{2.3}$$

Now one can also consider the equation of motion for the operator j(z) [10]:

$$\frac{\delta j(z)}{\delta\varphi(u)} = -i\theta(u-z)[j(u),j(z)] + \Lambda_2(z,u). \tag{2.4}$$

Then

$$N_Q(A(x)A(y)) = {}^1/_2\{A(x),A(y)\} - {}^1/_2 D^{(1)}(x-y) + \frac{i}{2}\int dz\,du\,[D^{ret}(y-u)\Lambda_2(u,z)D^{ret}(z-x) + D^{adv}(y-u)\times\Lambda_2(u,z)D^{adv}(z-x)] + \frac{1}{2}\int dz\,du D(x-z)D(y-u)[j(u),j(z)]\times\{\theta(y-u)\theta(u-z)\theta(z-x) - \theta(x-z)\theta(z-u)\theta(u-y)\}. \tag{2.5}$$

We have thus derived a relation between $N_Q(A(x)\,A(y))$ and $^1/_2\{A(x),A(y)\}$, and it is expressed in terms of both the retarded current commutator and the operator Λ_2. We now make precise the meaning of the Dyson T_D-product of the currents, or, what amounts to the same from the viewpoint of the definition, the meaning of the retarded current commutator. Namely, we shall interpret it as a formal product of the commutator [j(u), j(z)] with the θ-function $\theta(u-z)$, defined by its Fourier representation. In [2,11] we have established (for the simplest case of a vacuum expectation value) that this prescription can be formulated in terms of definite rules which make precise the method of analytic continuation into the complex plane.

In the case x = y in which we are interested, the last term of (2.5) will consist of a closed chain of θ-functions, which vanishes identically for the adopted definition of retarded commutators. Regarding the terms containing Λ_2, they yield for x = y

$$\frac{i}{2}M_2(x,x) = \frac{i}{2}\int dz\,du\,D(x-u)D(z-x)\times\{\theta(x-u)\Lambda_2(u,z)\theta(z-x) + \theta(u-x)\Lambda_2(u,z)\theta(x-z)\}. \tag{2.6}$$

It is clear that if the operator $\Lambda_2(u,z)$ would contain only $\delta(u-z)$ without its derivatives the integrations over u or z could be done and one would again obtain closed chains of θ-functions so that the contribution of terms with Λ_2 to (2.7) would vanish.[2)] However, in general, Λ_2 is not a local, but a quasilocal operator, which may contain derivatives of any order of the delta function. We shall see below that already if it contains the second time derivative the corresponding term in (2.5) will yield nonvanishing results. Therefore for $x \to y$ we obtain the final expression from (2.5):

$$N_Q(A^2(x)) = A^2(x) - {}^1/_2 D^{(1)}(0) + {}^1/_2 iM_2(x,x). \tag{2.7}$$

In other words, the relation between $N_Q(A^2(x))$ and $A^2(x)$ has the form of a Wick theorem with the contraction

$$\mathfrak{D}^-(x,x) = {}^1/_2 i\{iD^{(1)}(0) + M_2(x,x)\}, \tag{2.8}$$

which now depends on the current-like operator, and not the currents themselves.

Similarly one can establish a relation between

$$N_Q(A(x)A(y)A(z)) \text{ and } {}^1/_6\{A(x),A(y),A(z)\}.$$

The starting point will again be the generalized Yang-Feldman equation [2,9] (now for the N_Q-product of three fields) and the subsequent manipulations follow the same plan as for the case of two fields. The result, which we write directly for x = y = z, effecting the obvious symmetrization, is

$$N_Q(A^3(x)) = A^3(x) - {}^3/_2 D^{(1)}(0)A(x) + 2i\int dy\,dz\,D^{ret}(x-y)\frac{\delta j(z)}{\delta\varphi(y)}D^{ret}(z-x)A(x) + iA(x)\int dy\,dz\,D^{ret}(x-y)\frac{\delta j(z)}{\delta\varphi(y)}D^{ret}(z-x) + \int dy\,dz\,du\,D^{ret}(x-y)D^{ret}(x-z)\frac{\delta^2 j(u)}{\delta\varphi(y)\delta\varphi(z)}D^{ret}(u-x). \tag{2.9}$$

Separating here, as in the case for $N_Q(A^2)$ the terms

2)Here, of course, it is of the essence that we have chosen x = y before integrating over u or z. This is in agreement with the known circumstance that one may not carry out computations in a local theory using as intermediate regularization a simple separation of the variables in a local lagrangian since after removing this kind of regularization there remain some additional terms.

involving the retarded commutators and those with the operators (the first again yield zero, if one takes into account the adopted definition of the T_D-product) we are led to the final result

$$N_Q(A^3(x)) = A^3(x) + 2i\mathfrak{D}^-(x,x)A(x) + iA(x)\mathfrak{D}^-(x,x) + {}^1/_3 M_3(x,x,x), \tag{2.10}$$

where

$$M_3(x,x,x) = \int dy\,dz\,du\,[D^{ret}(x-y)D^{ret}(x-z)\Lambda_3(y,z,u)D^{ret}(u-x) + D^{ret}(x-y)D^{ret}(x-u)\Lambda_3(y,z,u)D^{ret}(z-x) + D^{ret}(x-z)D^{ret}(x-u)\Lambda_3(y,z,u)D^{ret}(y-x)]. \tag{2.11}$$

Thus we see that already the relation between the ordinary and quasinormal third power of the Heisenberg field $A(x)$ no longer has the form required by the Wick theorem. Nevertheless one may expect that a theorem of the Wick type is valid for higher powers, but with a more complicated right hand side.

3. A WICK THEOREM FOR THE POWERS OF HEISENBERG OPERATORS AND THE FORM OF THE CONTRACTION $\mathfrak{D}^-(x, x)$

One should not be surprised that in the general case the Wick theorem allowing to expand quasinormal products into ordinary ones, since its derivation was based on the fact that the elementary contraction was a c-number. One might therefore think that the situation will change if one restricts one's attention to theories in which the elementary contraction, i.e., essentially M_2, will not involve operators.

More precisely, one can see from (2.11) that it suffices to require: a) commutativity of $A(x)$ and M_2 and b) the vanishing of the operator M_3:

$$[A(x), M_2(x,x)] = 0, \quad M_3(x,x,x) = 0, \tag{3.1}$$

and then (2.11) could be written in the form

$$N_Q(A^3(x)) = A^3(x) + 3i\mathfrak{D}^-(x,x)A(x), \tag{3.2}$$

which would mean that the Wick theorem holds for the third power of $A(x)$, in the same manner as for the square. After this it is legitimate to expect that in the general case for the expansion of $N_Q(A^n(x))$ in terms of ordinary powers of the field $A(x)$ the following theorem will hold[3)].

Wick theorem. The power $N_Q(A^n(x))$ can be expanded in terms of $A^m(x)$ in the same manner as the power $:\varphi^n(x):$ is expanded in terms of the powers $\varphi^m(x)$, the contractions $\mathfrak{D}^-(x, x)$ having the form (2.8).

Then

$$N_Q(A^n(x)) = A(x)N_Q(A^{n-1}(x)) + i(n-1)\mathfrak{D}^-(x,x)N_Q(A^{n-2}(x)). \tag{3.3}$$

There arises the problem to characterize the class of theories for which the formulated Wick theorem is valid. It is easy to see that M_3 vanishes not only for $\Lambda_3 \equiv 0$, but also if Λ_3 does not contain terms with derivatives of the delta function. Moreover, one can show [8] that $M_3 = 0$ even when Λ_3 contains no more than three derivatives with respect to time of the delta functions in each argument. Only starting from fourth-order derivatives will the corresponding integral begin to be different from zero. Thus one might expect that the class of theories for which the theorem holds is at any rate not wider than the class of theories which we discovered earlier [8], and where the consequences of the Bogolyubov [1] and Lehmann [12] 4) axioms coincide.

We now discuss the contraction $\mathfrak{D}^-(x, x)$. If Λ_2 does not contain derivatives $\mathfrak{D}^-(x,x) = -\tfrac{1}{2}D^{(1)}(0)$. In a "pure renormalization" model [14] when 5) λ_2^R is a c-number of the form

$$\lambda_2^R(x,y) = -\frac{1-Z_3}{\sqrt{Z_3}}\hat{N}\,\delta(x-y)\frac{-K_x-K_y}{2}, \tag{3.4}$$

we obtain, substituting (3.4) into (2.8)

$$\mathfrak{D}^-(x,x) = D^-(0) + \left(1-\frac{1}{Z_3}\right)\frac{1}{2}[D^{ret}(0) + D^{adv}(0)] = \frac{1}{Z_3}D^-(0) + \left(1-\frac{1}{Z_3}\right)D^c(0), \tag{3.5}$$

i.e., also a c-number.

Since in genuinely renormalizable theories the operator Λ_2 has the representation [8]

$$\Lambda_2(x,y) = \lambda_2^R(x,y) + \Lambda_2^0(x,y), \tag{3.6}$$

where λ_2^R is of the form (3.4) and Λ_2^0 does not contain derivatives (possibly with the exception of the first derivative), $\mathfrak{D}^-(x, x)$ will be a c-number, i.e., in such theories the first condition (3.1) will be automatically satisfied.

Finally, for the case of the Lehmann class of theories the contraction $\mathfrak{D}^-(x, x)$ can turn out to be an operator, since in this class Λ_2 may contain operator terms with derivatives. It may however happen that the first contraction $\mathfrak{D}^-(x, x)$ behaves effectively as a c-number with respect to the operator $A(x)$.

4. GENERALIZED FUNCTIONAL DIFFERENTIATION AND PROOF OF WICK'S THEOREM

It is known [2,10] that the current-like operators Λ_ν satisfy the equations of motion

$$\frac{\delta\Lambda_\nu(x_1,\ldots,x_\nu)}{\delta\varphi(y)} + i\theta(y^0-x_i^0)[j(y),\Lambda_\nu(x_1,\ldots x_\nu)] = \Lambda_{\nu+1}(x_1,\ldots,x_\nu,y). \tag{4.1}$$

It is convenient to introduce the concept of generalized functional derivative of an operator $L(x)$:

$$\frac{\delta^* L(x)}{\delta^*\varphi(y)} \equiv \left(\frac{\delta}{\delta\varphi(y)} + i\theta(y^0-x^0)[j(y),\ldots]\right)L(x). \tag{4.2}$$

The derivative $\delta^*/\delta^*\varphi(y)$ acts on an ordinary

3) It is natural that in going over to higher order terms we shall be led to additional conditions of the type (3.1), terms which will be made concrete in the sequel. It is however clear from the outset that in order for such a Wick theorem to be valid it is necessary that the contributions from higher current-like operators $\Lambda\nu$ vanish, since otherwise one would be compelled to introduce in addition to contractions of pairs of operators A(x) of the form (2.8) also "contractions" corresponding to triples, quadruples, etc. of fields A(x). In other words in place of the Wick theorem one would have to apply to the powers of the Heisenberg fields something like the R-operation[7] known from perturbation theory.

4) In all justice this class of theories should be named after Lehmann since it is exactly the class in which the Lehmann reduction formula [12,13] holds.

5) The integral operators $\hat{N}$, $\hat{I}$, D^{adv}, $(\widehat{-K\delta})$, the function $\delta(x-y)$ as well as the rules for differentiating such operators have been described in detail in [14]. In particular, in the case where one may set $\hat{I} = 1$, the operator $\hat{N} = 1/Z_3^{1/2}$.

product of operators according to the Leibnitz formula, owing to the distributivity of the retarded commutator. With this notation the equations (4.2) take the form

$$\Lambda_{\nu+1}(x_1,\ldots,x_\nu,y)=\frac{\delta^*\Lambda_\nu(x_1,\ldots,x_\nu)}{\delta^*\varphi(y)}, \tag{4.3}$$

i.e., the higher order current-like operators Λ_ν are obtained by iterated application of generalized differentiation to the lowest order operator $\Lambda_1(x)\equiv j(x)$. Then if j(x(is subject to a dynamical law of the type (1.2) we can derive the whole chain of current-like operators by means of generalized functional differentiation, after learning how to take the variations of individual terms in $j(x) = f[A(x)]$, and in particular, after obtaining the expression for $\delta^*A(x)/\delta^*\varphi(y)$. In this connection we formulate a lemma.

Lemma 1. In all field theories

$$\frac{\delta^*A(x)}{\delta^*\varphi(y)}=\delta(x-y)-\theta(x^0-y^0)\int dzD^{adv}(x-z)\Lambda_2(z,\ y) \tag{4.4}$$
$$-\theta(y^0-x^0)\int dz\,D^{ret}(x-z)\Lambda_2(z,y).$$

In order to prove it we apply (4.2) to the Yang-Feldman relation for A(x) and utilize the solvability condition [1,10] for the current

$$\frac{\delta^*A(x)}{\delta^*\varphi(y)}=\delta(x-y)-\theta(x^0-y^0)\int dzD^{adv}(x-z)\frac{\delta j(z)}{\delta\varphi(y)} \tag{4.5}$$
$$-\theta(y^0-x^0)\int dzD^{ret}(x-z)\frac{\delta j(y)}{\delta\varphi(z)}.$$

If we now use the equation of motion (2.4), the considerations of Sec. 2 show that the terms involving the commutators of the currents cancel and what is left is (4.4).

One can also consider the generalized functional derivative of a linear operator of a more general form

$$B(x)=\int dz\,\Phi(x-z)A(z)=\hat{\Phi}_xA(x), \tag{4.6}$$

where $\Phi(x - z)$ is a c-function. The calculation in (4.5) is repeated here and

$$\frac{\delta^*B(x)}{\delta^*\varphi(y)}=\hat{\Phi}_x\delta(x-y)-\theta(x^0-y^0)\,\hat{\Phi}\hat{D}^{adv}\frac{\delta j(x)}{\delta\varphi(y)} \tag{4.7}$$
$$-\theta(y^0-x^0)\,\hat{\Phi}\hat{D}^{ret}\frac{\delta j(y)}{\delta\varphi(x)}.$$

One cannot, however, make a transition to an expression of the type (4.4), owing to the nonlocality of $\Phi(x - z)$. For us the case when $\Phi(x - z)$ is a quasilocal operator will be of particular interest, e.g., $\Phi(x - z) = -K_x\delta(x - z)$.

Considering the generalized functional derivatives of quasinormal powers of the Heisenberg fields A(x) we prove another lemma.

Lemma 2. In all field theories

$$N_Q(A^n(x))=A(x)N_Q(A^{n-1}(x))+i(n-1)D^c(0)N_Q(A^{n-2}(x)) \tag{4.8}$$
$$-i\int dz\,D^{adv}(x-z)\frac{\delta^*}{\delta^*\varphi(z)}N_Q(A^{n-1}(x)).$$

For the proof we consider the expression

$$T_W(:\varphi(x)\varphi^{n-1}(x):S)=\varphi(x)T_W(:\varphi^{n-1}(x):S)+$$
$$+i(n-1)D^c(0)T_W(:\varphi^{n-2}(x):S)-i\int dz\,D^{adv}(x-z)\frac{\delta T_W(:\varphi^{n-1}(x):S)}{\delta\varphi(z)}, \tag{4.9}$$

where we have used the ordinary Wick theorem to transform the fields $\varphi(x)$. Multiplying both sides from the right by S^+ and replacing the field $\varphi(x)$ in the first term by means of Eq. (1.1), as well as subjecting the term with the functional derivative to a transformation, we obtain

$$N_Q(A^n(x))=A(x)N_Q(A^{n-1}(x))+i(n-1)D^c(0)N_Q(A^{n-2}(x))$$
$$-i\int dz\,D^{adv}(x-z)\left\{\frac{\delta}{\delta\varphi(z)}+i[j(z),\ldots]\right\}N_Q(A^{n-1}(x)). \tag{4.10}$$

It is easy to see that if one inserts $1\equiv\theta(x^0-z^0)+\theta(z^0-x^0)$ in the curly bracket in front of the commutator, then taking into account the form of $D^{adv}(x - z)$ it will become the generalized functional derivative, and thus lemma 2 will be proven.

In order to transform (4.8) further it is necessary to make an assumption about the generalized derivative of $N_Q(A^{n-1}(x))$. We first differentiate (4.8) directly and obtain that in all field theories

$$\frac{\delta^*N_Q(A^n(x))}{\delta^*\varphi(y)}=\frac{\delta^*A(x)}{\delta^*\varphi(y)}N_Q(A^{n-1}(x))+A(x)\frac{\delta^*N_Q(A^{n-1}(x))}{\delta^*\varphi(y)}$$
$$+i(n-1)D^c(0)\frac{\delta^*N_Q(A^{n-2}(x))}{\delta^*\varphi(y)} \tag{4.11}$$
$$-i\int dz\,D^{adv}(x-z)\frac{\delta^*\delta^*}{\delta^*\varphi(y)\delta^*\varphi(z)}N_Q(A^{n-1}(x)).$$

If one now takes into account the expression (4.4) for $\delta^*A(x)/\delta^*\varphi(y)$ and the expressions derived in the Lagrangian formalism [8] for the operators Λ_ν in concrete theories, it is most natural to make the following assumption relative to the form of (4.11).

Lemma 3. In the class of renormalizable field theories we have:

$$\frac{\delta^*N_Q(A^n(x))}{\delta^*\varphi(y)}=nN_Q(A^{n-1}(x))\frac{\delta^*A(x)}{\delta^*\varphi(y)}. \tag{4.12}$$

Lemma 3 is easily proved by induction, making use of (4.11), lemma 2 and the preceding considerations regarding the vanishing of M_2 and M_3 in the absence of sufficiently high-order derivatives of (cf.[15]).

We now turn to the proof of Wick's theorem formulated in the preceding section. According to lemma 2, the left-hand side of (3.3) is equal to the right-hand side of (4.9). Substituting the expression for the generalized derivative from lemma 3, we obtain

$$N_Q(A^m(x))=A(x)N_Q(A^{m-1}(x)) \tag{4.13}$$
$$+i(m-1)\left\{D^c(0)-\int dz\,D^{adv}(x-z)\frac{\delta^*A(x)}{\delta^*\varphi(z)}\right\}N_Q(A^{m-2}(x)),$$

and taking into account lemma 1, this proves formula (3.3). Thus the Wick theorem is proved in renormalizable theories for $N_Q(A^n(x))$ with the contraction $\mathfrak{D}^-(x, x)$ of the form (3.5).

As regards the Lehmann class of theories, one could advance arguments in favor of the idea that this class might contain nonrenormalizable theories for which (4.12) remains valid.

5. SOME THEOREMS ON CURRENT-LIKE OPERATORS

Making use of the formalism of generalized functional derivatives we now consider several concrete forms of current-like operators which can be encountered in renormalizable theories, i.e., in theories for which the Wick theorem has been established for $N_Q(A^n(x))$, and lemma 3 holds. We show that in such theories this formalism allows one to determine the whole sequence of operators Λ_ν, i.e., to describe the dynamics completely. Of course, for this we shall have to assume several concrete dynamical laws of the type of Eq. (1.2).

Before formulating the initial assertions and results we discuss the question of what types of terms may enter the expression (1.2) for the current in renormalizable theories. As an example we consider the theory of a scalar self-interacting field, in which the discussion by means of the Lagrangian method [5] yields the following form for the current operator $j(x)$:

$$j(x) = -4gZ_1N_Q(A^3(x)) + Z_3\delta m^2 A(x) - (1 - Z_3)(-\widehat{K\delta})A(x), \quad (5.1)$$

where $(-\widehat{K\delta})$ is the operator defined in [14]. The conclusions which will be drawn now can be extended without difficulty to any renormalizable theory in which a scalar field participates in the interaction.

In order to derive the higher order current-like operators it is necessary to learn how to form generalized functional derivatives of the powers $N_Q(A^n(x))$ without derivatives, and in particular of $A(x)$ itself. In addition the same procedure needs to be applied to the only operator involving derivatives in (5.1), namely $(-\widehat{K\delta})A(x)$. The latter problem is simpler, since this term is linear in $A(x)$, i.e., it has the form of the operator $B(x)$ discussed above (cf. (4.6)).

In other words, in renormalizable theories there are always derivative terms in the current $j(x)$, but these terms refer only to the renormalization of the field. The latter circumstance implies [8] that the operator Λ_2 will have the form (3.6) with derivatives occurring only in the c-number part λ_2^R. At the same time it is exactly the problem of the presence and number of derivatives in the current-like operators which plays a decisive role in the structure of a field theory. We now prove the necessary theorems.

Theorem 1. Assume that in a renormalizable theory the current-like operator $\Lambda_\nu(x_1, \ldots, x_\nu)$ (or a term entering into this operator) has the form

$$\Lambda_\nu(x_1, \ldots, x_\nu) = \sum_{\alpha=0}^{n} C_\alpha N_Q(A^\alpha(x_1))\,\delta(x_1 - x_2) \ldots \delta(x_{\nu-1} - x_\nu). \quad (5.2)$$

Then a Λ_ν of the form (5.2) contributes to $\Lambda_{\nu+1}$, according to (4.3), a term of the form

$$\Lambda_{\nu+1}(x_1, \ldots x_\nu, x_{\nu+1}) = \quad (5.3)$$
$$= \sum_{\alpha=0}^{n} \alpha C_\alpha N_Q(A^{\alpha-1}(x_1))\,\frac{\delta^* A(x_1)}{\delta^*\varphi(x_{\nu+1})}\,\delta(x_1 - x_2) \ldots \delta(x_{\nu-1} - x_\nu).$$

The proof of theorem 1 follows directly from lemma 3.

Since the functional derivative of a c-number vanishes, (3.6) implies that in renormalizable theories all Λ_ν starting from $\nu = 3$, are determined by using theorem 1 applied only to the operator Λ_2^0. It remains to determine the form of $\delta^*A(x)/\delta^*\varphi(y)$ in such theories and to indicate a method of determining the operator Λ_2^0 from a known current $j(x)$.

The first question is solved by the following theorem.

Theorem 2. In renormalizable field theories

$$\frac{\delta^* A(x)}{\delta^*\varphi(y)} = \frac{\hat{N}}{\sqrt{Z_3}}\,\delta(x - y). \quad (5.4)$$

Returning to (4.4) we see that only the term Λ_2^R contributes to (5.4), the contribution being the same as in the pure renormalization model [14]. Establishing first that

$$\hat{D}_x^{adv}\lambda_2^R(x, y) = -\frac{1 - Z_3}{2\sqrt{Z_3}}\,\hat{I}\,\hat{N}\delta(x - y) - \frac{1 - Z_3}{2\sqrt{Z_3}}\,D^{adv}(x - y)\hat{N}(-\underset{\rightarrow}{K_y}), \quad (5.5)$$

we find

$$\frac{\delta^* A(x)}{\delta^*\varphi(y)} = \frac{(1 + \sqrt{Z_3})\hat{N} - (1 - \sqrt{Z_3})}{2\sqrt{Z_3}}\,\delta(x - y) \quad (5.6)$$
$$+ \frac{1 - Z_3}{2\sqrt{Z_3}}\left\{\bar{D}(x - y)\,\hat{N}(-\underset{\rightarrow}{K_y}) - \frac{\varepsilon(x^0 - y^0)}{2}\,D(x - y)\,\hat{N}(-\underset{\rightarrow}{K_y})\right\},$$

where in the last term the operator $(-K_y)$ operates on $\epsilon(x^0 - y^0)$, yielding

$$D(x - y)\,\hat{N}(-\underset{\rightarrow}{K_y})\,\frac{\varepsilon(x^0 - y^0)}{2} = \bar{D}(x - y)\,\hat{N}(-\underset{\rightarrow}{K_y}) \quad (5.7)$$
$$+ D(x - y)\,\hat{N}\delta'(x^0 - y^0) - 2D(x - y)\,\hat{N}\delta(x^0 - y^0)\,\frac{\partial}{\partial y^0}.$$

Collecting all terms we obtain (5.4).

We now consider an operator $B(x)$ of the form (4.6) with $\Phi_x = (-\widehat{K\delta})_x$.

Lemma 4. In all field theories

$$\frac{\delta^*(-\widehat{K\delta})A(x)}{\delta^*\varphi(y)} = -K\delta(x - y) - \theta(x^0 - y^0)(-\widehat{K\delta})\hat{D}^{adv}\Lambda_2(x, y)$$
$$- \theta(y^0 - x^0)(-\widehat{K\delta})\,\hat{D}^{ret}\Lambda_2(x, y). \quad (5.8)$$

In order to prove (5.8) we consider the expression (4.7). A relatively laborious computation (cf. [15]) shows that for an arbitrary function f

$$\theta(x^0 - y^0)(-\widehat{K\delta})_x\hat{D}^{adv}f(x, y) = {}^1/_2\,\theta(x^0 - y^0)\,\hat{I}f(x, y)$$
$$+ \frac{1}{2}(-\underset{\leftarrow}{K_x})\,\theta(x^0 - y^0)\,\hat{D}^{adv}f(x, y) + \left(\underset{\leftarrow}{\frac{\partial}{\partial x^0}}\right)\delta(x^0 - y^0)\,\hat{D}^{adv}f(x, y)$$
$$- \frac{1}{4}\left(\underset{\leftarrow}{\frac{\partial}{\partial x^0}}\right)\delta(x^0 - y^0)\,\hat{D}^{adv}f(x, y) + \frac{1}{4}\,\delta(x^0 - y^0)\,\hat{D}^{adv}f(x, y)\left(\underset{\rightarrow}{\frac{\partial}{\partial y^0}}\right)$$
$$- \frac{1}{4}\,\delta(x^0 - y^0)\,\hat{\dot{D}}^{adv}f(x, y) + \frac{1}{4}\,\delta(x^0 - y^0)\,\hat{D}^{adv}\,\frac{\partial f(x, y)}{\partial y^0}. \quad (5.9)$$

This implies that in this case in (4.7) one can replace $\delta j(x)/\delta\varphi(y)$ by $\Lambda_2(x, y)$, since the retarded current commutator again gives a vanishing expression, owing to a closed chain of θ-functions.

The concrete form of (5.8) for the case of interest is given by the following theorem.

Theorem 3. In a pure renormalization model [14]

$$\frac{\delta^*(-\widehat{K\delta})A_R(x)}{\delta^*\varphi(y)} = \frac{\hat{N}}{\sqrt{Z_3}}\,\delta(x - y)\left(\frac{-K_x - K_y}{2}\right). \quad (5.10)$$

Indeed, taking into account the fact that λ_2^R has the form (3.4) and utilizing lemma 4 we obtain the desired result, after using the transformations described in detail in [15].

It is easy to understand that owing to (3.6) the contribution from λ_2^R to the expression (5.8) will in the present theory coincide with (5.10). However if the contribution of λ_2^0 to (4.4) vanishes (cf. (5.4)), the contribution of λ_2^0 to (5.8) does not vanish. We shall compute the contribution and find the final expression for the operator Λ_2 for renormalizable theories in the following section.

6. A LINEAR EQUATION FOR Λ_2

We can now carry out the last step and derive an equation for Λ_2 on the basis of the theorems proved above. The exclusive difficulty of solving (4.1) directly resides in the nonlineaity of the equations. We shall manage to avoid this difficulty by passing to generalized functional derivatives, i.e., to equations of the type (4.3). Knowing the expression (1.2) for the current $j(x)$ and using theorems 1–3, we obtain for the operator Λ_2 in a theory with $j(x)$ of the form (5.1) the linear equation

$$\Lambda_2(x, y) = \frac{\delta^* j(x)}{\delta^* \varphi(y)} = [-12gZ_1N_Q(A^2(x)) + Z_3\delta m^2]\frac{\delta^* A(x)}{\delta^* \varphi(y)} - (1-Z_3)\frac{\delta^*(-\hat{K}\delta)A(x)}{\delta^*\varphi(y)}. \tag{6.1}$$

Substituting into (6.1) the expressions (5.4), (5.8) and taking into account (3.6) and (5.10) we obtain

$$\begin{aligned}\Lambda_2(x, y) = {} & [-12gZ_1N_Q(A^2(x)) + Z_3\delta m^2]\frac{\hat{N}}{\sqrt{Z_3}}\delta(x-y) \\ & - \frac{1-Z_3}{2\sqrt{Z_3}}\hat{N}\delta(x-y)(-K_x - K_y) \\ & - (1-Z_3)[-\theta(x^0-y^0)(-\hat{K}\delta)\hat{D}^{adv}\Lambda_2^0(x, y) \\ & - \theta(y^0-x^0)(-\hat{K}\delta)\hat{D}^{ret}\Lambda_2^0(x, y)].\end{aligned} \tag{6.2}$$

It is easy to see that in a pure renormalization model [14] (i.e., when $\Lambda_2^0 = 0$) the latter equation becomes an identity

$$\lambda_2^R(x, y) = -\frac{1-Z_3}{2\sqrt{Z_3}}\hat{N}\delta(x-y)(-K_x - K_y). \tag{6.3}$$

In other words, this solution leads to the same expression (cf. (3.4)) for Λ_2^R as the other independent methods [8,14]. In the general case, substituting into the left-hand side of (6.2) an expression for Λ_2 of the form (3.2) and reducing similar terms, we are led to a linear equation for the operator Λ_2^0 only:

$$\begin{aligned}\Lambda_2^0(x, y) = {} & [-12gZ_1N_Q(A^2(x)) + Z_3\delta m^2]\frac{\hat{N}}{\sqrt{Z_3}}\delta(x-y) \\ & - (1-Z_3)[-\theta(x^0-y^0)(-\hat{K}\delta)\hat{D}^{adv}\Lambda_2^0(x, y) \\ & - \theta(y^0-x^0)(-\hat{K}\delta)\hat{D}^{ret}\Lambda_2^0(x, y)].\end{aligned} \tag{6.4}$$

In principle this equation can be solved directly. For this it is necessary either to combine all terms involving Λ_2^0 in the left-hand side and then to divide both sides of the equation by the operator obtained in this way (in the spirit of [14]), or to use iterations, utilizing as a zero order approximation the first term of the right-hand side of (6.4). In either case it is desirable to simplify first the integro-differential operator in the square brackets of eq. (6.4).

For this purpose we consider it in more detail, replacing for convenience the θ-functions by ϵ-functions:

$$\begin{aligned}& -\frac{1+\varepsilon(x^0-y^0)}{2}(-\hat{K}\delta)\hat{D}^{adv}\Lambda_2^0(x, y) \\ & -\frac{1-\varepsilon(x^0-y^0)}{2}(-\hat{K}\delta)\hat{D}^{ret}\Lambda_2^0(x, y) = -\frac{1}{2}\hat{I}\Lambda_2^0(x, y) \\ & -\frac{1}{2}(-\underleftarrow{K_x})\hat{\tilde{D}}\Lambda_2^0(x, y) + \frac{1}{2}\frac{\varepsilon(x^0-y^0)}{2}(-\underleftarrow{K_x})\hat{D}\Lambda_2^0(x, y).\end{aligned} \tag{6.5}$$

In the last term the operator $(-\underleftarrow{K}_x)$ can also act on $\epsilon(x^0 - y^0)$. After doing this the terms involving $\widehat{D}$ disappear from (6.5) and as a result we obtain that the operator we are interested in has the form

$$-\frac{1}{2}\hat{I}\Lambda_2^0(x, y) + \frac{1}{2}\delta'(x^0-y^0)\hat{D}\Lambda_2^0(x, y) = -\frac{1+\hat{I}}{2}\Lambda_2^0(x, y), \tag{6.6}$$

where account has been taken of the fact that Λ_2^0 does not contain derivatives, and use has been made of the rules of differentiation listed in [14].

Substituting (6.6) into the fundamental equation (6.4), combining the terms with Λ_2^0 and dividing by the appropriate operator, we finally obtain for Λ_2^0 in such a theory

$$\Lambda_2^0(x, y) = \frac{2}{(1+Z_3) - (1-Z_3)\hat{I}} \times [-12gZ_1N_Q(A^2(x)) + Z_3\delta m^2]\frac{\hat{N}}{\sqrt{Z_3}}\delta(x-y). \tag{6.7}$$

If it is not necessary to differentiate this expression further [14], one may set $\hat{I} \equiv 1$, so that the current-like operator $\Lambda_2(x, y)$ of the form (3.6) becomes

$$\Lambda_2(x, y) = \left(\frac{1}{Z_3} - 1\right)K\delta(x-y) - \frac{12gZ_1}{Z_3^2}N_Q(A^2(x))\delta(x-y) + \frac{\delta m^2}{Z_3}\delta(x-y), \tag{6.8}$$

in agreement with previously derived expressions of the Lagrangian method [5,8].

It is obvious that all higher Λ_ν in this theory can be obtained from (6.7) in a simple manner by applying theorems 1 and 2. In the case $\hat{I} \equiv 1$ one also obtains the expressions already derived earlier:

$$\Lambda_3(x, y, z) = -\frac{24gZ_1}{Z_3^3}A(x)\delta(x-y)\delta(x-z), \tag{6.9}$$

$$\Lambda_4(x, y, z, u) = -\frac{24gZ_1}{Z_3^4}\delta(x-y)\delta(x-z)\delta(x-u), \tag{6.10}$$

and all higher Λ_ν vanish. Thus the number of nontrivial operators Λ_ν, in a renormalizable theory with $j(x)$ of the form (5.1) turns out to be finite. Naturally, all these results can be carried over without undue difficulty to other renormalizable theories.

7. CONCLUSION

In this investigation we have succeeded in establishing the relation between the powers $N_Q(A^n(x))$ (which are easily defined in the axiomatic method [2,9]) and ordinary powers of the Heisenberg fields $A(x)$. It turned out that the relation between these objects depends in an essential manner on the character of the current-like operators Λ_ν and can be reduced to a Wick theorem for a quite definite class of theories, characterized by the number of derivatives. In renormalizable theories the corresponding contractions turn out to be c-numbers.

The establishment of such a connection allows one to determine the operators Λ_ν successively, starting with the expression (1.2) for the current in renormalizable theories. This is achieved by introducing the operation of generalized functional differentiation. It is remarkable that this operation allows one to "linearize," in a manner of speaking, the nonlinear equations of form (4.1) for the Λ_ν only in a definite class of theories. The examples considered in Secs. 5 and 6 show the effectiveness of this method. In particular we have succeeded in providing a constructive example of a nontrivial theory with a finite number of nonvanishing Λ_ν.

In recent months there have appeared a number of papers noting the contradictions inherent in the traditional concepts of current algebra, and the necessity of adding to the corresponding relations of terms involving the derivatives of delta functions.[16] In the light of the preceding discussion one can guess the source of such difficulties (the corresponding investigations have not considered explicitly counterterms, in particular such including derivatives). It should be expected at the same time that the equal-time commutators should be related to the corresponding operator Λ_2. Therefore neglecting the latter might lead to a vanishing of the equal-time commutator. In order that this does not happen and the theory be nontrivial it is necessary that Λ_2 contain both an operator and a c-number part, and this is realized in our method.

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