INTERACTION OF HIGH-FREQUENCY AND LOW-FREQUENCY WAVES IN NONLINEAR DISPERSIVE MEDIA

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Approximate solutions are given for the equations that describe the interaction of high-frequency and low-frequency waves propagating in nonlinear dispersive media under the assumption of a weak nonlinear interaction. The number of high-frequency waves that participate in the interaction is arbitrary but only one low-frequency wave is considered. Resonance and nonresonance wave interactions are considered, as is the case of a weakly inhomogeneous medium. The results that are obtained are applied to the interaction of light with plasma waves.

1. INTRODUCTION

THE present work is devoted to an investigation of the nonlinear interaction of waves that propagate in dispersive media. The assumptions as to the nature of the nonlinear interaction are the following.

We assume that in some medium there can propagate (in the linear approximation) two waves, low-frequency and high-frequency, and that the interaction between these waves is such that one high-frequency wave can decay into a low-frequency wave and another high-frequency wave, or that it can combine with a low-frequency wave to form a new high-frequency wave. Thus, if initially only one high-frequency wave with frequency ω_0 and one low-frequency wave with frequency ν are excited, then by virtue of the nonlinear interaction there will be an infinite number of waves at the combination frequencies $\omega_n = \omega_0 + n\nu$, where n is an arbitrary integer.

The problem of determining the redistribution of energy between the various harmonics can be solved approximately under the following assumptions: 1) the nonlinear interaction is so weak that the amplitudes of the waves change only slightly in distances of the order of a wavelength (in other words, the notion of a wave with a definite frequency and wave vector holds); 2) the high-frequency ω is considerably higher than the low-frequency ν.

The basic equations are derived in Sec. 2. Section 3 is devoted to an investigation of the symmetry properties of the equations that are obtained and to the conservation relations and the consequences that follow from these. The fourth and fifth sections contain an investigation and solution of a particular problem. In the sixth section we extend the results to the case of weakly inhomogeneous media. As an example we consider the interaction of transverse electromagnetic waves and plasma waves in an isotropic plasma (Sec. 7).

2. BASIC EQUATIONS

Let us consider a uniform nonlinear dispersive medium in which two waves can propagate (we denote these by u and v); the equations that describe the propagation of these waves are assumed to be of the following form:

$$\frac{\partial^2 u}{\partial t^2} - c_1^2\frac{\partial^2 u}{\partial x^2} = \int\limits_{-\infty}^{\infty} G_1(x-x', t-t')u(x',t')\,dx'\,dt' + \mathcal{H}_1(u,v), \tag{2.1a}$$

$$\frac{\partial^2 v}{\partial t^2} - c_2^2\frac{\partial^2 v}{\partial x^2} = \int\limits_{-\infty}^{\infty} G_2(x-x', t-t')v(x',t')\,dx'\,dt' + \mathcal{H}_2(u,u), \tag{2.1b}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are quadratic forms in u and v given by the relations

$$\mathcal{H}_1(u,v) = \frac{\delta}{\delta u}\mathcal{L}(u,u,v), \quad \mathcal{H}_2(u,u) = \frac{\delta}{\delta v}\mathcal{L}(u,u,v). \tag{2.2}$$

Here, $\mathcal{L}(u, u, v)$ is the sum of the cubic terms of the Lagrangian which describes the nonlinear interaction between the waves. The dispersive properties of the medium are described by the kernels $G_1(z, \tau)$ and $G_2(z, \tau)$. The dispersion relations, as determined from the linear theory, are of the following form:

$$\omega^2 - c_1^2k^2 + g_1(k,\omega) = 0, \quad \nu^2 - c_2^2\varkappa^2 + g_2(\varkappa,\nu) = 0, \tag{2.3}$$

where

$$g_{1,2}(k,\omega) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} G_{1,2}(z,\tau)e^{-i(\omega\tau+kz)}\,dz\,d\tau.$$

As is evident from Eq. (2.2), the nonlinear interaction is a three-wave interaction in which two of the interacting waves are of type u and the other of type v.

We shall solve this problem with two different sets of boundary conditions:

a) At the boundary of the medium (x = 0) two waves are excited: type u with frequency ω_0 and amplitude a_0 and type v with frequency ν and amplitude b_0:

$$u(0,t) = a_0e^{i\omega_0 t}, \quad v(0,t) = b_0e^{i\nu t}. \tag{2.4}$$

b) At the boundary of the medium two type u waves are excited with frequencies ω_0 and ω_1 and amplitudes a_0 and a_1:

$$u(0,t) = a_0e^{i\omega_0 t} + a_1e^{i\omega_1 t}, \quad v(0,t) = 0. \tag{2.5}$$

Because of the nonlinear interaction, as the waves propagate along the x-axis there appear u-waves with combination frequencies

$$\omega_n = \omega_0 + n\nu \tag{2.6}$$

and v-waves with frequencies $m\nu$. The problem consists of determining the dependence of the amplitudes of the u and v waves on the coordinate x for the specified boundary conditions.

In this work we shall limit our analysis to the case in which of all the low-frequency waves, only one is excited with appreciable amplitude, the frequency ν (this situation is realized if, for example, v-waves with frequency $m\nu$ ($m > 1$) are highly damped in the medium).

We now seek the solution of (2.1) in the form

$$u(x,t)=\mathrm{Re}\,\{a_n(x)k_n^{-1/2}e^{(i\omega_n t-ik_n x)}\},\quad v(x,t)=\mathrm{Re}\,\{b(x)\varkappa^{-1/2}e^{i\nu t-i\varkappa x}\},\quad k_n=k(\omega_n), \tag{2.7}$$

where the relations between the frequencies and wave vectors are given by (2.3).

Substituting Eq. (2.7) in Eq. (2.1) and taking account of Eq. (2.2) we obtain

$$k_n^{-1}a_n''-2ia_n'=h(n-1,n)\,a_{n-1}b\exp(-i\Delta_n x)+h(n,n+1)\,a_{n+1}b^*\exp(i\Delta_{n+1}x),$$
$$\varkappa^{-1}b''-2ib'=\sum_n h(n-1,n)\,\overset{*}{a}_{n-1}a_n\exp(i\Delta_n x), \tag{2.8}$$

where

$$\Delta_n=k_{n-1}+\varkappa-k_n,$$

$$(k_nk_{n-1}\varkappa)^{-1/2}h(n-1,n)\,a_{n-1}b\exp[-i(k_n+\varkappa)x]=\mathscr{H}_1(a_{n-1}e^{-ik_nx},\ be^{-i\varkappa x}). \tag{2.9}$$

By virtue of the assumption as to the weakness of the nonlinear interaction we can neglect the second derivatives of the amplitude in Eq. (2.8).[1)] As a result we obtain the following system of equations:

$$-2ia_n'=h(n-1,n)\,a_{n-1}b\exp(-i\Delta_n x)+h(n,n+1)\,a_{n+1}b^*\exp(i\Delta_{n+1}x), \tag{2.10}$$

$$-2ib'=\sum_n h(n-1,n)\,\overset{*}{a}_{n-1}a_n\exp(i\Delta_n x). \tag{2.11}$$

In the case being considered, by virtue of the inequalities

$$\omega_n\gg\nu,\qquad k_n\gg\varkappa \tag{2.12}$$

it is reasonable to replace the dispersion relation $k(\omega)$ by the linear approximation $k(\omega)\approx k_0+k'(\omega_0)(\omega-\omega_0)$. Then the detuning is found to be independent of m:

$$\Delta_n=\Delta. \tag{2.13}$$

In place of Eq. (2.11) it is convenient to use the following relation, which follows from (2.10), (2.11) and (2.13):

$$b''-i\Delta b'+\Omega^2 b=0, \tag{2.14}$$

where

$$4\Omega^2=|a_-|^2h^2(n_-,n_-+1)-|a_+|^2h^2(n_+-1,n_+)+\sum_{n_-<n<n_+}|a_n|^2[h^2(n,n+1)-h^2(n-1,n)]. \tag{2.15}$$

Here, k_- and k_+ are the limits of the allowable values of the wave vectors for the u-waves so that the u-waves with $k < k_-$ and $k > k_+$ cannot propagate in the medium. In the majority of practical cases the volume occupied by the medium is bounded and the wave conversion process occurs over finite distances L, so that only a finite number of waves N(L) can actually be excited. Hence the boundaries of the spectrum can be neglected if the wave vectors of the initial u-waves remain sufficiently far from the boundaries:

$$(k_0-k_-)/\varkappa,\qquad (k_+-k_0)/\varkappa>N(L). \tag{2.16}$$

In this case it is convenient to replace Eq. (2.15) by the expression

$$4\Omega^2=\sum_n|a_n|^2[h^2(n,n+1)-h^2(n-1,n)]. \tag{2.17}$$

Considerable interest attaches to the case in which the detuning Δ is small because it is only for conditions of resonance nonlinear coupling that one expects any important redistribution of energy between wave modes. In the nonresonance case in which Δ is not small only a few waves can be excited. For this reason, in the nonresonance case one expects that only a few equations will remain out of the infinite system in Eq. (2.10). For the case in which the amplitudes of all waves are zero except for three or four, equations of the form (2.10) and (2.11) have been investigated in nonlinear optics and in the theory of nonlinear wave interactions in a plasma (cf.[1-3]). A paper by Danilkin[4] has considered the interaction of high-frequency electromagnetic waves with electron plasma waves in a uniform isotropic plasma and has obtained a solution of equations of the form in (2.10) and (2.11) which is a particular case of the solutions obtained in the present work (cf. Sec. 4).

3. CONSERVATION LAWS AND SYMMETRY OF THE EQUATIONS OF MOTION

Two integrals of the motion follow from Eqs. (2.10) and (2.11):

$$E=\sum_n|a_n|^2\omega_n+|b|^2\nu\equiv E_u+E_v, \tag{3.1}$$

$$I=\sum_n|a_n|^2. \tag{3.2}$$

The first of these is the total energy of the system while the second is the total action of oscillators with frequencies $\{k_n\}$. The second integral is essentially an adiabatic invariant.

Furthermore, for the case of exact resonance ($\Delta = 0$) the equations for the amplitude are invariant under the transformations

$$x\to-x,\quad b\to-b. \tag{3.3}$$

A very useful feature follows: if the amplitude b(x) vanishes at certain points $x = x_1, x_2$ then b(x) and, consequently $a_n(x)$ are periodic functions with period $2|x_2 - x_1|$.[2)]

On the basis of these properties of the equations of motion we can draw certain conclusions as to the nature of the redistribution of energy between the waves that are excited.

If the band of allowed values of wave vector $\{k_n\}$ is bounded from below ($k_n \geq k_-$) the total energy of the u-waves is bounded from below:

$$E_u\geqslant Ik_-, \tag{3.4}$$

and consequently

$$E_v\leqslant E-Ik_-. \tag{3.4'}$$

If the band of allowable values of the wave vector $\{k_n\}$

[1)]This procedure is equivalent to the first approximation in the Krylov-Bogolyubov method.

[2)]We note that Eq. (2.1) contains second derivatives which are not invariant against the substitution (3.3) when $\Delta = 0$ so that the indicated invariants and the conclusions that follow are actually approximate.

is bounded from above ($k_n \lesssim k_+$) then the total energy of the u-waves is bounded from above:

$$E_u \leqslant Ik_+, \tag{3.5}$$

and consequently $E_V \geq E - Ik_+$.

If $k_- \leq k \leq k_+$, then

$$Ik_- \leqslant E_u \leqslant Ik_+. \tag{3.6}$$

4. RESONANCE CASE

We now consider the case in which $\Delta = 0$, that is to say, the exact resonance condition is satisfied. By virtue of the inequalities in (2.12) the coefficients $h(n-1, n)$ are weak functions of n and this makes it possible to find an approximate solution for the problem.

a) **Zeroth approximation.** In the zeroth approximation in κ/k_n the coefficients $h(n-1, n)$ are independent of n so that Eqs. (2.10) and (2.14) assume the following form:

$$-2ia_n' = h(ba_{n-1} + b^*a_{n+1}), \tag{4.1}$$

$$b'' + \Omega^2 b = 0, \quad 4\Omega^2 = h^2[|a_-|^2 - |a_+|^2]$$
$$(a_n = 0 \quad \text{for } n = n_- - 1, \quad n = n_+ + 1). \tag{4.2}$$

The general solution of Eqs. (4.1) is given by the following expressions:

$$a_n(x) = \sum_{-\infty}^{\infty} a_m{}^0 y_{mn}(x), \tag{4.3}$$

where

$$y_{mn}(x) = i^{(m-n)} e^{i(m-n)\beta} J_{m-n}(h|B|),$$
$$B(x) = \int_0^x b(z)\,dz, \quad \beta(x) = \arg B(x), \tag{4.4}$$

and $J_n(x)$ is the Bessel function. In this case, for an unbounded band of allowed values we find

$$a_m{}^0 = a_m(0). \tag{4.5}$$

For a semi-bounded band ($a_n = 0$ for $n = l - 1$ or $n = l + 1$)

$$a_m{}^0 = \pm \sum_{k=l}^{\pm\infty} a_k(0)[\delta(k-m) - \delta(k+m-2l)] \tag{4.6}$$

and for a finite range of allowed values of $\{k_n\}$ ($a_n = 0$ for $n = l - 1$ and $n = l + N + 1$)

$$a_m{}^0 = \sum_{s=-\infty}^{\infty} \sum_{k=l}^{l+N} a_k(0)[\delta(k-m-2sN) - \delta(k+m-2l-2sN)]. \tag{4.7}$$

It is evident from these expressions that as $|B(x)|$ increases a larger and larger number of high-frequency waves are excited and the squares of the moduli of the amplitudes of the excited waves are equilibrated on the average.

As far as the amplitudes of the low-frequency wave are concerned we note that these depend sensitively on the form of the region of allowed values of wave vectors $\{k_n\}$ and initial conditions, as can be seen easily from Eqs. (4.2) and Eqs. (4.3)—(4.7). Thus, for the case of an unbounded band of allowed values of $\{k_n\}$ the quantity $\Omega = 0$ and the solution of equation (4.2) becomes the following:

$$b(x) = b_0 + b_0'x, \quad B(x) = b_0x + {}^1/_2 b_0'x^2, \tag{4.8}$$

for the boundary conditions in (2.4) we have $b_0' = 0$ and for the boundary conditions in (2.5) $b_0 = 0$ and $b_0' = ha_0^* a_1$ and the amplitudes of the low-frequency waves increase linearly with x. If one takes account of the limitations on the regions of allowed values of $\{k_n\}$, it follows from the energy conservation relation (3.1) that infinite growth of b(x) is not possible. In the case of semi-bounded systems bounded from below for allowed values of $\{k\}$ we find $\Omega^2 \geq 0$ and consequently b(x) vanishes many times. By virtue of the symmetry of Eq. (3.3) this means that b(x) and, consequently $a_n(x)$, are periodic functions. If the band of allowable values of $\{k_n\}$ is bounded from above then b(x) increases exponentially. For the case of a finite range of allowed values of $\{k_n\}$ the nature of the variation of b(x) is determined only by the initial conditions. In accordance with the energy conservation relation (2.1) we find that b(x) is a bounded function. Consequently either b(x) vanishes at least two times and the wave conversion process is periodic in x or it does not vanish more than once, in which case $|B(x)|$ increases with x and the conversion process, in accordance with Eqs. (4.3)—(4.7), leads to an equilibration (on the average) of the energy distribution between the high-frequency waves over the entire range $l \leq n \leq l + N$.

c) **WKB approximation.** In order to solve the equations for the u-waves (2.10) taking account of the dependence of the coefficients $h(n-1, n)$ on n we make use of the WKB method in conjunction with a Laplace transform. The general solution of this equation is given in the form of (4.3) where the coefficients a_m^0 are determined from Eqs. (4.5)—(4.7); the functions $y_{mn}(x)$ are found to be given by the following expressions after some simple but tedious calculations:

$$y_{mn}(x) = \frac{2ie^{i(m-n)\beta}}{(h_m h_n)^{1/2}} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\exp(\mu|B|)\prod_{l=m}^{n}\rho_l}{[(\rho_n{}^2-1)(\rho_m{}^2-1)]^{1/2}}\,d\mu. \tag{4.9}$$

Here,

$$\rho_n(\mu) + \rho_n{}^{-1}(\mu) = -2i\mu/h_n, \quad h_n = {}^1/_2[h(n-1, n) + h(n, n+1)],$$
$$B(x) = \int_0^x b(z)\,dz, \quad \beta(x) = \arg B(x). \tag{4.10}$$

When n = m,

$$y_{mm}(x) = J_0(h_m|B|). \tag{4.11}$$

In the case of a semi-bounded band and a finite range of allowed values of $\{k_n\}$ the coefficients h_n are determined by relations of the form in (4.6) and (4.7):

$$h_n = \pm \sum_{k=l}^{\pm\infty} h_k[\delta(k-n) - \delta(k+n-2l)] \tag{4.12}$$

and

$$h_n = \sum_{s=-\infty}^{\infty} \sum_{k=l}^{l+N} h_k[\delta(k-n-2sN) - \delta(k+n-2l-2sN)].$$

In deriving Eq. (4.9) and the equations of motion (2.10) we have omitted terms proportional to $\beta'(x)$. In neglecting the dependence of the coefficients $h(n-1, n)$ on n these terms cancel each other and for the boundary conditions being treated here (2.4) and (2.5), it follows from Eq. (2.14) that $\beta'(x) = 0$. It can be shown that the role of the omitted terms is unimportant in the more general case. The point is that if B(x) is an unbounded increasing function then $\beta(x)$ changes slowly and $\beta'(x)$ is negligibly small. However, if B(x) and all of the wave conversion processes are periodic then terms of order $\beta'(x)$ cannot have a noticeable effect on the process within the length corresponding to one period. As is evident from Eq. (4.9), the functions $y_{mn}(x)$ are asymmetric with respect to the transformation $n \to 2m - n$ if the coefficients $h(n-1, n)$ depend on n. This means that the process of exciting the u-waves for the boundary conditions in (2.4) is asymmetric with respect to $n = 0$. For example

$$y_{\pm 1,0} = J_1(\bar{h}_{\pm 1}|B|), \quad \bar{h}_{\pm 1} = (h_0 h_{\pm 1})^{1/2}.$$

It then follows that for small values of $|B|$ ($h|B| \ll 1$) the ratio of the moduli of the amplitudes $|a_1/a_{-1}|$ is equal to $|\bar{h}_1/\bar{h}_{-1}|$; for large values of $|B|$ ($h|B| \gg 1$) the ratio $|a_1/a_{-1}| \sim (|\bar{h}_{-1}/\bar{h}_1|)^{1/2}$. Thus, when $h_n > h_{n-1}$ waves at higher frequencies predominate; when $h_n < h_{n-1}$ waves at lower frequencies predominate.

In the equation for the amplitude of the v-wave, if we take account of the dependence of the coefficients $h(n-1, n)$ on n in Ω^2 contributions are also obtained from terms that are proportional to the square of the amplitudes of the unbounded waves

$$4\Omega^2 = |a_-|^2 h^2(n_-, n_- + 1) - |a_+|^2 h^2(n_+ - 1, n_+) + \sum_{n_- < n < n_+} |a_n|^2 [h^2(n, n+1) - h^2(n-1, n)]. \tag{4.13}$$

If the dependence of the coefficients $h(n-1, n)$ on n is weak the difference in the square brackets in Eq. (4.13) can be regarded as independent of n. Denoting this quantity by q, we find that (4.13) is replaced by

$$4\Omega^2 = |a_-|^2 [h^2(n_-, n_- + 1) - q] - |a_+|^2 [h^2(n_+ - 1, n_+) + q] + qI, \tag{4.14}$$

where I is the integral of the motion (3.2).

The form of Ω^2 becomes particularly simple when the limit of allowable values of $\{k_n\}$ can be regarded as infinite: $4\Omega^2 = qI$. In this case the behavior of the amplitudes of the v-wave is determined by the sign of q. When $q > 0$, with the boundary conditions in (2.4) we find

$$b(x) = b_0 \cos \Omega x, \quad B(x) = (b_0/\Omega) \sin \Omega x. \tag{4.15}$$

In this case $b(x)$ and $a_n(x)$ are periodic functions in x with period $2\pi/\Omega$. Under these conditions, since the maximum number of appreciably excited u-waves is approximately equal to $2 \max h|B|$, as follows from Eq. (4.5), then

$$N_{max} \approx 2h \left| \frac{b_0}{\Omega} \right| = 2h \frac{|b_0|}{q|a_0|^2}. \tag{4.16}$$

When $q > 0$ and for the boundary conditions in (2.5)

$$b(x) = b_0' \sin \Omega x, \quad B(x) = \frac{b_0'}{\Omega}(1 - \cos \Omega x), \quad b_0' = a_0^* a_1 h(0, 1), \tag{4.17}$$

so that in this case

$$N_{max} \approx 4h(0, 1) \frac{|b_0'|}{\Omega} = \frac{4h^2(0, 1)|a_0^* a_1|}{q[|a_0|^2 + |a_1|^2]}. \tag{4.18}$$

When $q < 0$, with the boundary conditions in Eqs. (2.4) and (2.5)

$$b(x) = b_0 \operatorname{ch}|\Omega|x, \quad B(x) = \frac{b_0}{|\Omega|} \operatorname{sh}|\Omega|x \tag{4.19}$$

and

$$b(x) = b_0' \operatorname{sh}|\Omega|x, \quad B(x) = \frac{b_0'}{|\Omega|}[\operatorname{ch}|\Omega|x - 1].$$

In this case the amplitude of the v-wave and $|B(x)|$ both increase exponentially in x. Taking account of the boundaries of the region of allowable values of $\{k_n\}$, in accordance with Eq. (3.4′), we find that $b(x)$ becomes a bounded function and that its behavior is extremely sensitive to the boundary conditions.

5. NONRESONANCE CASE

We now consider the case in which $\Delta \neq 0$, that is to say, the case in which the exact resonance condition is not satisfied. If the transformation

$$b(x) \to b_1(x) = b(x) \exp(-i\Delta x) \tag{5.1}$$

is made we find that the equations for the amplitude of the u-waves (2.10) are of the same form as the equations for the resonance case so that the solution is given by Eqs. (4.3) and (4.9)–(4.12) if the following substitutions are made in the latter:

$$B(x) \to B_1(x) = \int_0^x b_1(z)\,dz, \quad \beta(x) \to \beta_1(x) = \arg B_1(x). \tag{5.2}$$

We find that the equation for the amplitude of the v-wave (2.14) differ somewhat in form from the case investigated earlier. This difference leads, first of all, to a violation of the symmetry of Eqs. (2.10) and (2.14) with respect to the transformation in (3.3) so that now the periodicity of the conversion of the u-waves is no longer connected with the periodicity of $b(x)$ and vice versa. For simplicity we limit the analysis to the case of an infinite band of allowable values of $\{k_n\}$. In this case, in the equation [cf. Eq. (2.14)]

$$b'' - i\Delta b' + \Omega^2 b = 0 \tag{5.3}$$

the quantity Ω^2 can be regarded as a constant, as in the preceding section,

$$4\Omega^2 = qI \tag{5.4}$$

so that the integration of Eq. (5.3) can be carried out trivially

$$b(x) = c_1 e^{i\lambda_1 x} + c_2 e^{i\lambda_2 x}, \tag{5.5}$$

where

$$\lambda_{1,2} = {}^1/_2[\Delta \pm (\Delta^2 + 4\Omega^2)^{1/2}], \quad c_1 = \frac{ib'(0) + \lambda_2 b(0)}{\lambda_2 - \lambda_1}, \quad c_2 = \frac{ib'(0) + \lambda_1 b(0)}{\lambda_1 - \lambda_2}. \tag{5.6}$$

It is evident from this expression and from Eq. (5.1) that $b(x)$ and $b_1(x)$ are oscillatory functions and that the amplitude of the oscillations either remains constant (when $4\Omega^2 > -\Delta^2$) or grows exponentially (when $4\Omega^2 < -\Delta^2$). In similar fashion we find the behavior of $B_1(x)$:

$$B_1(x) = \frac{c_1'}{i\lambda_1'}(e^{i\lambda_1' x} - 1) + \frac{c_2'}{i\lambda_2'}(e^{i\lambda_2' x} - 1), \tag{5.7}$$

where

$$\lambda_{1,2}' = \lambda_{1,2} - \Delta, \quad c_{1,2}' = c_{1,2}(\lambda_1', \lambda_2').$$

Thus, Eqs. (4.9)–(4.12) together with Eqs. (5.2) and (5.5)–(5.7) represent a solution of the problem. In order to investigate in greater detail the effect of the detuning of the resonance on the wave-conversion process, we consider $\Delta^2 \gg 4\Omega^2$, in which case the detuning plays the decisive role in the wave conversion. Under these conditions

$$\lambda_1 \approx \Delta, \quad \lambda_2 \approx -\frac{\Omega^2}{\Delta}, \quad \lambda_1' \approx \frac{\Omega^2}{\Delta}, \quad \lambda_2' \approx -\Delta. \tag{5.8}$$

For the boundary conditions (2.4)

$$b(x) = b_0 e^{-i\Omega^2 x/\Delta}, \quad B_1(x) = \frac{ib_0}{\Delta}(e^{-i\Delta x} - 1), \tag{5.9}$$

$$|B_1| = 2\left|\frac{b_0}{\Delta}\sin\frac{\Delta x}{2}\right|, \quad \beta_1(x) = \arg b_0 - \frac{\Delta x}{2}.$$

From these expressions and from Eq. (4.3), (4.9), and (4.11) it is evident that the amplitude of the v-wave oscillates slowly and that the process of conversion of the high-frequency waves is periodic with period $2\pi/\Delta$, where the maximum number of u-waves that can be excited is given by

$$N_{max} \approx 2\max(h|B_1|) = 4h|b_0/\Delta|. \tag{5.10}$$

For the boundary conditions in (2.5)

$$b(x) = \frac{ib_0'}{\Delta}(e^{-i\Omega^2 x/\Delta} - e^{i\Delta x}), \quad b_0' = h(0,1)a_0^* a_1, \tag{5.11}$$

$$B_1(x) = b_0'[\Omega^{-2}(e^{-i\Omega^2 x/\Delta} - 1) - \Delta^{-2}(e^{-i\Delta x} - 1)]. \tag{5.12}$$

For small values of x ($\Omega^2 x/\Delta \ll 1$) both terms in the rectangular brackets in Eq. (5.12) make comparable contributions. When $x > \Delta^{-1}$ the second term can be neglected so that

$$B_1(x) = \frac{b_0'}{\Omega^2}(e^{i\Omega^2 x/\Delta} - 1), \quad |B_1| = 2\left|\frac{b_0'}{\Omega^2}\sin\frac{\Omega^2 x}{2\Delta}\right|, \quad \beta_1(x) = \arg b_0' - \Omega^2 x/2\Delta. \tag{5.13}$$

In this case the amplitude of the v-wave oscillates with frequency Δ while the period associated with the conversion of u-waves is very large: $4\pi\Delta/\Omega^2$ where the maximum number of significantly excited u-waves is

$$N_{max} \sim 4h|b_0'/\Omega^2|. \tag{5.14}$$

In reviewing the behavior of the wave-conversion process in the nonresonance case we may note the following.

a) If a u-wave and a v-wave are excited at the boundary x = 0 [the condition in Eq. (2.4)], since waves at the combination frequencies ω_n are formed with wave vectors that differ from the characteristic $k(\omega_n)$, where the magnitude of the detuning $n\Delta$ increases with the number of the combination wave, the conversion of the u-waves is periodic. The value of the period and the maximum number of appreciably excited waves is inversely proportional to the magnitude of the detuning Δ. Under these conditions the amplitude of the v-wave oscillates with frequency $\Omega^2/\Delta \ll \Delta$.

b) If two u-waves are excited at the boundary [the condition in Eq. (2.5)] then there can be a very large (as compared with the preceding case) number of u-waves at the combination frequencies by virtue of the fact that the amplitude of the v-wave b(x) contains a term which is proportional to $e^{i\Delta x}$, indicating the excitation of a v-wave with wave vector $\kappa' = \kappa + \Delta$ which satisfies the resonance condition $k_n = k_{n-1} + \kappa'$.

6. GENERALIZATION TO THE CASE OF INHOMOGENEOUS MEDIA

If the medium is inhomogeneous its dispersion and nonlinear properties depend on the coordinate x. In this case even the solution of the linear approximation to the problem of wave propagation becomes difficult. Under the assumption that the inhomogeneity is weak (that is to say the properties of the medium do not change greatly in a distance of the order of a wavelength) an approximate solution to the linear problem can be obtained from the WKB approximation:

$$u_\omega(x,t) = a_\omega k^{-1/2}(\omega, x)\exp\left[i\omega t + i\int_0^x k(\omega, z)\,dz\right], \tag{6.1}$$

$$v_\nu(x,t) = b_\nu \varkappa^{-1/2}(\nu, x)\exp\left[i\nu t + i\int_0^x \varkappa(\nu, z)\,dz\right].$$

The equations for the amplitudes a_ω and b_ν are of the same form as Eqs. (2.10) and (2.11) but now the coefficients $h(n-1, n)$ depend on x:

$$-2ia_n' = h(n-1, n; x)\,b a_{n-1}\exp\left\{-i\int_0^x \Delta(z)\,dz\right\} + h(n, n+1; x)\,b^* a_{n+1}\exp\left\{i\int_0^x \Delta(z)\,dz\right\}, \tag{6.2}$$

$$-2ib' = \sum_n h(n-1, n; x)\,a_{n-1}^* a_n \exp\left\{i\int_0^x \Delta(z)\,dz\right\}. \tag{6.3}$$

Here, as before, $\Delta(x) = k_n(x) - k_{n-1}(x) - \kappa(x)$.

The difficulties that arise in the investigation of these equations in general form are so substantial that we are forced to consider certain particular cases in which the solutions are especially simple. Under the assumption that the coefficients $h(n-1, n; x)$ can be written in the form of a product of two functions, one of which depends only on n and the other only on x, we find that the variables in Eq. (6.2) can be separated. We shall limit our analysis here to this case. Then

$$h(n-1, n; x) = h(n-1, n)\varphi(x). \tag{6.4}$$

Transforming to the new variable

$$z = \int_0^x \varphi(y)\,dy, \tag{6.5}$$

and using Eqs. (6.2) and (6.3) we find

$$-2ia_n' = h(n-1, n)\,b_1 a_{n-1} + h(n, n+1)\,b_1^* a_{n+1}, \tag{6.6}$$

$$b'' - i\Delta b' - \Omega^2 b = 0, \quad b_1(z) = b(z)\exp\left[-i\int_0^z \Delta(y)\,dy\right]. \tag{6.7}$$

The solution of the first of these equations is given by the relations in (4.3) and (4.9)–(4.12). As far as the second equation is concerned we find that under the assumption of an unbounded band of allowable values of $\{k_n\}$ the coefficient Ω^2 is a constant:

$$4\Omega^2 = qI; \quad q = h^2(n, n+1) - h^2(n-1, n).$$

Although the solution of this equation in general form is not known, the investigation in any given case is not particularly complicated. For example, when the resonance detuning is important $[\Delta(z) \gg \Omega^2]$ the solution of the equation becomes

$$b(z) = b_0 + \frac{b_0'}{i\Delta(0)}\left[\exp\left\{i\int_0^z \Delta(y)\,dy\right\} - 1\right]. \tag{6.8}$$

Qualitatively, the wave conversion process in this case is not very different from that investigated in the preceding section.

7. INTERACTION OF HIGH-FREQUENCY ELECTROMAGNETIC WAVES WITH PLASMA WAVES

As an example we consider the interaction of high-frequency transverse electromagnetic waves with plasma waves in an isotropic plasma. The particular solution of this problem is contained in papers by Danilkin.[4,5]

Assume that at the boundary of a plasma (at the plane x = 0) there is excited an electromagnetic wave with frequency ω and a plasma wave with frequency ν:

$$\mathbf{u}(\mathbf{r}, t)|_{x=0} = \mathbf{u}_0 \exp\left[i(\omega t + k_{0y}y + k_{0z}z)\right],$$
$$\mathbf{v}(\mathbf{r}, t)|_{x=0} = \mathbf{v}_0 \exp\left[i(\nu t + \varkappa_y y + \varkappa_z z)\right]. \tag{7.1}$$

As a result of the nonlinear interaction, during the course of the propagation of the waves there will appear waves at the combination frequencies $\omega_n = \omega_0 + n\nu$ with wave vectors $k_n = k(\omega_n)$ where $k_{ny} = k_{0y} + n\kappa_y$ and $k_{nz} = k_{0z} + n\kappa_z$. If damping is neglected, dispersion relations are of the following form:

$$\omega^2 = \omega_e^2 + c^2k^2, \qquad \nu^2 = \omega_e^2 + 3v_T^2\varkappa^2, \tag{7.2}$$

where $\omega_e = (4\pi e^2\rho m^{-1})^{1/2}$ is the plasma frequency, m is the mass, e is the charge v_T is the mean thermal velocity and ρ is the electron density.

We note that neglect of the damping of the plasma waves is valid only when $\nu \gg \kappa v_T$. When $\nu \sim \kappa v_T$ the Landau damping becomes important so that the neglect of the damping is allowable only for long plasma wavelengths. Under these conditions $\nu \approx \omega_0$. Our analysis is restricted to these waves.

The original equations and the derivation of the equations for the amplitudes of the interacting waves can be found in the papers by Danilkin.[4,5] We choose the following amplitude normalization:

$$\mathbf{u}_n = \mathbf{e}a_n\left[\frac{e\omega_e\varkappa}{2mc^2\nu^2}\frac{\omega_n^2}{k_n\cos\theta_n}\right]^{1/2}, \tag{7.3}$$
$$\mathbf{v}_\nu = \frac{\boldsymbol{\varkappa}}{\varkappa}b_\nu\left[\frac{e\omega_e^2}{6mv_T^2}\frac{1}{\varkappa\cos\varphi}\right]^{1/2},$$

where θ_n and φ are the angles formed by the wave vectors k_n and κ with the x-axis; then these equations can be written in the following form:

$$-2ia_n' = h(n-1, n)a_{n-1}b\exp(-i\Delta_n x) + h(n+1, n)a_{n+1}b^*\exp(i\Delta_{n+1}x), \tag{7.4}$$

$$-2ib' = \sum_{n=n_-}^{n=n_+} h(n-1, n)a_{n-1}^* a_n \exp(i\Delta_n x),$$

where

$$h(n-1, n) = \frac{\varkappa}{2\sqrt{6}\,\nu^2}\frac{\omega_e^3}{c^2v_T}\left(\frac{e}{m}\right)^{3/2}\left[\cos\theta_{n-1}\cos\theta_n\cos\varphi k_{n-1}k_n\varkappa\right]^{-1/2}, \tag{7.5}$$
$$\Delta_n = k_n - k_{n-1} - \varkappa.$$

If the frequency of the electromagnetic wave ω is much higher than the plasma frequency then $\omega_0 \gg \nu$ so that $k \gg \kappa$ and the condition in (2.12) is satisfied. Furthermore, under these conditions the dispersion relation (7.2) does not differ from the linear relation up to accuracy of order (ω_e^2/ω_0^2). Since the system of equations in (7.5) coincides with a system in (2.10) and the conditions are satisfied for which the analysis applies to the latter, we can apply all of the results obtained in the earlier sections. In particular, if $k_{0y} = k_{0z} = \kappa_y = \kappa_z = 0$, then $\theta_n = \varphi = 0$. If $\kappa = \nu/c \approx \omega_e/c$ the resonance condition is satisfied and $\Delta_n = 0$. In this case the amplitudes $a_n(x)$ are determined by the relations in (4.3), (4.7) and (7.9).[3)] We find that the amplitude of the plasma wave grows exponentially if, in accordance with Eqs. (4.2), (4.14) and (4.19), at the initial stage we can neglect the limitation from below on the allowable values of $\{k_n\}$:

$$b(x) = b_0\,\mathrm{ch}|\Omega|x, \qquad \Omega^2 = -\frac{1}{96}\left[\frac{\varkappa}{\nu^2}\frac{\omega_e^3}{c^2v_T}\left(\frac{e}{m}\right)^3\right]^2\frac{|a_0|^2}{k_0^3}.$$

In this case

$$B(x) = b_0\Omega^{-1}\,\mathrm{sh}\,|\Omega|x.$$

If $\kappa \neq \nu/c$ the resonance condition is not satisfied, $\Delta_n = \Delta \neq 0$ and the solution for the amplitude of the plasma wave is given by Eqs. (5.4)–(5.6).

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[1] S. A. Akhmanov and R. V. Khokhlov, Problemy nelineinoi optiki (Problems of Nonlinear Optics), Fizmatgiz, 1964.

[2] N. Bloembergen, Nonlinear Optics, Benjamin, New York, 1965.

[3] V. N. Tsytovich, Nelineinye effekty v plazme (Nonlinear Effects in Plasmas), Nauka, 1967.

[4] I. S. Danilkin, Zh. Tekh. Fiz. 36, 266 (1966) [Sov. Phys.-Tech. Phys. 11, 193 (1966)].

[5] I. S. Danilkin, Zh. Tekh. Fiz. 35, 667 (1965) [Sov. Phys.-Tech. Phys. 10, 524 (1965)].

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[3)] The paper by Danilkin[4] contains a solution for the resonance case in the zeroth approximation (cf. Sec. 3) without considering any limitation on the allowable values of k_n.