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INVESTIGATION OF THE STABILITY OF AN ISOTROPIC UNIVERSE WITH A COSMOLOGICAL CONSTANT

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The stability of an isotropic world with a cosmological constant Λ is considered. The nature of the variation of rotational perturbations, and of gravitational and acoustic waves with an increase in the radius of curvature is similar to the case with $\Lambda = 0$, i.e., stability exists in an expanding universe. With respect to the perturbations of the density and the associated perturbations of the metric, it turns out that in the case when the value of Λ is close to the critical value Λ_c they increase rapidly near the critical radius of curvature.

RECENTLY Petrosian, Salpeter, and Szekeres [1] in studying the distribution of quasars with respect to the red shift in the case of emission spectra and Kardoshev [2] in the case of absorption spectra have come to the conclusion that the evolution of the universe must be described by the Einstein equations with a cosmological constant Λ

$$R_i^k - {}^1/_2 R\delta_i^k - \Lambda\delta_i^k = \varkappa T_i^k. \tag{1}$$

In this connection it is of interest to investigate the stability of an isotropic model of the universe with a nonzero Λ-term.

For an isotropic model with positive curvature the metric can be written in the form

$$ds^2 = a^2(\eta)\,[d\eta^2 - d\chi^2 - \sin^2\chi(\sin^2\theta d\varphi^2 + d\theta^2)], \tag{2}$$

where χ, φ, θ are four-dimensional spherical coordinates, while the variable η is related to the time t and the curvature a by the equation $dt = a d\eta$ (the velocity of light is $c = 1$).

Basically the model with positive curvature is of interest. Transition to a model with negative curvature is accomplished by the replacement

$$a \to ia, \quad \chi \to i\chi, \quad \eta \to i\eta. \tag{3}$$

During the later stages of the expansion where the Λ-term is essential we shall use the equation of state $dp/d\epsilon = u^2(a)$ assuming that $u \ll 1$. In this case the equations for the unperturbed metric can be obtained under the assumption $p = 0$:

$$\frac{da}{d\eta} \equiv \dot{a} = \sqrt{P(a)}, \quad P(a) = \frac{2}{3}a_c a - a^2 + \frac{\Lambda a^4}{3}. \tag{4}$$

Here $a_c = \kappa M/4\pi^2$, M is the total mass of the universe.

The nature of the variation of a with time depends in an essential manner on the value of the constant Λ. In particular, for $\Lambda = a_c^{-2}$ there exists a special solution of equation (4): $a = a_c = \text{const}$ corresponding to Einstein's stationary model. For $\Lambda > \Lambda_c = a_c^{-2}$ the radius of curvature a increases monotonically with time having an inflection point at $a = a_c$ (Lemaitre model).

It is assumed in [1,2] that this particular case is the one realized in our universe, with the value of Λ only slightly greater than Λ_c

$$0 < \Delta = \Lambda/\Lambda_c - 1 \ll 1.$$

An investigation of the stability of the isotropic model with $\Lambda = 0$ has been carried out by E. Lifshitz [3].

The equations for the perturbation of the metric $\sigma g_{ik} \equiv h_{ik}$

$$\delta R_i^k - {}^1/_2\delta R\delta_i^k = \varkappa\delta T_i^k \tag{5}$$

do not explicitly contain the Λ-term (dependence on Λ is contained only in the unperturbed function $a(\eta)$). Therefore we can use the equations for the perturbations h_α^β obtained by E. Lifshitz ($\alpha, \beta = 1, 2, 3$).

Scalar, vector and tensor solutions of (5) are possible leading respectively to perturbations of the density, perturbations of the velocity and to gravitational waves.

1. We consider perturbations of the first type which depend on the scalar four-dimensional spherical function $Q^{(n)}$ 1). In order to obtain this solution we represent the perturbation of the metric in the form

$$h_\alpha^\beta = \lambda(\eta)P_\alpha^\beta + \mu(\eta)Q_\alpha^\beta,$$

$$Q_\alpha^\beta = \delta_\alpha^\beta Q^{(n)}, \quad P_\alpha^\beta = \frac{1}{n^2-1}Q^{(n);\beta}_{;\alpha} + Q_\alpha^\beta, \quad P_\alpha^\alpha = 0.$$

Here λ and μ are two new functions for which we obtain from (5) the following equations

$$\begin{gathered}\ddot{\lambda} + 2\frac{\dot{a}}{a}\dot{\lambda} - \frac{n^2-1}{3}(\lambda+\mu) = 0,\\ \ddot{\mu} + 2\frac{\dot{a}}{a}\dot{\mu}\Big(1 + \frac{3}{2}u^2\Big) + \frac{n^2-4}{3}(\lambda+\mu)(1+3u^2) = 0.\end{gathered} \tag{6}$$

In (6) we go over from the variable η to the independent variable a with the aid of the transformations

$$\dot{a} = \sqrt{P(a)}, \quad \dot{\lambda} = \dot{a}\lambda' = \lambda'\sqrt{P}, \quad \ddot{\lambda} = \lambda''P + \frac{\lambda'P'}{2}$$

etc. Primes denote differentiation with respect to a.

In this case (6) will go over into

$$\begin{gathered}\lambda'' + \lambda'\Big(\frac{2}{a} + \frac{P'}{2P}\Big) - \frac{n^2-1}{3P}(\lambda+\mu) = 0,\\ \mu'' + \mu'\Big[\frac{2}{a}\Big(1 + \frac{3}{2}u^2\Big) + \frac{P'}{2P}\Big] + \frac{n^2-4}{3}(\lambda+\mu)(1+3u^2) = 0.\end{gathered} \tag{7}$$

Perturbations of the density and of the velocity are expressed in terms of the functions $\lambda(a)$ and $\mu(a)$ by the formulas

1)The most symmetric function $Q^{(n)}$ has the form $Q^{(n)} = \sin n\chi/\sin\chi$. For details cf. [4]

$$\frac{\delta\varepsilon}{\varepsilon}=\frac{a}{6a_c}\left[(n^2-4)(\lambda+\mu)+\frac{3P}{a}\mu'\right]Q^{(n)},$$

$$\delta v_\alpha=\frac{a}{6a_c}[(n^2-4)\lambda'+(n^2-1)\mu']P_\alpha,$$

$$P_\alpha=\frac{1}{n^2-1}Q^{(n)}{}_{;\alpha}.$$

Equations (7) possess two first integrals corresponding to fictitious perturbations of the metric, i.e., to perturbations which can be eliminated by a transformation of the coordinate system:

1) $\lambda=-\mu=\text{const}$;

2) $\lambda=(n^2-1)\int\frac{da}{a\sqrt{P}}\equiv\lambda_0;\quad \mu=-(n^2-1)\int\frac{da}{a\sqrt{P}}+\frac{3\sqrt{P}}{a^2}\equiv\mu_0.$

With the aid of these two first integrals we can reduce the order of equations (7) by making the substitution

$$\lambda+\mu=(\lambda_0+\mu_0)\int\frac{\xi da}{\sqrt{P}},$$

$$\lambda'-\mu'=(\lambda_0'-\mu_0')\int\frac{\xi da}{\sqrt{P}}+\frac{\zeta}{a\sqrt{P}},$$

We obtain a system of two equations of the first order with respect to the functions $\xi(a)$ and $\zeta(a)$:

$$\xi'+\xi\left(\frac{P'}{P}-\frac{2}{a}\right)=\frac{u^2}{2}\frac{\zeta}{\sqrt{P}},\tag{8}$$

$$\zeta'+\frac{\zeta}{a}=\frac{\xi}{\sqrt{P}}\left[-2(n^2-1)+\frac{3}{2}\frac{P'}{a}-\frac{6P}{a^2}\right].$$

Here, wherever possible, we have dropped $u^2\ll 1$.

In solving the system of equations (8) we consider two cases. For $un\ll 1$ one can set $u=0$, and this corresponds to the equation of state $p=0$ for dustlike matter. For $un\gg 1$ ($n\gg 1$) the perturbations have the form of acoustic waves propagating with the velocity u and having a wave length of the order of a/n.

2. We consider first the case $p=0$. Then the solution of equations (7) together with the first integrals has the form

$$\lambda=C_1(n^2-1)\frac{a_c\sqrt{P}}{a^2}\int\frac{a^2da}{P^{3/2}}+C_2(n^2-1)a_c^2\int\frac{da}{a^2\sqrt{P}}$$
$$+Aa_c\int\frac{da}{a\sqrt{P}}+B,\quad \mu=-C_1(n^2-4)\frac{a_c\sqrt{P}}{a^2}\int\frac{a^2da}{P^{3/2}}$$
$$-C_2(n^2-1)a_c^2\int\frac{da}{a^2\sqrt{P}}-Aa_c\int\frac{da}{a\sqrt{P}}+\frac{3Aa_c\sqrt{P}}{(n^2-1)a^2}-B.$$

In the case of dustlike matter $p=0$ one can choose the reference system in such a manner that it turns out to be comoving and synchronous [5], i.e., $\delta v_\alpha=0$. From this condition we determine A:

$$A=-C_2(n^2-1).$$

Moreover, we set $B=0$. With the aid of the perturbations of the metric obtained in this manner we obtain the perturbations of the density. As a result we obtain

$$\lambda=C_1(n^2-1)\frac{a_c\sqrt{P}}{a^2}\int\frac{a^2da}{P^{3/2}}+C_2(n^2-1)a_c\int\frac{(a_c-a)}{a^2\sqrt{P}}da,$$

$$\mu=-C_1(n^2-4)\frac{a_c\sqrt{P}}{a^2}\int\frac{a^2da}{P^{3/2}}-C_2(n^2-1)a_c$$
$$\times\int\frac{(a_c-a)}{a^2\sqrt{P}}da-\frac{3C_2a_c\sqrt{P}}{a^2};$$

$$\frac{\delta\varepsilon}{\varepsilon}=\left[C_1(n^2-4)\frac{\sqrt{P}}{2a^2}\int\frac{a^3da}{P^{3/2}}-C_2(n^2-4)\frac{a_c\sqrt{P}}{2a^2}\right]Q^{(n)},\quad \delta v_\alpha=0.\tag{9}$$

One can perform a transition to a model with negative curvature directly in formulas (9) by carrying out the replacement (3) and, in addition, by replacing

$$n\to in,\quad P(a)=\frac{2a_ca}{3}+a^2+\frac{\Lambda a^4}{3}.$$

The formulas obtained above, in principle, give a solution of the problem, and the integrals in (9) can be reduced to elliptic integrals. The dependence of the radius of curvature on η and on t is defined by the formulas

$$\eta=\int\frac{da}{\sqrt{P}},\quad t=\int\frac{ada}{\sqrt{P}}.\tag{10}$$

In the case $\Lambda=0$, i.e., $P(a)=(2/3)a_ca-a^2$, as can be easily shown, the solutions obtained above coincide with the solutions of E. Lifshitz. As can be seen from (6) the stationary model is unstable. In (9) this corresponds to a singularity at $\Lambda=\Lambda_c$, $a=a_c$.

We investigate in greater detail the dependence of the perturbations of the density on the radius of curvature in the case $0<\Delta\ll 1$. We introduce the notation $x=a/a_c$. Then we have

$$\frac{\delta\varepsilon}{\varepsilon}=3C_1(n^2-4)Q^{(n)}\frac{[x(x-1)^2(x+2)+\Delta x^4]^{1/2}}{2x^2}\times\int_0^x\frac{x^3dx}{[x(x-1)^2(x+2)+\Delta x^4]^{3/2}}.\tag{11}$$

The lower limit of the integral has been set equal to zero, and this corresponds to neglecting the term proportional to C_2 which is bounded during the expansion process.

The first terms of the expansion of (11) in powers of $\Delta^{1/2}$ have the form:

$$\frac{\delta\varepsilon}{\varepsilon}(x\ll 1)=\frac{3}{10}C_1(n^2-4)Q^{(n)}x,$$

$$\frac{\delta\varepsilon}{\varepsilon}(1)=\frac{\sqrt{3}}{2}C_1(n^2-4)Q^{(n)}\frac{1}{\Delta^{1/2}},$$

$$\frac{\delta\varepsilon}{\varepsilon}(\infty)=\sqrt{3}C_1(n^2-4)Q^{(n)}\frac{1}{\Delta}.$$

Near $x=1$ the behavior of $\delta\epsilon/\epsilon$ is represented by the following expansion ($\delta x\ll\Delta^{1/2}$):

$$\frac{\delta\varepsilon}{\varepsilon}(1+\delta x)=\frac{\sqrt{3}(n^2-4)C_1Q^{(n)}}{2\Delta^{1/2}}\times\left[1+\sqrt{3}\frac{\delta r}{\Delta^{1/2}}+\frac{3}{2}\left(\frac{\delta x}{\Delta^{1/2}}\right)^2-\sqrt{3}\left(\frac{\delta x}{\Delta^{1/2}}\right)^2+\dots\right],$$

in the case of large x

$$\frac{\delta\varepsilon}{\varepsilon}(x\gg 1)=3C_1(n^2-4)Q^{(n)}\left[\frac{1}{\sqrt{3}\Delta}-\frac{1}{x^2}\right].$$

For the "amplification coefficient" of the perturbations of the density which arose long before the critical radius we obtain

$$K=\frac{\delta\varepsilon}{\varepsilon}(\infty)\Big/\frac{\delta\varepsilon}{\varepsilon}(x\ll 1)=\frac{10}{\sqrt{3}}\frac{1}{\Delta x}.$$

Its value in order of magnitude is greater by a factor Δ^{-1} than in the case when the Λ-term is equal to zero [3,4].

3. It makes sense to consider the equation of state $dp/d\epsilon=u^2(a)$ only for large values of n, such that $un\gtrsim 1$. In this case equations (8) will go over into

$$\xi'+\xi\left(\frac{P'}{2P}-\frac{2}{a}\right)=\frac{u^2}{2}\frac{\zeta}{\sqrt{P}},$$

$$\zeta'+\frac{\zeta}{a}=-\frac{2n^2}{\sqrt{P}}\xi.\tag{12}$$

For $un \gg 1$ the solution of the system (12) has the form:

$$\xi = in\sqrt{\frac{au}{P}}(C_1 e^{i\Phi} - C_2 e^{-i\Phi}),$$

$$\zeta = -2n^2\sqrt{\frac{a}{uP}}(C_1 e^{i\Phi} + C_2 e^{-i\Phi}).$$

Here

$$\Phi = n\int u\,d\eta = n\int \frac{u\,da}{\sqrt{P}} \gg 1.$$

Correspondingly, the perturbations of the density and of the velocity in this case will be given by:

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{n^2 Q^{(n)}}{6a_c}\frac{1}{\sqrt{ua}}(C_1 e^{i\Phi} + C_2 e^{-i\Phi}),$$

$$\delta v_\alpha = \frac{in^3 P_\alpha}{6a_c}\sqrt{\frac{u}{aP}}(C_1 e^{i\Phi} - C_2 e^{-i\Phi}).$$

4. We consider rotational perturbations. The perturbations of the metric are proportional to the tensor S_α^β with zero trace $S_\alpha^\alpha = 0$, so that there exists no scalar. As a result of this the perturbations of the density are equal to zero, the perturbations of the velocity are proportional to the vector S^α ($S_\alpha^\beta = S^\beta_{;\alpha} + S_\alpha^{;\beta}$). We represent the perturbations of the metric in the form

$$h_\alpha^\beta = \sigma(a) S_\alpha^\beta.$$

Then Eqs. (5) reduce to the single equation for σ

$$\sigma'' + \sigma'\left(\frac{2}{a} + \frac{P'}{2P}\right) = 0, \qquad (13)$$

which has a first integral corresponding to fictitious perturbations of the metric $\sigma = \text{const}$.

The solutions of (13) have the form

$$\sigma(a) = C\int \frac{da}{a^2\sqrt{P}}.$$

The perturbations of the velocity are given by

$$\delta v^\alpha = \frac{a\sigma'\sqrt{P}}{4a_c} S^\alpha = \frac{C}{4a_c a} S^\alpha.$$

Thus, the perturbations of the velocity are stable in an expanding universe. The dependence on the radius of curvature agrees exactly with the case $\Lambda = 0$ [3,4] as should have been expected as a result of the conservation of the moment of momentum of the perturbation [5].

5. We now consider gravitational waves. Perturbations of the metric can be represented in the form

$$h_\alpha^\beta = \nu(a) G_\alpha^\beta.$$

Perturbations of the density and of the velocity are identically equal to zero, since the corresponding scalar and vector do not exist $G_\alpha^\alpha = 0$ and $G^\beta_{\alpha;\beta} = 0$.

For the perturbations of the metric we obtain from (5) the equation

$$\nu'' + \nu'\left(\frac{P'}{2P} + \frac{2}{a}\right) + (n^2 - 1)\frac{\nu}{P} = 0. \qquad (14)$$

For $\Lambda = 0$ the equation has been investigated in [3,4].

We consider the case $n \gg 1$. We seek a solution of the form $\nu(a) = C(a)\exp[i\Phi(a)]$. Then the solution of equation (14) has the form

$$\nu(a) = \frac{C_1}{a}e^{in\eta} + \frac{C_2}{a}e^{-in\eta}, \quad \eta = \int \frac{da}{\sqrt{P}}.$$

The solution corresponds to gravitational waves propagated with the velocity of light. The amplitude of the perturbations of the metric decreases with increasing radius of curvature like a^{-1}, and thus the perturbations of the metric are stable in an expanding universe. The nature of damping agrees with the one qualitatively predicted in [5].

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