SUPERCONDUCTIVITY AND QUASI-ONE-DIMENSIONAL (THREAD-LIKE) STRUCTURES

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A model of a metal is considered which is one-dimensional with respect to all electron properties, except oscillations of the electron density. As a consequence all previous objections[2,3] against the possible existence of superconductivity of a peculiar one-dimensional type[1,6] become invalid. With respect to magnetic properties our metal is a superconductor of the second kind with a critical field $H_{c_2} \sim T_c/\mu$ (T_c is the transition temperature and μ the Bohr magneton) which considerably exceeds the thermodynamic field H_c. (Even at low transition temperatures of the order of 5–10°K, the field H_{c_2} will be of the order of hundreds of thousands of Oe.)

1. INTRODUCTION

SEVERAL years ago, Little[1] advanced the idea that one possible way of increasing the temperature of a superconducting transition is to use long polymer chains. Figure 1 shows one of the possible chains of this type; R denotes here some definite monomer. If the distances between the neighboring carbon atoms turn out to be equal (i.e., the case of resonance), then the chain of carbon atoms will constitute a one-dimensional metal with one free electron per atom (with half-filled band). If, in addition, the monomer R has one or several excited electronic states lying in the optical region, then attraction is produced between the electrons in the chain —C=C—C=, and according to Little's estimates the magnitude of this attraction can exceed in certain cases the Coulomb repulsion. The presence of attraction of such intensity should, in Little's opinion, cause a transition of the conduction electrons into a superconducting state even at temperatures on the order of hundreds or even several thousand degrees.

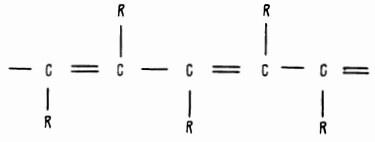

FIG. 1

However, soon after the publication of Little's paper, Ferrel and Rice[2] pointed out that a one-dimensional system cannot go over into a superconducting state at all. Later Hohenberg proved this statement with absolute rigor[3]. Physically this impossibility is due to the destructive action of the free-electron density fluctuations. On the other hand, Little's mechanism is not effective in ordinary more or less close-packed metals, in which there are no electronic excited states at all in the optical region.

In this connection, it is natural to consider a metallic system in which the atoms of the "metal" (the latter may be, for example, carbon atoms) are packed with the minimum density, so as to leave room for the action of the Little mechanism, and, on the other hand, the fluctuations of the electron density have the ordinary three-dimensional (plasma) character. Such a metal can be visualized as made up of regularly arranged polymer chains of the type shown in Fig. 1. The bonds between the metallic chains —C=C—C= will be realized in this case via several lateral monomers, thus greatly hindering the transition of the electrons from one metallic chain to another. Therefore the conductivity retains the usual metallic character along the chains, and is either nonexistent at all in a transverse direction, or is exceedingly small (the latter corresponds to the fact that the effective mass of the transverse motion of the electrons is exponentially small compared with the longitudinal mass). At the same time, the long-wave fluctuations of the conduction-electron density, by virtue of the long-range nature of the Coulomb interaction between the electrons from different chains, will remain practically the same as in ordinary metals, and by the same token will not destroy the superconductivity (Sec. 3).

A real analog of substances of the type described above are crystals of tetracyan-quinone-dimethane (TCNQ) or its compounds (see, for example,[4]). The conductivity of certain substances of this class reaches 100 $ohm^{-1}cm^{-1}$, and its anisotropy is of the order of 1:3000. It should be noted, to be sure, that (TCNQ) itself, judging from the character of the temperature dependence of its conductivity, is a semiconductor. However, by introducing a sufficient amount of donors, the character of the conductivity can in principle be changed to metallic. We note also that, as shown by Zavadovskiĭ[5], the introduction of impurities in a quasi-one-dimensional system, even in amounts that are comparable with the number of conduction electrons, does not influence at all its superconducting properties.

2. THE MODEL

To investigate the feasibility of superconductivity in systems of the type described above, we shall consider a simple model of a quasi-one-dimensional metal. It can be represented as consisting of regularly arranged chains of metallic atoms (conducting filaments). The distances between the filaments exceeds by several times (not necessarily by one order of magnitude!) the distances between the neighboring metallic atoms within the limits of one chain. It is clear that the Fermi surface of such a metal will be a corrugated plane, and the

amplitude of the corrugation will be exponentially small compared with the Fermi energy ϵ_0. In other words, the character of the motion of the electron along the filament corresponds to the weak-coupling limit, while the motion across the filaments corresponds to the strong-coupling limit. For concreteness we shall use in the estimates the following dispersion law:

$$\varepsilon(\mathbf{p}) = v(|p_z| - p_0) + \alpha(\cos ap_x + \cos ap_y) \tag{1}$$

with $\alpha \ll vp_0 \equiv \epsilon_0$. The case when there is no conductivity across the filaments obviously corresponds to $\alpha = 0$. According to the foregoing, the distance between filaments, a, coincides in order of magnitude with $1/p_0$ (in our case of one electron per atom, $p_0 = \pi/2c$, where c—distance between the atoms in the filament).

If it turns out that α is much smaller than the temperature T_c of the superconducting transition, then, in the logarithmic approximation, all the results of Bychkov, Gor'kov, and one of the authors[6], obtained for the case of a one-dimensional metal, remain valid for our model, the only exception being that the "one-dimensional" interaction constant g is replaced by λ/a^2, where λ is the three-dimensional constant. In fact, in the theory of a superconducting transition in a quasi-one-dimensional system, just as in[6], an important role is played not only by the Cooper diagram C (Fig. 2a), but also by the zero-sound diagram Z (Fig. 2b) with a longitudinal-momentum transfer p_z equal to $2p_0$. In the calculation of the diagram C (in the technique using finite temperatures) the logarithmic integration in it is always cut off at a temperature T, while in the diagram Z the cut off is at the larger of the two quantities T or α. If $T \gg \alpha$, then it is possible to neglect the quantity α throughout in the integrands of C and Z, and the integration is over the entire range of variation of the momenta p_z and p_y. This yields the already mentioned factor $1/a^2$ at the interaction constant λ.

The vanishing of the temperature T_c of the superconducting transition in the case of a purely one-dimensional metal occurs in approximations that follow the logarithmic approximation. However, the theory of[6] is so complicated, that the transition to the next-higher approximations can hardly be effected. We therefore confine ourselves, following Ferrel[2], to an estimate of the influence of the electron-density fluctuations on T_c.

Unfortunately, we have no proof that the Ferrel formula (2) takes correct quantitative account of the presence of fluctuations in all cases. Moreover, this certainly is not the case if the contribution of these fluctuations is not large, as is the case, for example, in three-dimensional systems. We think, however, that formula (2) gives a qualitatively correct asymptotic dependence of T_c on the system dimensions L when the fluctuations exert a destructive action. In particular, it can be verified that the dependence of T_c on L, which follows from Ferrel's results[2], does not contradict the Hohenberg inequalities.

3. FLUCTUATIONS

According to Ferrel[2], the transition temperature T_c (or, which is the same, the superconducting gap Δ), with allowance for the fluctuations of the electron density, is given by the formula

FIG. 2

$$T_c = \text{const} \cdot \exp\{-\langle\varphi^2(0)\rangle/2\}, \tag{2}$$

where $\varphi(\mathbf{r})$—operator of the phase of the oscillations, connected with the velocity operator $\mathbf{v}$ by the ordinary relation

$$m\mathbf{v} = \operatorname{grad}\varphi, \tag{3}$$

m—mass of the free electron. With the aid of the continuity equation[1)]

$$i\partial n/\partial\tau + n_0 \operatorname{div}\mathbf{v} = 0, \tag{4}$$

where n_0—average density of the conduction electron, we can express the mean-square fluctuations of the Fourier components of the phase $\varphi_{K\omega}$ in terms of the density fluctuations $n_{K\omega}$:

$$\langle\varphi_{K\omega}\varphi_{K\omega}^*\rangle = \frac{\omega^2 m^2}{n_0^2 K^4}\langle n^{K\omega} n^{K\omega *}\rangle. \tag{5}$$

From this we get for $\langle\varphi^2(0)\rangle$ (when T = 0)

$$\langle\varphi^2\rangle = \frac{m^2}{(2\pi)^4 n_0^2}\int d\omega d^3K\,\frac{\omega^2}{K^4}\langle n_{K\omega} n_{K\omega}^*\rangle. \tag{6}$$

In the calculation of $\langle nn^*\rangle$ it is necessary to take into account the Coulomb interaction between the filaments. Let us calculate, for example, the spectrum of the plasma waves (assuming that the conditions of perturbation theory $e^2/v \ll 1$ are formally satisfied). To this end we sum, as usual, the sequence of diagrams of Fig. 3, where the broken line represents the ordinary Coulomb potential $V_0(\mathbf{r}) = -e^2/r$. Its Fourier component coincides, obviously with the known expressions for the system of uniformly charged filaments. In different limiting cases, they take the form

$$V_0(\mathbf{K}) = \begin{cases} -4\pi e^2/K^2, & Ka \ll 1, \\ -2e^2a^2\ln(a/\rho_0), & K \sim 1/a, \\ -2e^2a^2\ln(1/q\rho_0), & q \gg 1/a. \end{cases}$$

Here and throughout we denote by $\mathbf{k}$ and q respectively the transverse and longitudinal components of the vector $\mathbf{K}$, and by ρ_0 the "radius" of the filament. Inasmuch as in our model $\ln(a/\rho_0)$ and $\ln(1/p_0\rho_0)$ are small, we can use for the estimates the ordinary three-dimensional expression for all $\mathbf{K}$.

The screened Coulomb potential at $Ka \ll 1$ is given by

$$V = -4\pi e^2\Big/\Big(k^2 + q^2 + \varkappa^2\frac{v^2q^2}{\omega^2 + v^2q^2}\Big). \tag{7}$$

where κ—Debye momentum, which has the usual order of magnitude $\kappa^2 \sim e^2/va^2 \sim e^2mp_0$. Formula (7) is valid at not too small longitudinal momenta $vq \gg \alpha$. At still smaller momenta q, the expression for V has the usual three-dimensional form. From (7) follows a formula for the frequency of the plasma oscillations

$$\bar{\omega}^2 = \omega_0^2q^2/K^2 + v^2q^2 = \omega_0^2\cos^2\theta + v^2q^2, \tag{8}$$

$\omega_0 = v\kappa$. Thus, unlike the one-dimensional case, in our model there exists a finite albeit angle-dependent gap.

1)We use throughout an imaginary time $-i\tau$ and imaginary frequencies $i\omega$ (cf., e.g.[7]).

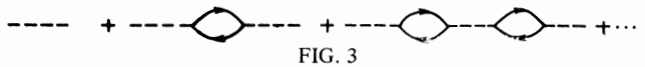

FIG. 3

FIG. 4

This circumstance affects also the character of the long-wave fluctuations of the electron density. Noting that $N \equiv \langle nn^* \rangle$ is connected with the screening potential V by the obvious relation

$$V = V_0 + V_0 N V_0,$$

we obtain directly

$$\langle \varphi^2 \rangle = \frac{m^2}{(2\pi)^4 n_0^2} \int d\omega\, d^3K \frac{\omega^2}{K^4 V_0^2} (V - V_0).$$

Taking the integral over ω and over the angles, we get

$$\langle \varphi^2 \rangle = \frac{m^2 v}{8\pi^3 n_0^2} \frac{\omega_0^2}{e^2} \int dK \sqrt{\varkappa^2 + K^2}. \tag{9}$$

The main region of integration in (9) lies, if the condition $e^2 \ll v$ is satisfied, at large values of the momentum, $K \gg \kappa$. In order of magnitude, formula (9) yields (when $K \lesssim p_0$)

$$\langle \varphi^2 \rangle \sim \frac{m^2 v \omega_0^2}{n_0^2 e^2} p_0^2 \sim 1.$$

It is clear that this result should not change also for $e^2 \sim v$, when $\kappa \sim p_0$.

The transition to the limiting case of a single filament occurs when the distance between filaments a tends to infinity. The essential region of integration will now be $q \gtrsim 1/a \to 0$. In this region $V_0 = 2e^2a^2 \ln q\rho_0$, and the screening potential and the spectrum of the plasma oscillations coincide with those of one charge filament with radius ρ_0. The result of the calculations of $\langle \varphi^2 \rangle$ is given by Ferrel[2].

We note that although the transverse-motion energy α from (1) does vanish as $a \to \infty$, the vanishing of α itself still does not denote at all that we are dealing with a single filament. When $\alpha = 0$ there occur no jumps from one filament to another, but, as follows from the preceding calculations, the effect of the Coulomb screening does not disappear.

A second dangerous circumstance for the theory is the infrared catastrophe mentioned in[6], connected with the fact that in the case when the dependence of the electron energy on the transverse momentum is neglected, the electron can generate, with adherence to the conservation laws, an arbitrary number of electron-hole pairs or, what is the same, an arbitrary number of long-wave quanta of electron-density oscillations. This circumstance is expressed mathematically in the fact that the correction of order n to the self-energy $\Sigma^{(n)}$ had in[6] the form

$$\Sigma^{(n)} \sim \frac{(p - p_0)^n}{(i\omega - v(p - p_0))^{n-1}}. \tag{10}$$

This form of $\Sigma^{(n)}$ denotes that the exact electron Green's function $\mathfrak{G}$ has an essential singularity in place of the usual single-particle pole at $\omega = -iv(p - p_0)$. This explains by the same token the results of Tomonaga[8] and Gaudin[9], who have shown that the spectrum of the low-lying excitations of a one-dimensional Fermi system is limited to the ordinary density oscillations.

Allowance for the Coulomb interaction between the filaments changes the situation radically. (Physically this is connected with the fact that the emission of oscillation quanta with energy $\overline{\omega}$ [from (8)] is now forbidden by the conservation laws.) In fact, expressions of the type (10) in any arbitrary diagram result from the integration over the region in which the longitudinal momenta of all the lines are close to each other and to p_0. This means that the singularity is determined only by the interaction with small transverse, i.e., Coulomb interactions. Replacing the bare Coulomb potential by a screened potential, it is easy to verify that the singularities of interest to us are now given by diagrams of the type of Fig. 4, where the wavy line denotes the screen potential (7). Calculations show that singularities of the type (10) go over into

$$\Sigma^{(n)} \sim (i\omega - v(p - p_0)) \ln^{n-1} (i\omega - v(p - p_0)). \tag{11}$$

Corrections of this type fit in naturally within the scheme of the logarithmic approximation developed in[6]. It can be verified, in particular, that expression (11) contains the interaction constant raised to the power n (in the form e^{2n}). Thus, the relative corrections to the Green's function are of the order of $g(gL)^{n-1}$, where L is a large logarithm, and can be discarded with logarithmic accuracy.

4. MAGNETIC PROPERTIES

The magnetic field influences a superconductor in two ways: via the orbital motion, and by tending to break the antiparallel spins of the Cooper pair. We begin with the influence on the orbital motion. In Appendix I we calculate with logarithmic accuracy the zero-sound diagram Z and the Cooper diagram C without allowance for spin. It is shown there that the field does not act on Z at all, and that C depends on the orientation of the field relative to the filaments. In particular, in a field parallel to the filaments we have

$$C \sim \ln \frac{\varepsilon_0}{\max\{T, a\sqrt{eHa^2/c}\}} \tag{12a}$$

and in a perpendicular field

$$C \sim \ln \frac{\varepsilon_0}{\max\{T, \sqrt{aeHa/c}\}}. \tag{12b}$$

It follows therefore that the orbital critical fields are given by the expressions

$$H_{c2}^{\parallel} \sim \frac{T_c^2 c}{\alpha^2 e a^2} \sim \frac{T_c}{\mu} \frac{T_c \varepsilon_0}{\alpha^2}, \qquad H_{c2}^{\perp} \sim \frac{T_c^2 c}{\alpha e a} \sim \frac{T_c}{\mu} \frac{T_c}{\alpha}, \tag{13}$$

where μ—Bohr magneton. In the transformations we used, in order of magnitude, the relation $a \sim 1/p_0$.

On the other hand, it is physically obvious that the magnetic field breaks the spins of the Cooper pair even in much weaker fields, $H \sim T_c/\mu$. It is therefore sufficient to consider only the action of the field on the spin. The values of the diagrams C and Z now depends on the

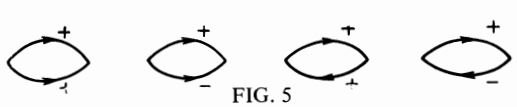

FIG. 5

spin indices of the Green's functions. We shall denote the orientation relative to the magnetic field by the symbols + and −. It is easy to verify that two of the diagrams of Fig. 5 ($C_{++}(C_{--})$ and Z_{+-}) are not affected by the field, and two others, as expected, are proportional to

$$C_{+-}, Z_{++}(Z_{--}) \sim \ln \frac{\varepsilon_0}{\max\{T, \mu H\}} \tag{14}$$

To find the transition curve we turn to the previously obtained equations for the vertex part[6]. It is clear that when $\mu H < T$ the magnetic field has no connection with them and the old answer is retained.

$$T_c = \varepsilon_0 e^{-\pi v/|g|}, \quad H < T_c/\mu. \tag{15}$$

In the region $\mu H > T$, the systems (9) and (10) of[6] are altered. Without showing the derivations, we present the results.

The form of the equations depends on the spin indices and on the channels (Cooper and zero-sound). We introduce several symbols (all the undefined symbols coincide with those given in[6]; we recall only that we calculate the vertex part with the following disposition of the momenta: $\Gamma_{\alpha\beta\gamma\delta}(p_0, -p_0; -p_0, p_0)$). The quantities $\Gamma_{++++}(\Gamma_{----})$ and $\Sigma_{++++}(\Sigma_{----})$ will be denoted by Γ_1 and Σ_1, and the quantities $\Gamma_{+--+}(\Gamma_{-++-})$ and $S_{+--+}(S_{-++-})$ by Γ_2 and S_2. In addition

$$t = \frac{1}{2\pi v}\ln\frac{\varepsilon_0}{T}, \quad h = \frac{1}{2\pi v}\ln\frac{\varepsilon_0}{\mu H}.$$

1. Cooper Channel

a) $\eta < \xi < h$, $\eta < h < \xi < t$. In these regions, the equations retain their earlier form[6]:

$$\Gamma_1(\eta, \xi) = \Sigma_1(\eta) + \Sigma_1(\eta)\int_\eta^\xi \Gamma_1(\zeta, \xi)\,d\zeta + \int_0^\eta \Sigma_1(\zeta)\Gamma_1(\zeta, \xi)\,d\zeta, \tag{16a}$$

b) $h < \eta < \xi < t$:

$$\Gamma_1(\xi) = \Sigma_1(h) + \Sigma_1(h)\Gamma_1(\xi)(\xi - h) + \int_0^h \Sigma_1(\zeta)\Gamma_1(\zeta, \xi)\,d\zeta. \tag{16b}$$

2. Zero-Sound Channel

a) $\xi < \eta$, $h < t$. In this region the equations retain their earlier form[6]:

$$\Gamma_2(\eta, \xi) = S_2(\xi) - S_2(\xi)\int_\xi^\eta \Gamma_2(\eta, \zeta)\,d\zeta - \int_0^\xi S_2(\zeta)\Gamma_2(\eta, \zeta)\,d\zeta. \tag{17a}$$

b) $h < \xi < \eta < t$:

$$\Gamma_2(\eta) = S_2(h) - S_2(h)\Gamma_2(\eta)(\eta - h) - \int_0^h S_2(\zeta)\Gamma_2(\eta, \zeta)\,d\zeta. \tag{17b}$$

The quantities Γ_{+-+-} and Γ_{-+-+} depend only on h in the region of interest to us, $h < \xi$ and $\eta < t$, and coincide with the corresponding quantities $\Gamma(h, h)$ of[6].

Equations (16) and (17) can be easily solved (Appendix II). We present the results for $\xi = \eta = t$:

$$\Gamma_1(t) = \frac{\Gamma_1(h)}{1 - \Gamma_1(h)(t - h)}, \tag{18}$$

$$\Gamma_2(t) = \frac{\Gamma_2(h)}{1 + \Gamma_2(h)(t - h)}. \tag{19}$$

Here $\Gamma(h)$ coincides with the corresponding quantities $\Gamma(h, h)$ from[6], and equal

$$\Gamma_1(h) = -\frac{g^2 h}{1 + 2gh}, \tag{20}$$

$$\Gamma_2(h) = -g\frac{1 + gh}{1 + 2gh}. \tag{21}$$

The quantity $\Gamma_{+-+-}(h, h) \equiv \Gamma_3(h)$ equals[6]:

$$\Gamma_3(h) = \frac{g}{1 + 2gh}. \tag{22}$$

The poles of expressions (18), (19), and (22) describe the possible transition curves. We start with the case on the traction. As already mentioned, when $\mu H < T_c$ ($|g|h > 1$) the transition always takes place at a temperature T_c (15). When $|g|h < 1$, i.e., $\mu H > T_c$, the vertex parts Γ_1 and Γ_2 have no poles at all (we recall that expressions (18) and (22) are valid only when $t > h$). The quantity $\Gamma_3(h)$ has a pole at $h = 1/2|g|$, i.e., when $H_{c2} = T_c/\mu$. Thus, the critical magnetic field does not depend on the temperature (with logarithmic accuracy) and equals $H_{c2} = T_c/\mu$. Since it exceeds greatly the thermodynamic field $H_c \sim T_c\sqrt{mp_0}$ ($H_c/H_{c2} \sim ev^{1/2}/c$), our metal is a superconductor of the second kind with parameter $\kappa \sim c/e\sqrt{v}$. The state diagram in terms of the variables H and T is shown in Fig. 6.

It is interesting to note that a transition takes place also in the case of repulsion in a magnetic field. We shall explain its nature in a special article, and confine ourselves here only to a determination of the transition curve, with logarithmic accuracy. When $\mu H < T$, as follows from[6], the vertex parts have no poles. When $\mu H > T$, a pole appears in Γ_2. The equation of the corresponding transition curve is

$$T_c = \mu H \exp\left\{-\frac{2\pi v}{g}\frac{1 + (g/\pi v)\ln(\varepsilon_0/\mu H)}{2 + (g/\pi v)\ln(\varepsilon_0/\mu H)}\right\}; \tag{23}$$

in weak fields

$$T_c \approx \mu H e^{-2\pi v/g}, \tag{23a}$$

and in strong fields

$$T_c \approx (\mu H)^{3/2}\varepsilon_0^{-1/2}e^{-\pi v/g}. \tag{23b}$$

The diagram of state is shown schematically in Fig. 7.

APPENDIX I

Estimate of Diagrams in a Magnetic Field

The Green's function of the electron in the magnetic field has in the quasi-classical approximation the form (see, for example,[7])

$$\mathfrak{G}_H(\mathbf{r}, \mathbf{r}') = \exp\left\{i\frac{e}{c}(\mathbf{r} - \mathbf{r}')\mathbf{A}\left(\frac{\mathbf{r} + \mathbf{r}'}{2}\right)\right\}\mathfrak{G}(\mathbf{r}, \mathbf{r}'), \tag{A.1}$$

where A is the vector potential and $\mathfrak{G}$ the Green's function without the field. The latter is of the form (see[2])

$$\mathfrak{G}(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3p \frac{\psi_{\mathbf{p}}(\mathbf{r})\psi_{\mathbf{p}}^*(\mathbf{r}')}{i\omega - v(|p_z| - p_0) - a(\cos ap_x + \cos ap_y)} \tag{A.2}$$

$$\omega = (2n + 1)\pi T.$$

Here $\psi_{\mathbf{p}}(\mathbf{r})$ is the Bloch function of the electron, corre-

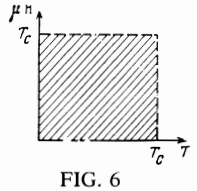

FIG. 6

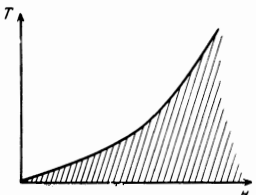

FIG. 7

sponding to the energy (1). The longitudinal-motion wave function can be taken in our model in the form of a plane wave $\psi_{\parallel} = \exp(ip_z z)$. The transverse-motion function, in accordance with the strong-coupling approximation, is conveniently written in the form of an expansion in Wannier functions $w_1(\rho)$:

$$\psi_{\perp}(\rho) = a \sum_{\mathbf{l}} \exp(-i\mathbf{p}_{\perp}\mathbf{l}a)\, w_{\mathbf{l}}(\rho). \qquad (A.3)$$

Here $\mathbf{l}$—number of filaments: $\mathbf{l} = (l_x, l_y)$, l_x, $l_y = \ldots, -1, 0, 1, \ldots$. In our approximation we can neglect, with exponential accuracy, the overlap integrals of the functions on different filaments $\mathbf{l}$.

We change over in (A.2) to the Wannier representation in terms of the transverse coordinates. After integration over the momenta, we get:

$$\mathfrak{G}_H(\mathbf{l},\mathbf{l}';\, z,z') = -\frac{i}{v}\operatorname{sign}\omega\cdot\exp\Big\{i\frac{e}{c}(z-z')A_z + i\frac{e}{c}a(\mathbf{l}-\mathbf{l}')\mathbf{A}\Big\}$$
$$\times\exp\Big\{-\frac{|z-z'||\omega|}{v} + ip_0|z-z'|\operatorname{sign}\omega - \frac{i\alpha|z-z'|}{v}\operatorname{sign}\omega\Big\}$$
$$\times\exp\Big\{i(l_x - l_x' + l_y - l_y')\frac{\pi}{2}\Big\}$$
$$\times J_{l_x-l'_x}\Big(-\frac{\alpha|z-z'|}{v}\Big)J_{l_y-l'_y}\Big(-\frac{\alpha|z-z'|}{v}\Big). \qquad (A.4)$$

Here J—Bessel function; the vector potential in (A.4) is taken at the point $(\mathbf{A}(\mathbf{l}+\mathbf{l}')/2, (z+z')/2)$. In the derivation we used the well known formula

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\varphi + ix\sin\varphi}\, d\varphi.$$

The Cooper diagram C has in the Wannier representation the form

$$C = \frac{T}{v^2a^2}\sum_{\omega}\int d(z-z')\sum_{\mathbf{l}-\mathbf{l}'}\mathfrak{G}_H(\mathbf{l},\mathbf{l}';\, z,z';\, \omega)\mathfrak{G}_H(\mathbf{l},\mathbf{l}';\, z,z';\, -\omega), \qquad (A.5)$$

and the zero-sound diagram

$$Z = \frac{T}{v^2a^2}\sum_{\omega}\int d(z-z')\sum_{\mathbf{l}-\mathbf{l}'}\mathfrak{G}_H(\mathbf{l},\mathbf{l}';\, z,z';\, \omega)\mathfrak{G}_H(\mathbf{l}',\mathbf{l};\, z',z;\, \omega). \qquad (A.6)$$

We see that Z does not depend on the field.

In a longitudinal field, C can be written in the form

$$C = \frac{T}{v^2a^2}\sum_{\omega}\int dz\sum_{l_1,l_2} e^{-2|z\omega|/v}\exp\Big\{i\frac{e}{c}Ha^2l_1l_2\Big\}J_{l_1}^2\Big(\frac{\alpha z}{v}\Big)J_{l_2}^2\Big(\frac{\alpha z}{v}\Big) \qquad (A.7)$$

Summing over l_2 and ω in (A.7), we get

$$C = \frac{T}{v^2a^2}\sum_{l}\int dz\,\mathrm{sh}^{-1}\Big(\frac{2\pi|z|T}{v}\Big)J_l^2\Big(\frac{\alpha z}{v}\Big)J_0\Big(2\frac{\alpha z}{v}\sin\frac{eHa^2l}{2c}\Big). \qquad (A.8)$$

The sum over l (at large $\alpha z/v$) converges rapidly, and the main contribution is made by terms in the narrow region near $l = \alpha|z|/v$. Using also the fact that $\Sigma J_l^2(x) = 1$, we rewrite (A.8) in the form

$$C \approx \frac{T}{v^2a^2}\int dz\,\mathrm{sh}^{-1}\Big(\frac{2\pi|z|T}{v}\Big)J_0\Big(\frac{\alpha^2eHa^2}{cv^2}z^2\Big). \qquad (A.8')$$

The integral in (A.8′) diverges at the lower limit. As usual, it must be cut off at distances $|z| \sim v/\epsilon_0$. The logarithmic integration takes place in the region $v/\epsilon_0 \ll |z| \ll v/T$. If $T \gg \alpha\sqrt{eHa^2/c}$, then it is necessary to choose T as the upper limit; on the other hand, if the inverse inequality holds, then the integral is cut off at $|z| \sim v/\alpha\sqrt{eHa^2/c}$. This leads to the result given in the text.

In a transverse field we get in lieu of (A.8′)

$$C = \frac{T}{v^2a^2}\sum_{l}\int\frac{dz}{\mathrm{sh}(2\pi|z|T/v)}J_l^2\Big(\frac{\alpha z}{v}\Big)\cos\Big(\frac{e}{c}Halz\Big). \qquad (A.9)$$

Similar arguments lead in this case to formula (12b).

APPENDIX II

Solution of Eq. (16)

It follows from (16b) that Γ_1 does not depend on η when $\eta > h$. Taking this into account, we rewrite (16a) in the region $\eta < h$ in the form

$$\Gamma_1(\eta,\xi) = \Sigma_1(\eta)(1+\Gamma_1(\xi)(\xi-h))$$
$$+\Sigma_1(\eta)\int_{\eta}^{h}\Gamma_1(\zeta,\xi)\,d\zeta + \int_{0}^{\eta}\Sigma_1(\zeta)\Gamma_1(\zeta,\xi)\,d\zeta. \qquad (A.10)$$

We make the substitution $\Gamma_1(\eta,\xi) = f(\eta,\xi)(1+\Gamma_1(\xi)(\xi-h))$. Then

$$f(\eta,\xi) = \Sigma_1(\eta) + \Sigma_1(\eta)\int_{\eta}^{h} f(\zeta,\xi)\,d\zeta + \int_{0}^{\eta}\Sigma_1(\zeta)f(\zeta,\xi)\,d\zeta. \qquad (A.11)$$

This equation coincides with the corresponding equation in the absence of a magnetic field when $\xi = h$. Therefore

$$\Gamma_1(\eta,\xi) = \Gamma_1(\eta,h)(1+\Gamma_1(\xi)(\xi-h)), \qquad (A.12)$$

where $\Gamma_1(\eta, h)$ is the vertex part, calculated in[6]. We now substitute (A.12) in (16b):

$$\Gamma_1(\xi) = \Big(\Sigma_1(h) + \int_0^h\Gamma_1(\zeta,h)\Sigma_1(\zeta)\,d\zeta\Big)(1+\Gamma_1(\xi)(\xi-h)). \qquad (A.13)$$

But the expression

$$\Sigma_1(h) + \int_0^h\Sigma_1(\zeta)\Gamma_1(\zeta,h)\,d\zeta$$

is none other than $\Gamma_1(h, h) \equiv \Gamma_1(h)$ defined in the text. Thus,

$$\Gamma_1(\xi) = \Gamma_1(h)(1+\Gamma_1(\xi)(\xi-h)), \qquad (A.14)$$

from which follows formula (18).

Equations (17) are solved in analogous manner.

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