RENORMALIZATION OF THE FIELD OPERATOR IN AXIOMATIC FIELD THEORY

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It is shown that the renormalization of the field within the framework of axiomatic field theory does not reduce to the multiplication of the field, current, or charge by certain numerical factors. The latter are in fact integro-differential operators which are different on and off the energy shell. Only on account of such specific peculiarities does it become possible to satisfy the integrability condition in renormalized theory. The spectral representations are defined more accurately. In particular, one-particle contributions to the spectral density (in the presence of subtractions) are consistently taken into account with the help of the unitarity condition. The apparatus developed here permits one to relate the two traditional methods of introducing the renormalization constants: the Hamiltonian method and the axiomatic method of Lehmann. It is shown that the formulas for the Heisenberg operators themselves correspond to the formulas for their vacuum expectation values in the unrenormalized as well as in the renormalized theory.

1. INTRODUCTION

THERE are two methods of introducing the concept of the renormalized field in the axiomatic approach. One method was first formulated by Lehmann.[1] This method is not essentially based on the concept of a field. The renormalization constant Z_3 appears here in the renormalization of the Green's function and is expressed through an integral over the spectral density:

$$Z_3^{-1} = 1 + \int_{(2m)^2}^{\infty} d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2}$$

The constant $Z_3^{-1/2}$ is introduced purely formally from the consideration that if $\langle 0|T(A(x)A(y))|0\rangle$ acquires the factor Z_3^{-1} by the renormalization, then this must be equivalent to the multiplication of each field by $Z_3^{-1/2}$.

This is a completely satisfactory method of introducing and defining the constant Z_3 which renormalizes the Green's function or the commutator, but the taking of the square root is not a satisfactory operation, as was shown in[2,3]. Indeed, if we admit some elements of the usual Hamiltonian theory, i.e., consider not only quantities of the Green's function type but also fields and currents connected by the Yang-Feldman equations, it is easily shown that such a "numerical" renormalization of the current operator is inconsistent with the unitarity condition.

On the other hand it is important in a number of cases (as an example it suffices to mention the current algebra) to know the exact expressions for the renormalized currents and fields. Therefore, a renormalization procedure was developed in[2,3] within the framework of the axiomatic theory,[4] which is much closer to the standard perturbation theory[5] and, in particular, makes explicit use of the perturbation theoretical form of the counter term for the field renormalization in the Lagrangian, but is otherwise not based on the Lagrangian formalism. It turned out that in order to avoid contradictions with unitarity in the strong sense (i.e., off the energy shell) and with the existence of the usual connection between field and current (Yang-Feldman equation), the renormalization transformation must be of the operator type and different for terms on and off the energy shell.

We shall construct a theory starting from the S matrix by the so-called axiomatic method,[1,2] and we shall assume, as usual, that the scattering matrix and all Heisenberg operators are given by infinite Wick polynomials in the asymptotic fields:

$$S = \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int dx_1 \ldots dx_\nu \Phi^{(\nu)}(x_1, \ldots, x_\nu) : \varphi(x_1) \ldots \varphi(x_\nu) :. \quad (1)$$

When the S matrix actually depends not only on $\varphi(x_i)$ but also on the derivatives $\partial^{(\alpha)}\varphi(x_i)$ one must of course assume that the differential or integro-differential operators are contained in the coefficient functions $\Phi^{(\nu)}(x_1, \ldots, x_\nu)$. In this case the investigation is much more complicated: one must be very careful every time a multiplication by a θ function occurs in the theory, and in particular, one must strictly distinguish between the Dyson and Wick T products.[2,3] Indeed, then the θ function is subject to an effective differentiation which leads to the appearance of additional terms proportional to δ functions.

It is usually said that the interaction contains no derivatives in renormalizable theories.[1)] On the other hand, it is known from perturbation theory that the field renormalization introduces terms in the S matrix which are proportional to derivatives even if the original interaction did not contain such terms. It is clear that something similar must happen in axiomatic theory.

Recently we have considered[2)] a model of an "interaction with derivatives" which was constructed such that the entire interaction was reduced merely to a renormalization of the Heisenberg field. For a consistent description of the field renormalization in the axiomatic approach a special mathematical technique had to be developed. In the present paper we wish to consider, on

[1)]Except, perhaps, the first derivative, as in scalar electrodynamics.

[2)]We use the results and notation of [2,3] without explanation.

the basis of the results of[2], the renormalization of the S matrix in the general case, when the S matrix contains, besides the "renormalizing" interaction studied in[2], also some fundamental interaction which leads to real scattering processes. Generally speaking, we shall not specify this interaction more precisely and assume only that it does not contain derivatives. The derivatives enter in the "renormalized" S matrix through the field renormalization, which we shall carry out explicitly, using the results of[2].

2. COMPOSITE S MATRIX

The first part of the problem can be formulated in a more general form without specifying the form of the two parts of the interaction. Assume that there are two types of interaction 0 and R containing one and the same field, and which are known separately. How can one describe the composite interaction which includes the two original ones?

In the Lagrangian formalism the Lagrangians for the interactions 0 and R are simply added. But when the S matrix is taken as the starting quantity, correspondence arguments suggest the composition law

$$S = T_W(S_0 S_R), \tag{2}$$

where S_0 and S_R are the scattering matrices describing the separate interactions and S is the composite S matrix including both original interactions.

The Heisenberg current operator corresponding to the full S matrix,

$$j(x) = i\frac{\delta S}{\delta\varphi(x)}S^+ \tag{3}$$

can be written, using (2),

$$j(x) = J_0(x) + J_R(x), \tag{4}$$

where

$$J_0(x) = iT_W\left(\frac{\delta S_0}{\delta\varphi(x)}S_R\right)S^+, \quad J_R(x) = iT_W\left(S_0\frac{\delta S_R}{\delta\varphi(x)}\right)S^+. \tag{5}$$

In addition, the theory also contains currents corresponding to the separate original interactions,

$$j_0(x) = i\frac{\delta S_0}{\delta\varphi(x)}S_0^+, \quad j_R(x) = i\frac{\delta S_R}{\delta\varphi(x)}S_R^+. \tag{6}$$

The problem consists in the determination of the connection between the current j(x) and the currents $j_0(x)$ and $j_R(x)$. To solve this problem we introduce the idea of the out-current, i.e., we assume that in analogy to the connection between the Heisenberg and asymptotic fields,

$$A(x) = T_W(\varphi(x)S)S^+ = \varphi(x) - \int d\zeta D^{adv}(x-\zeta)j(\zeta), \tag{7}$$

we can write[3,6]

$$j(x) = T_W(j^{out}(x)S)S^+. \tag{8}$$

For the S matrices S_R and S_0 this connection has the form

$$j_0(x) = T_W(j_0^{out}(x)S_0)S_0^+, \quad j_R(x) = T_W(j_R^{out}(x)S_R)S_R^+. \tag{9}$$

We note now that we can write, using (6) and (9),

$$\frac{\delta S_0}{\delta\varphi(x)} = -iT_W(j_0^{out}(x)S_0), \quad \frac{\delta S_R}{\delta\varphi(x)} = -iT_W(j_R^{out}(x)S_R). \tag{10}$$

Substituting (10) in (5) and using the composition law (2), we can write

$$J_0(x) = T_W(T_W(j_0^{out}(x)S_0)S_R)S^+$$
$$= T_W(j_0^{out}(x)T_W(S_0S_R))S^+ = T_W(j_0^{out}(x)S)S^+, \tag{11}$$
$$J_R(x) = T_W(j_R^{out}(x)S)S^+. \tag{12}$$

If we assume that the interactions corresponding to the matrices S_0 and S_R are known one should regard the expressions for the corresponding out-currents as known, too. The latter can in general be written in the form of certain finite polynomials in normal products of the asymptotic fields:

$$j_i^{out}(x) = \sum_\nu \frac{1}{\nu!}\int dy_1\dots dy_\nu f_i^{(\nu)}(x, y_1, \dots, y_\nu) : \varphi(y_1)\dots\varphi(y_\nu):, \quad i = 0, R. \tag{13}$$

Substituting now the expressions for the out-currents (13) in the corresponding formulas (9), we obtain

$$j_0(x) = \sum_\nu \frac{1}{\nu!}\int dy_1\dots dy_\nu f_0^{(\nu)}(x, y_1, \dots, y_\nu)N_{Q_0}(A_0(y_1)\dots A_0(y_\nu)), \tag{14}$$

$$j_R(x) = \sum_\nu \frac{1}{\nu!}\int dy_1\dots dy_\nu f_R^{(\nu)}(x, y_1, \dots, y_\nu)N_{Q_R}(A_R(y_1)\dots A_R(y_\nu)), \tag{15}$$

where N_{Q_0} and N_{Q_R} are quasi-normal products,[2,7] formed with the matrices S_0 and S_R, respectively.

Formulas (14) and (15) are the dynamic laws for the 0 and R theories separately, which we know. It is remarkable that (13) also determines simultaneously the dynamical law for the composite S matrix. Indeed, substituting (13) in (11) and (12), we find

$$J_0(x) = \sum_\nu \frac{1}{\nu!}\int dy_1\dots dy_\nu f_0^{(\nu)}(x, y_1, \dots, y_\nu)N_Q(A(y_1)\dots A(y_\nu)), \tag{16}$$

$$J_R(x) = \sum_\nu \frac{1}{\nu!}\int dy_1\dots dy_\nu f_R^{(\nu)}(x, y_1, \dots, y_\nu)N_Q(A(y_1)\dots A(y_\nu)), \tag{17}$$

where N_Q are now quasi-normal products formed with the full S matrix (2), and the coefficient functions $f_0^{(\nu)}$ and $f_R^{(\nu)}$ are the same as in the corresponding formulas (14) and (15).

Thus the currents J_0 and J_R are the same functions (or functionals) as the currents j_0 and j_R but with a different functional argument. Symbolically this may be written

$$j_0(x) = \hat{F}_0[A_0], \quad j_R(x) = \hat{F}_R[A_R], \tag{18}$$

and

$$J_0(x) = F_0[A], \quad J_R(x) = \hat{F}_R[A], \tag{19}$$

where the functional dependence on the corresponding Heisenberg fields is understood in the sense of quasi-normal products.

Since the Yang-Feldman relation (7) gives a second connection between the current j_i, the Heisenberg field A_i, and the asymptotic field φ, the dynamic law [one of (18) or (19)] together with the corresponding Yang-Feldman relation allows one to determine the full solution of the system, expressing the field and the current through the asymptotic field $\varphi(x)$.

7. RENORMALIZATION OF THE FIELD OPERATORS

We shall now assume that S_0 is a matrix corresponding to some actual unrenormalizable interaction without derivatives and S_R is a matrix leading to a renormalization of the field operators such as the one considered in[2] as a model.

The dynamical law (16) has now the explicit form

$$j_R(x) = (Z-1)(-\widehat{K\delta})A_R(x), \tag{20}$$

and we at once write, using the results of the preceding section,

$$J_R(x) = (Z-1)(-\widehat{K\delta})A(x). \tag{21}$$

Equation (20) together with the Yang-Feldman relation (7) for A_R is a complete system of equations for $j_R(x)$ and $A_R(x)$. The solutions of this system, found in[2], have the form

$$A_R(x) = \hat{N}\{\varphi(x) + (1-\sqrt{Z})\hat{D}^{adv}\hat{N}(-K\varphi(x))\}, \tag{22}$$

$$j_R(x) = -(1-\sqrt{Z})\hat{N}\{\varphi(x)(-\underset{\rightarrow}{K_x}) + (-K_x\varphi(x))\} - (1-\sqrt{Z})^2\hat{N}\hat{D}^{adv}\hat{N}(-K\varphi(x))(-\underset{\rightarrow}{K_x}). \tag{23}$$

These solutions allow one to find expressions for $A(x)$ and $J_R(x)$ in the following manner. Let us consider the Yang-Feldman relation (9) for the full theory. Substituting for $j(x)$ the sum of $J_0(x)$ and $J_R(x)$ and introducing the notation $a_0(x) = \varphi(x) - \hat{D}^{adv}J_0(x)$, we find

$$A(x) = a_0(x) - \hat{D}^{adv}J_R(x). \tag{24}$$

We note now that the system of two equations (24) and (21) is obtained from the system for j_R and A_R by the replacement

$$\varphi \to a_0, \quad A_R \to A, \quad j_R \to J_R.$$

Hence, the solutions $A(x)$ and $J_R(x)$ can be obtained from the solutions (22) and (23) by the same replacement:

$$A(x) = \hat{N}\{a_0(x) + (1-\sqrt{Z})\hat{D}^{adv}\hat{N}(-K_x a_0(x))\},$$

$$J_R(x) = -(1-\sqrt{Z})\hat{N}\{a_0(x)(-\underset{\rightarrow}{K_x}) + (-K_x a_0(x))\} - (1-\sqrt{Z})^2\hat{N}\hat{D}^{adv}\hat{N}(-K_x a_0(x))(-\underset{\rightarrow}{K_x}).$$

Substituting the expression for $a_0(x)$ in these formulas, we have

$$A(x) = A_R(x) - \hat{N}\hat{D}^{adv}\hat{N}J_0(x) \tag{25}$$

and

$$J_R(x) = j_R(x) + (\hat{N}-1)J_0(x) + (-\underset{\leftarrow}{K_x})(1-\sqrt{Z})\hat{N}\hat{D}^{adv}\hat{N}J(x).$$

From the last expression we obtain the following form for the full current:

$$j(x) = j_R(x) + \{1 + (-K_x)(1-\sqrt{Z})\hat{N}\hat{D}^{adv}\}\hat{N}J_0(x). \tag{26}$$

Having obtained these particular solutions, we can proceed to establish the renormalization relations. We understand this in the following way. The "unrenormalized" system is described by the equations

$$j_0(x) = \hat{F}_0[A_0], \quad A_0(x) = \varphi(x) - \hat{D}^{adv}j_0(x).$$

Even with the simplest form of the functional $\hat{F}_0$, nobody has yet been able to solve these equations. However, our task is only to find a connection between these "unrenormalized" solutions and the solutions of the renormalized solutions, determined by the equations

$$J_0(x) = \hat{F}_0[A], \quad A(x) = A_R(x) - \hat{N}\hat{D}^{adv}\hat{N}J_0(x).$$

It has been shown in[2] that the simple "numerical" field renormalization leads to contradictions.

In order to understand the character of the transformation which we want to determine, we consider first the trivial case (which does not lead to an interaction) where the connection $j_0 = \hat{F}_0(A_0)$ is linear:

$$j_0(x) = g_1 A_0(x). \tag{27}$$

Since the "vertex" may in general not be a number but may contain operators of the type $\hat{I}$ or $\hat{N}$ (even in the unrenormalized theory without derivatives), we shall write the law (27) in the form

$$j_0(\Gamma_1, \varphi|x) = \hat{\Gamma}_1 A_0(\Gamma_1, \varphi|x), \tag{28}$$

where we have also indicated what the functional arguments of j_0 and A_0 are. Substituting (28) in the Yang-Feldman relation we obtain

$$A_0(\Gamma_1, \varphi|x) = \varphi(x) - \hat{D}^{adv}\hat{\Gamma}_1 A_0(\Gamma_1, \varphi|x). \tag{29}$$

The linear character of (29) allows us at once to write down the solution in the form

$$A_0(\Gamma_1, \varphi|x) = \{1 + \hat{D}^{adv}\hat{\Gamma}_1\}^{-1}\varphi(x). \tag{30}$$

As usual, the inverse operator is understood in the sense of a formal expansion in a power series. For the current we obtain

$$j_0(\Gamma_1, \varphi|x) = \hat{\Gamma}_1\{1 + \hat{D}^{adv}\hat{\Gamma}_1\}^{-1}\varphi(x). \tag{31}$$

In virtue of what has been said above, the law (29) allows us to write for $J_0(x)$

$$J_0(\Gamma_2, \varphi|x) = \hat{\Gamma}_2 A(\Gamma_2, \varphi|x). \tag{32}$$

We emphasize that, since the current J_0 and the field A are constructed with the help of the matrix S and not S_0, the vertex is somewhat altered, as noted in the functional arguments in (32). Substituting (32) in the Yang-Feldman relation and treating it in the same way as (29), we obtain

$$A(\Gamma_2, \varphi|x) = \{1 + \hat{N}\hat{D}^{adv}\hat{N}\hat{\Gamma}_2\}^{-1}A_R(x). \tag{33}$$

We make the transition from $\varphi(x)$ to $A_R(x)$ in two stages, so that

$$A_R(x) = \hat{N}\psi(x), \quad \psi(x) = \{1 + (1-\sqrt{Z})\hat{D}^{adv}\hat{N}(-\underset{\rightarrow}{K})\}\varphi(x). \tag{34}$$

We note that the field $\psi(x)$, by definition, differs from the free out-field only off the energy shell. Going over to the field $\psi(x)$ in the argument of (33), we obtain

$$A(\Gamma_2, \varphi|x) = \hat{N}\{1 + \hat{D}^{adv}\hat{N}\hat{\Gamma}_2\hat{N}\}^{-1}\psi(x). \tag{35}$$

Finally (32) yields for the current $J_0(x)$

$$J_0(\Gamma_2, \varphi|x) = \hat{\Gamma}_2\hat{N}(1 + \hat{D}^{adv}\hat{N}\hat{\Gamma}_2\hat{N})^{-1}\psi(x). \tag{36}$$

Let us now compare the "renormalized" solutions (35), (36) with the "unrenormalized" solutions (30), (31). Clearly, when we introduce the vertex renormalization in the form

$$\hat{\Gamma}_1 = \hat{N}\hat{\Gamma}_2\hat{N}, \tag{37}$$

then the solutions of the two types are connected by the relations

$$A(\Gamma_2, \varphi|x) = \hat{N}A_0(\Gamma_1, \psi|x), \quad J_0(\Gamma_2, \varphi|x) = \hat{N}^{-1}j_0(\Gamma_1, \psi|x). \tag{38}$$

Thus the renormalization transformation has been reduced 1) to the multiplication of the field by the operator $\hat{N}$ and of the current by the factor $1/\hat{N}$, 2) to the replacement of the vertex $\hat{\Gamma}_1$ by $\hat{\Gamma}_2$ according to (37), and finally and very importantly, 3), to a transition to another functional argument: $\varphi \to \psi$. The first two points differ from the usual renormalization procedure only in that the numerical factors are replaced by operators, which is connected with the rules for treating derivatives. The third point, the transition to the new field $\psi(x)$, is connected with the peculiar divergence of the S matrix off the energy shell. The divergence corresponding to the transition $\varphi \to \psi$ is a special divergence which guarantees that unitarity and causality are satisfied in the strong sense, i.e., for the S matrix off the energy shell (2).

We can now go over to the consideration of actual interactions. Since the original S matrix corresponds to an unrenormalized theory with a derivative-free interaction, the operator $\hat{F}_0$ acting on $A_0(x)$ must in general give a polynomial in (quasi-normal) powers of $A_0(x)$. The degree of this polynomial must at least be two for real processes (scattering) to occur. We restrict ourselves to this lowest nontrivial degree, since the generalization to higher degrees will be obvious.

Thus we assume that

$$j_0(\Gamma_1, \varphi\,|\,x) = \hat{F}_0[A_0] = gN_{Q_0}(A_0^2(\Gamma_1, \varphi\,|\,x)) = -\hat{\Gamma}_1 \Big\langle \begin{matrix} A_0(\Gamma_1, \varphi\,|\,x) \\ A_0(\Gamma_1, \varphi\,|\,x) \end{matrix} \;. \tag{39}$$

We have written (39) in the form of a "structural formula" which makes the "valence" of the vertex explicit. This is necessitated by the circumstance already noted in the consideration of the trivial "bivalent" vertex, that $\hat{\Gamma}$ is an operator which is "dressed" on each line of the vertex.

Using the explicit expression (39) for the current we rewrite the Yang-Feldman equation in the unrenormalized theory:

$$A_0(\Gamma_1, \varphi\,|\,x) = \varphi(x) - \hat{D}^{adv} - \hat{\Gamma}_1 \Big\langle \begin{matrix} A_0(\Gamma_1, \varphi\,|\,x) \\ A_0(\Gamma_1, \varphi\,|\,x) \end{matrix} \tag{40}$$

and in the renormalized theory:

$$A(\Gamma_2, \varphi\,|\,x) = \varphi(x) - \hat{N}\hat{D}^{adv}\hat{N} - \hat{\Gamma}_2 \Big\langle \begin{matrix} A(\Gamma_2, \varphi\,|\,x) \\ A(\Gamma_2, \varphi\,|\,x) \end{matrix} \;. \tag{41}$$

Now one should not think that these equations can be solved explicitly, but one can establish a renormalization correspondence on the basis of our experience with the bivalent model.

Since we are here dealing with a three-valent vertex, it is natural to expect that one has

$$-\hat{\Gamma}_1 \Big\langle = -\hat{N} - \hat{\Gamma}_2 \Big\langle \begin{matrix} \hat{N}- \\ \hat{N}- \end{matrix} \;. \tag{42}$$

instead of (37).

The relation between the fields and the currents in the two theories retains the previous form:

$$A(\Gamma_2, \varphi|x) = \hat{N}A_0(\hat{\Gamma}_1, \psi|x), \quad J_0(\Gamma_2, \varphi|x) = \hat{N}^{-1}j_0(\hat{\Gamma}_1, \psi|x). \tag{43}$$

Indeed, using (34) we rewrite (41) in the form

$$A(\Gamma_2, \varphi\,|\,x) = \hat{N}\left\{\psi(x) - \hat{D}^{adv}\hat{N} - \hat{\Gamma}_2 \Big\langle \begin{matrix} A(\Gamma_2, \varphi\,|\,x) \\ A(\Gamma_2, \varphi\,|\,x) \end{matrix} \right\}, \tag{44}$$

and substituting (42) and (43) in the right-hand side of (44) we obtain

$$A(\Gamma_2, \varphi\,|\,x) = \hat{N}\left\{\psi(x) - \hat{D}^{adv}\hat{N} - \hat{\Gamma}_2 \Big\langle \begin{matrix} \hat{N}A_0(\Gamma_1, \psi\,|\,x) \\ \hat{N}A_0(\Gamma_1, \psi\,|\,x) \end{matrix} \right\}$$
$$= \hat{N}\left\{\psi(x) - \hat{D}^{adv} - \hat{\Gamma}_1 \Big\langle \begin{matrix} A_0(\Gamma_1, \psi\,|\,x) \\ A_0(\Gamma_1, \psi\,|\,x) \end{matrix} \right\} = \hat{N}A_0(\Gamma_1, \psi\,|\,x). \tag{45}$$

Comparing the result (45) with (40), we see that the renormalization formulas (42) and (43) are correct.

If we go from the operators $\hat{N}$ to numbers, $\hat{N} \to 1/\sqrt{Z}$ and $\hat{\Gamma} \to g$, and leave out the terms which vanish on the energy shell, i.e., set $\psi = \varphi$ [cf. (34)], then the renormalization reduces to

$$A(g_2, x) = Z^{-1/2}A_0(g_1, x),$$
$$J_0(g_2, x) = Z^{1/2}j_0(g_1, x), \quad g_1 = Z^{-3/2}g_2. \tag{46}$$

4. INTEGRABILITY CONDITION

We must now convince ourselves that the integrability condition

$$\frac{\delta j(x)}{\delta\varphi(y)} - \frac{\delta j(y)}{\delta\varphi(x)} = i[j(x), j(y)], \tag{47}$$

which follows directly from the unitarity of the S matrix and the definition of the current as a variational derivative, is satisfied for the unrenormalized currents j_0 as well as for the renormalized currents j. It is quite obvious that the integrability condition cannot be satisfied simultaneously in both theories in the simplified version of renormalization (46).[4]

The full renormalized current j(x) has, according to (26) and (43), the form

$$j(\Gamma_2, \varphi\,|\,x) = j_R(\varphi\,|\,x) + \{1 + (-\overleftarrow{K})(1 - \sqrt{Z})\hat{N}\hat{D}^{adv}\}_x j_0(\Gamma_1, \psi\,|\,x). \tag{48}$$

In order to determine the variational derivative of the current we note that owing to (34)

$$\frac{\delta\psi(x)}{\delta\varphi(y)} = \delta(x-y) + (1 - \sqrt{Z})D^{adv}(x-y)\hat{N}(-\overrightarrow{K}_y). \tag{49}$$

Using this result we find

$$\frac{\delta j(\Gamma_2, \varphi\,|\,x)}{\delta\varphi(y)} = \frac{\delta j_R(\varphi|\,x)}{\delta\varphi(y)} + \{1 + (-\overleftarrow{K})(1 - \sqrt{Z})\hat{N}\hat{D}^{adv}\}_x$$
$$\cdot \frac{\delta j_0(\Gamma_1, \psi\,|\,x)}{\delta\psi(y)}\{1 + \hat{D}^{adv}\hat{N}(1 - \sqrt{Z})(-\overrightarrow{K})\}_y. \tag{50}$$

On the other hand, we can immediately compute the commutator of currents $j(\Gamma_2, \varphi|x)$ of the form (48) by first writing $j_R(\varphi|x)$ in the form

$$j_R(\varphi\,|\,x) = -(1 - \sqrt{Z})(-\overleftarrow{K})\hat{N}\psi(x) - (1 - \sqrt{Z})(-K\psi) \tag{51}$$

and noting that the second term on the right makes no contribution to the commutator. Omitting the details of a rather lengthy calculation we only quote the result:

$$[j(\Gamma_2, \varphi\,|\,x),\ j(\Gamma_2, \varphi\,|\,y)] = -i(1 - \sqrt{Z})^2(-\overleftarrow{K}_x)\hat{N}D(x-y)\hat{N}(-\overrightarrow{K}_y)$$
$$+ i\{1 + (-\overleftarrow{K})(1 - \sqrt{Z})\hat{N}\hat{D}^{ret}\}_x \frac{\delta j_0(\Gamma_1, \psi\,|\,y)}{\delta\psi(x)}$$
$$\times\{1 + \hat{D}^{ret}\hat{N}(1 - \sqrt{Z})(-\overrightarrow{K})\}_y$$
$$- i\{1 + (-\overleftarrow{K})(1 - \sqrt{Z})\hat{N}\hat{D}^{adv}\}_x \frac{\delta j_0(\Gamma_1, \psi\,|\,x)}{\delta\psi(y)}$$
$$\times\{1 + \hat{D}^{adv}\hat{N}(1 - \sqrt{Z})(-\overrightarrow{K})\}_y. \tag{52}$$

It is at once clear from the expression for the commutator of the currents and from formula (50) for the variational derivative of the current that the integrability condition (47) is satisfied by the renormalized current

$j(\Gamma_2, \varphi|x)$. That it is satisfied by the current $j_0(\Gamma_1, \varphi|x)$, i.e., by the unrenormalized current, is clear from the definition.

In conclusion we write down the expression for the commutator of the renormalized Heisenberg fields, which is obtained in an obvious manner:

$$[A(\Gamma_2, \varphi|x),\ A(\Gamma_2, \varphi|y)] = \hat{N}[\psi(x), \psi(y)]\hat{N} + i\hat{N}\hat{D}^{ret}\frac{\delta j_0(\Gamma_1, \psi|y)}{\delta\psi(x)}\hat{D}^{ret}\hat{N} - i\hat{N}\hat{D}^{adv}\frac{\delta j_0(\Gamma_1, \psi|x)}{\delta\psi(y)}\hat{D}^{adv}\hat{N} = \hat{N}[A_0(\Gamma_1, \psi|x),\ A_0(\Gamma_1, \psi|y)]\hat{N}, \quad (53)$$

in agreement with (43).

5. SPECTRAL REPRESENTATIONS

To determine the spectral representations one must know the retarded radiation operator, which satisfies the condition[8]

$$\frac{\delta j(x)}{\delta\varphi(y)} = i\theta(y^0 - x^0)[j(x), j(y)] + \Lambda_2(x, y), \quad (54)$$

where $\Lambda_2(x, y)$ is a current-like operator.[8] We introduce a few definitions:

$$\left\langle 0 \left| \frac{\delta j(x)}{\delta\varphi(y)} \right| 0 \right\rangle = -f^{adv}(x-y), \quad (55)$$

$$\langle 0 \mid [j(x), j(y)] \mid 0\rangle = -if(x-y), \quad (56)$$

$$\langle 0 \mid \Lambda_2(x, y) \mid 0\rangle = -\lambda_2(x-y) \quad (57)$$

and finally also[7]

$$f_D^{adv}(x-y) = -\theta(y_0 - x_0)f(x-y). \quad (58)$$

Taking the vacuum expectation value of (5) we find

$$f^{adv}(x-y) = f_D^{adv}(x-y) + \lambda_2(x-y). \quad (59)$$

We note that, although $f^{adv} \neq f_D^{adv}$,

$$f^{ret}(x) - f^{adv}(x) = f(x) = f_D^{ret}(x) - f_D^{adv}(x).$$

In order to obtain the spectral representations for $f^{adv}(x)$, we recall that the fact that $f(x)$ must be a relativistically invariant antisymmetric function requires that its representation has the form of the spectral integral

$$f(x) = \int_0^\infty d\zeta\, I(\zeta) D_\zeta(x), \quad (60)$$

where $D_\zeta(x)$ is the Pauli-Jordan function with the mass ζ.[9] Multiplying (60) formally by a θ function[3)] we find the spectral representation for $f_D^{adv}(x)$:

$$f_D^{adv}(x) = \int_0^\infty d\zeta\, I(\zeta) D_\zeta^{adv}(x). \quad (61)$$

Keeping the first two terms of the Taylor expansion of (61) we obtain the representation "with subtractions:"

$$f_D^{adv}(x) = f^{*adv}(x) + C_{0D}\delta(x) + C_{1D}(-K\delta(x)), \quad (62)$$

$$f^{*adv}(x) = \int_0^\infty d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2}\{(-K)(-K)D_\zeta^{adv}(x)\}, \quad (63)$$

$$C_{0D} = \int_0^\infty d\zeta \frac{I(\zeta)}{\zeta - m^2}, \qquad C_{1D} = -\int_0^\infty d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2}. \quad (64)$$

The operator $-K = -\Box + m^2$ is the Klein-Gordon operator with inverse sign and mass m.

Owing to the quasi-locality of the current-like operator Λ_2 it must be composed of δ functions and its derivatives. The assumed growth rate and covariance arguments allow us to write

$$\lambda_2(x) = \lambda_0\delta(x) + \lambda_1(-K\delta(x)). \quad (65)$$

Substituting (65) and (62) in formula (59) for $f^{adv}(x)$, we obtain the spectral representation in the form

$$f^{adv}(x) = f^{*adv}(x) + (C_{0D} + \lambda_0)\delta(x) + (C_{1D} + \lambda_1)(-K\delta(x)). \quad (66)$$

Finally, recalling the condition of stability of single-particle states $\langle 0|j(x)|\mathbf{k}\rangle = 0$ and using the spectral requirement, we conclude that the spectral density $I(\zeta)$ in (60) and (61) differs from zero only for values of ζ above the inelastic threshold, i.e., the integration in (6) to (64) and (66) begins at $\zeta = (2m)^2$ [or $\zeta = (3m)^2$ for pseudoscalar particles].

6. INVESTIGATION OF THE SINGLE-PARTICLE SINGULARITIES

We now turn to the spectral representations for fields. Using the Yang-Feldman formula (7) we easily see that[10]

$$[A(x), A(y)] = -iD(x-y) + i\hat{D}_x^{ret}\frac{\delta j(y)}{\delta\varphi(x)}\hat{D}_y^{ret} - i\hat{D}_x^{adv}\frac{\delta j(x)}{\delta\varphi(y)}\hat{D}_y^{adv}.$$

The vacuum expectation value of this commutator is, according to (55),

$$\langle 0 \mid [A(x),\ A(y)] \mid 0\rangle = -iD(x-y) - i\hat{D}^{ret}f^{ret}(x-y)\hat{D}^{ret} + i\hat{D}^{adv}f^{adv}(x-y)\hat{D}^{adv}. \quad (67)$$

Substituting here the spectral representation (66) for $f^{ret}(x)$ and the analogous representation for $f^{adv}(x)$ we can obtain a spectral representation for $\langle 0|[A(x), A(y)]|0\rangle$. However, from this representation one notes at once two defects of the formula (66) above. First, this formula contains Klein-Gordon operators whose action must be defined separately. Indeed, these may act in (67) "outside" on the functions $\hat{D}^{adv}$ standing to the right and left, or "inside" on the function $\hat{D}_\zeta^{adv}$ in the spectral representation of the currents (63). It is not obvious whether the final result is independent of these two possibilities. The second point which has to be made more precise in connection with (66) is that we have neglected, on account of the stability condition, the contribution of the single-particle states to the spectral integral. However, although $\langle 0|j(x)|\mathbf{k}\rangle$ is undoubtedly equal to zero, this does not mean that $D^{ret}(k)\langle 0|j(x)|\mathbf{k}\rangle$ vanishes. We must therefore investigate this question separately for formula (67) since this zero may be removed by a pole.

These two points are intimately connected. Indeed, since the Klein-Gordon operators with mass m do not yield zero when acting on a D function with mass ζ if only $\zeta \neq m^2$, it is easy to see that this result remains unaltered if the action of the operators $(-K)$ in (63) is defined in an arbitrary way. This is completely understandable, since in the p representation the question of

3)The formal character of the multiplication consists in the circumstance that one generally obtains a divergent expression. For a justification of this formal method cf. [7]. In the following we are considering a definite growth rate: n = 1.

the correct choice of the action of the operator (−K) reduces to the order of multiplication. Indeed, the source of this ambiguity is

$$\frac{1}{m^2-k^2-i\varepsilon}[(m^2-k^2)\delta(m^2-k^2)]$$
$$\neq\left[\frac{1}{m^2-k^2-i\varepsilon}(m^2-k^2)\right]\delta(m^2-k^2).$$

However, if one puts $\zeta \neq m^2$ in one of the factors, the ambiguity disappears and, for example, $(\zeta-k^2)^{-1}(m^2-k^2)\delta(m^2-k^2)=0$ for both orders of multiplication.

In other words, if we assume that the density $I(\zeta)$ does not contain terms with $\zeta = m^2$ [otherwise formulas (64) would lose their exact meaning], then the order of application of the Klein-Gordon operators in $f^{*adv}(x)$ can be regarded as immaterial. For definiteness we shall assume that they act outside, i.e., we set

$$f^{*adv}(x)=\int_{(2m)^2}^{\infty}d\zeta\,\frac{I(\zeta)}{(\zeta-m^2)^2}(\underleftarrow{-K})\,D_\zeta^{adv}(x)(\underrightarrow{-K}). \tag{68}$$

The corresponding contribution of the states with mass $M_n > 2m$ to the function f(x) is

$$f^*(x)=\int_{(2m)^2}^{\infty}d\zeta\,\frac{I(\zeta)}{(\zeta-m^2)^2}(-\underleftarrow{K})\,D_\zeta(x)(-\underrightarrow{K}).$$

Regarding the problem of the point $\zeta = m^2$ we establish first of all the formal expression for the contribution of the single-particle states to the spectral density $I(\zeta)$. Since $I(\zeta)$ is expressed in terms of matrix elements of the current commutator, we can easily single out the single-particle contribution, using the expansion in a complete system of states:

$$f^{(-)}(x-y)=i\sum_n\int d\mathbf{k}\,\langle 0|j(x)|\mathbf{k},n\rangle\langle\mathbf{k},n|j(y)|0\rangle$$
$$=\int d\zeta\,d\eta f^{adv}(x-\zeta)D^-(\zeta-\eta)f^{ret}(\eta-y) + [\text{contribution of states with } M_n\geq 2m].$$

From this we obtain the important formula

$$f(x-y)=\hat f^{adv}D(x-y)\hat f^{ret} + [\text{contribution of states with } M_n\geq 2m], \tag{69}$$

where the second term has the form (60).

We now introduce in $f^{adv}(x)$ a new term referring to the point $\zeta = m^2$ in the spectrum, which is not taken into account by (66) with the limit of integration $\zeta = (2m)^2$. We choose this term in analogy to (62) and (68):

$$C_{1m}\{(-\underleftarrow{K})\,D_{m^2}^{adv}(x-y)(-\underrightarrow{K})-(-K\delta(x-y))\}$$

with the as yet undetermined coefficient C_{1m}. We note that if the operators (−K) would here act inside on the function $D_{m^2}^{adv}$, the entire term would vanish. This corresponds to the addition of the (vanishing) number $C_{1m}(\zeta-m^2)^2\delta(\zeta-m^2)$ to the spectral density $I(\zeta)$ which, for an appropriate order of multiplication, can give a nonvanishing contribution if it is multiplied by a pole of the type $(\zeta-m^2)^{-2}$. However, it is more convenient not to introduce this term in the spectral density but to take it into account separately.

It is easy to see that if $f^{adv}(x)$ contains the term

$$(f^{adv}(x))_m=C_{1m}\{(-\underleftarrow{K})\,D_{m^2}^{adv}(x)(-\underrightarrow{K})-(-K\delta(x))\},$$

then f(x) will contain the term

$$(f(x))_m=C_{1m}(-\underleftarrow{K})D_{m^2}(x-y)(-\underrightarrow{K}).$$

Thus, supplementing the spectral representations (66) by these terms, we obtain the following complete spectral representations, which take account of the possible contribution from a term with $\zeta = m^2$ and of the fixed order of application of the Klein-Gordon operators:

$$f^{adv}(x-y)$$
$$=(\underleftarrow{-K})\left\{\int_{(2m)^2}^{\infty}\frac{d\zeta I(\zeta)}{(\zeta-m^2)^2}D_\zeta^{adv}(x-y)+C_{1m}D_{m^2}^{adv}(x-y)\right\}(\underrightarrow{-K})$$
$$-\left\{\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{(\zeta-m^2)^2}+C_{1m}-\lambda_1\right\}(-K\delta(x-y))$$
$$+\left\{\int_{(2m)^2}^{\infty}\frac{d\zeta I(\zeta)}{\zeta-m^2}+\lambda_0\right\}\delta(x-y), \tag{70}$$

$$f(x-y)=(\underleftarrow{-K})\left\{\int_{(2m)^2}^{\infty}\frac{d\zeta I(\zeta)}{(\zeta-m^2)^2}D_\zeta(x-y)+C_{1m}D_{m^2}(x-y)\right\}(\underrightarrow{-K}), \tag{71}$$

these satisfy the general requirements defining the retarded and advanced functions.

Having obtained the complete spectral representations (70) and (71), we can turn to (69), and equating the contribution of the single-particle state on the left and right-hand sides, we obtain the following very important equation, which has the meaning of a single-particle unitarity condition:

$$(\underleftarrow{-K})C_{1m}D_{m^2}(x-y)(\underrightarrow{-K})=\hat f^{adv}D_{m^2}(x-y)\hat f^{ret}. \tag{72}$$

Let us now analyze the right-hand side; using (70) and recalling $(\underrightarrow{-K})D_{m^2}=D_{m^2}(\underleftarrow{-K})=0$, we find

$$(-K)C_{1m}D(x-y)\,(-K)$$
$$=\frac{1}{4}(\underleftarrow{-K})\left\{\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{(\zeta-m^2)^2}+C_{1m}-\lambda_1\right\}^2D_{m^2}(x-y)(\underrightarrow{-K})$$
$$-\frac{1}{2}\left[\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{(\zeta-m^2)^2}+C_{1m}-\lambda_1\right]\left[\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{\zeta-m^2}+\lambda_0\right]$$
$$\times\left[(\underleftarrow{-K})D_{m^2}(x-y)+D_{m^2}(x-y)(\underrightarrow{-K})\right]$$
$$+\left[\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{\zeta-m^2}+\lambda_0\right]^2D_{m^2}(x-y). \tag{73}$$

From this equation we see first of all that in order to guarantee that the functions f and hence f^{adv} do not contain terms of a new structure, we must choose the free constant λ_0 such that

$$\lambda_0=-\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{\zeta-m^2}, \tag{74}$$

which corresponds to a definite renormalization of the mass. Comparing with (70) it is easy to see that this choice corresponds to the stability condition for single-particle states. For convenience we introduce the notation

$$\int_{(2m)^2}^{\infty}d\zeta\frac{I(\zeta)}{(\zeta-m^2)^2}-\lambda_1=1-Q,$$

and rewrite (73), using (74):

$$C_{1m}={}^1\!/_4(1-Q+C_{1m})^2.$$

Solving this quadratic equation,

$$C_{1m} = (1 \pm \sqrt{Q})^2, \tag{75}$$

we obtain the final spectral representations which are consistent with the requirements of single-particle unitarity and stability:

$$f^{adv}(x-y) = (\overleftarrow{-K}) \int_{(2m)^2}^{\infty} d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2} D_\zeta^{adv}(x-y)(\overrightarrow{-K})$$

$$+ (1 \pm \sqrt{Q})^2 (\overleftarrow{-K}) D_{m^2}^{adv}(x-y)(\overrightarrow{-K}) + 2(1 \pm \sqrt{Q})(-K\delta(x-y)), \tag{76}$$

$$f(x-y) = (\overleftarrow{-K}) \int_{(2m)^2}^{\infty} d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2} D_\zeta(x-y)(\overrightarrow{-K})$$

$$+ (1 \pm \sqrt{Q})^2 (\overleftarrow{-K}) D_{m^2}(x-y)(\overrightarrow{-K}). \tag{77}$$

These representations are a more precise form of the representations first obtained in[4].

7. RENORMALIZED AND UNRENORMALIZED THEORIES

We have carried out this investigation for the purpose of proving the following simple and natural assertion: When the unrenormalized currents and fields are substituted in the spectral representation one obtains a propagation function, or commutator, with the properties of the unrenormalized theory, and when the renormalized operators are substituted, one obtains a function with the properties of the renormalized theory. When this assertion is proved, the equivalence of the two methods of renormalization mentioned in the beginning is established. However, owing to the operator nature of the renormalizations, we had to make the spectral formulas more precise, which has been our concern in the preceding section.

We begin with the unrenormalized theory. From the meaning of its definition we must set

$$\lambda_1(\Gamma_1) = 0, \quad \lambda_0(\Gamma_1) = -\int_{(2m)^2} d\zeta \frac{I(\zeta)}{\zeta - m^2} \tag{78}$$

(the second formula was obtained in the preceding section).

By the current j(x), which has not yet been specified, we shall in this theory understand $j_0(\Gamma_1, \psi|x)$, and by the asymptotic field φ we shall understand $\psi(x)$. Accordingly we write applying for this case formula (67),

$$\langle 0|[A_0(\Gamma_1, \psi|x), A_0(\Gamma_1, \psi|y)]|0\rangle \equiv G_0(\Gamma_1, \psi|x-y) = -i\{D(x-y)$$
$$+ \hat{D}^{ret} f^{ret}(\Gamma_1|x-y)\hat{D}^{ret} - \hat{D}^{adv} f(\Gamma_1|x-y)\hat{D}^{adv}\}. \tag{79}$$

Substituting here the spectral representation (76) we obtain

$$G_0(\Gamma_1, \psi|x-y) = -i\{1 - (1 \pm \sqrt{Q})\hat{I}\} D_{m^2}(x-y)\{1 - (1 \pm \sqrt{Q})\hat{I}\}$$

$$- i\hat{I} \int_{(2m)^2}^{\infty} d\zeta \frac{I(\zeta)}{(\zeta - m^2)^2} D_\zeta(x-y)\hat{I}. \tag{80}$$

From this spectral representation we easily find the asymptotic limits which allow us to establish the connection with the renormalization method of Lehmann. On the energy shell, where $k^2 = m^2$, we evidently have

$$G_0(\Gamma_1, \psi|x-y) = -i\{1 - (1 \pm \sqrt{Q})\hat{I}\} D_{m^2}(x-y)\{1 - (1 \pm \sqrt{Q})\hat{I}\}.$$

On the other hand, in the other asymptotic limit, where $k^2 \to \infty$ and the D_ζ function practically ceases to depend on the mass, we obtain

$$G_0(\Gamma_1, \psi|x-y) = -i\{D(x-y) - (1 \pm \sqrt{Q})\hat{I}D(x-y)(1 - \hat{I})$$
$$- (1 - \hat{I})D(x-y)(1 \pm \sqrt{Q})\hat{I}\}.$$

If we go over to the numerical limit, setting $\hat{I} = 1$, we find

$$G_0(\Gamma_1, \psi|x-y) = \begin{cases} -iQD(x-y) & \text{for} \quad k^2 = m^2 \\ -iD(x-y) & \text{for} \quad k^2 \to \infty \end{cases} \tag{81}$$

In accordance with the usual limits of the field commutator[1] we can set $Q = Z_3$ at $k^2 = m^2$, where Z_3 is the renormalization constant. However, we can not yet identify this constant Z_3 with the constant Z which enters in the renormalized field and current. In order to establish this correspondence we must consider the renormalized theory, which we shall do below. In the limit $k^2 \to \infty$, on the other hand, the coefficient in (81) is equal to unity, which is natural for the unrenormalized theory.

We further note that in our scheme we can also define the renormalization of the field directly, considering the matrix element of the field $A_0(\Gamma_1, \psi|x)$, as was done by Källén;[11] in this case no ambiguity occurs.[12] Indeed, considering directly the matrix element of the field $A_0(\Gamma_1, \psi|x)$ between the vacuum and a single-particle state and using the Yang-Feldman formula [7] we obtain

$$\langle 0|A_0(\Gamma_1, \psi|x)|\mathbf{k}\rangle = \langle 0|\psi(x)|\mathbf{k}\rangle$$
$$- \int dy\, \hat{D}^{adv} \left\langle 0 \left| \frac{\delta j_0(\Gamma_1, \psi|x)}{\delta\psi(y)} \right| 0 \right\rangle \langle 0|\psi(y)|\mathbf{k}\rangle$$
$$= (1 + \hat{D}^{adv} f^{adv}) \langle 0|\psi(x)|\mathbf{k}\rangle. \tag{82}$$

Since owing to the presence of the matrix element $\langle 0|\psi(x)|\mathbf{k}\rangle$, we need consider the whole expression on the energy shell $k^2 = m^2$ only and since the fields $\varphi(x)$ and $\psi(x)$ coincide on the energy shell, we can write, using (76),

$$\langle 0|A_0(\Gamma_1, \psi(x)|\mathbf{k}\rangle = \{1 - (1 \pm \sqrt{Z_3})\hat{I}\}\langle 0|\varphi(x)|\mathbf{k}\rangle$$
$$= \mp\sqrt{Z_3}\langle 0|\varphi(x)|\mathbf{k}\rangle. \tag{83}$$

The last equation has the meaning of the limit for $\hat{I} = 1$. Thus we see that in this formalism the renormalization of the free lines with $\sqrt{Z_3}$ is obtained automatically without any ambiguities, as noted in[2]. It is seen from the formula for $G_0(x-y)$ that we must choose the solution with the plus sign in (75), since the solution with the minus sign leads to change of the sign of the norm.

Let us now consider the renormalized theory. Thus we must consider the fields $A(\Gamma_2, \varphi|x)$ and the currents $j(\Gamma_2, \varphi|x)$. However, since we are interested in the connection between the representations for the renormalized and unrenormalized fields, we shall first express everything in terms of the unrenormalized spectral density $I(\Gamma_1, \zeta)$. We take the vacuum expectation value of (50) [the first term in (50) is a c number[2]]:

$$f^{adv}(\Gamma_2, \varphi|x-y) = -\frac{\delta j_R(x)}{\delta\varphi(y)} + \{1 + (\overleftarrow{-K})(1 - \sqrt{Z}|\hat{N}\hat{D}^{adv}\}_x$$
$$\cdot f^{adv}(\Gamma_1, \psi|x-y)\{1 + \hat{D}^{adv}\hat{N}(1 - \sqrt{Z})(-K)\}_y. \tag{84}$$

Substituting here the spectral representation for $f^{adv}(\Gamma_1, \psi|x-y)$, we find the spectral representation for $f^{adv}(\Gamma_2, \varphi|x-y)$ expressed through the same spectral density $I(\Gamma_1, \zeta)$:

$$f^{adv}(\Gamma_2, \varphi|x-y) = (\overleftarrow{-K})\hat{N}\Big\{\int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1,\zeta)}{(\zeta-m^2)^2} D_\zeta^{adv}(x-y) + (\sqrt{Z} \pm \sqrt{Z_3}) D^{adv}(x-y)\Big\}\hat{N}(\overrightarrow{-K}) - (\sqrt{Z}\pm\sqrt{Z_3}) \cdot \{(\overleftarrow{-K})\hat{N}\delta(x-y) + \delta(x-y)\hat{N}(\overrightarrow{-K})\} \quad (85)$$

(here we have used the expression for $\delta j_R/\delta\varphi$ of[2]).

In the renormalized theory the spectral representation for f^{adv} must not contain the term with the Klein-Gordon operator $(m^2 - k^2)$ in first order. This condition forces us to set $Z = Z_3$, so that

$$f^{adv}(\Gamma_2, \varphi|x-y) = (\overleftarrow{-K})\hat{N}\int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1, \psi|\zeta)}{(\zeta-m^2)^2} D_\zeta^{adv}(x-y)\hat{N}(\overrightarrow{-K}), \quad (86)$$

$$Z = 1 - \int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1,\zeta)}{(\zeta-m^2)^2}. \quad (87)$$

This completes the proof that the two methods of introducing Z are completely equivalent when the exact rules for dealing with the Klein-Gordon operators are observed and the single-particle intermediate state is correctly taken into account.

To establish the contribution of the current-like operator Λ_2 we consider the function $f_D^{adv}(x)$. We note that (49) yields

$$f(\Gamma_2, \varphi|x-y) = (\overleftarrow{-K})\hat{N}\int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1,\zeta)}{(\zeta-m^2)^2} D_\zeta(x-y)\hat{N}(\overrightarrow{-K}). \quad (88)$$

Multiplying this f by $\theta(y^0 - x^0)$, we find f_D^{adv}. To this end we must compute

$$\theta(y^0-x^0)(\overleftarrow{-K})\hat{N}D_\zeta(x-y)\hat{N}(\overrightarrow{-K}).$$

Using[2]

$$\theta(x^0-y^0)(\overleftarrow{-K})\hat{N}D(x-y)\hat{N}(\overrightarrow{-K}) = (\overleftarrow{-K})\hat{N}D^{ret}(x-y)\hat{N}(\overrightarrow{-K}) - \frac{\hat{N}}{2\sqrt{Z}}\delta(\mathbf{x}-\mathbf{y})(\overrightarrow{-K_x} \overrightarrow{-K_y}),$$

we find

$$f_D^{adv}(\Gamma_2, \varphi|x-y) = (\overleftarrow{-K})\hat{N}\int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1,\zeta)}{(\zeta-m^2)^2} D_\zeta^{adv}(x-y)\hat{N}(\overrightarrow{-K}) - \frac{(1-Z)}{\sqrt{Z}}\hat{N}\delta(\mathbf{x}-\mathbf{y})\frac{\overrightarrow{-K_x} \overrightarrow{-K_y}}{2} - \frac{\hat{N}}{\sqrt{Z}}\lambda_0(\Gamma_1)\delta(x-y). \quad (89)$$

Comparing f^{adv} and f_D^{adv}, we find

$$\lambda_2(\Gamma_2, \varphi|x-y) = \frac{(1-Z)}{\sqrt{Z}}\hat{N}\delta(\mathbf{x}-\mathbf{y})\frac{\overrightarrow{-K_x} \overrightarrow{-K_y}}{2} + \frac{\hat{N}}{\sqrt{Z}}\lambda_0(\Gamma_1)\delta(x-y),$$

$$\lambda_2(\Gamma_2)(-K\delta(x-y)) = \lambda_{2R}(x-y) = -\langle 0|\Lambda_{2R}(x-y)|0\rangle,$$

$$\lambda_0(\Gamma_2) = \frac{\hat{N}}{\sqrt{Z}}\lambda_0(\Gamma_1). \quad (90)$$

Finally, computing the vacuum expectation value of the commutator of the renormalized fields, in analogy to the unrenormalized theory, we find

$$G(\Gamma_2, \varphi|x-y) = -iD(x-y) - i\hat{I}\hat{N}\int_{(2m)^2}^{\infty} d\zeta \frac{I(\Gamma_1,\zeta)}{(\zeta-m^2)^2} D_\zeta(x-y)\hat{N}\hat{I}. \quad (91)$$

For the asymptotic form of this expression we find

$$G(\Gamma_2, \varphi|x-y) = \begin{cases} -iD(x-y) & \text{for } k^2 = m^2, \\ -i\{D(x-y) + \hat{I}\hat{N}(1-Z)D(x-y)\hat{N}\hat{I}\} & \text{for } k^2 \to \infty. \end{cases} \quad (92)$$

In the numerical limit ($\hat{I} = 1$) the expression (92) goes over into $-iZ^{-1}D(x-y)$ for $k^2 \to \infty$, as it should be.

Comparing the earlier formula for $G_0(\Gamma_1, \psi|x-y)$ with (92), we obtain

$$G(\Gamma_2, \varphi|x-y) = \hat{N}G_0(\Gamma_1, \psi|x-y)\hat{N}, \quad (93)$$

which agrees with the analogous formula obtained above before taking the vacuum expectation value.

Turning now to the calculation of the single-particle matrix element, we have

$$\langle 0|A(\Gamma_2, \varphi|x)|\mathbf{k}\rangle = (1 + \hat{D}^{adv}\hat{f}^{adv}(\Gamma_2, \varphi))\langle 0|\varphi(x)|\mathbf{k}\rangle = \langle 0|\varphi(x)|\mathbf{k}\rangle, \quad (94)$$

since $\hat{f}^{adv}(\Gamma_2, \varphi)$ contains Klein-Gordon operators acting to both sides. Comparing with (83), we obtain

$$\langle 0|A(\Gamma_2, \varphi|x)|\mathbf{k}\rangle = \hat{N}\langle 0|A_0(\Gamma_1, \varphi|x)|\mathbf{k}\rangle, \quad (95)$$

again in agreement with the formula for the operators themselves.[3]

Finally, comparing (88) with the general formula for f(x) we obtain a relation between the spectral densities in the two theories, which has the form

$$I(\Gamma_2, \zeta) = Z^{-1}I(\Gamma_1, \zeta). \quad (96)$$

Replacing $I(\Gamma_1, \zeta)$ in (87) by $I(\Gamma_2, \zeta)$, we can express the constant Z directly through $I(\Gamma_2, \varphi)$:

$$\frac{1}{Z} = 1 + \int_{(2m)^2}^{\infty} \frac{d\zeta I(\Gamma_2,\zeta)}{(\zeta-m^2)^2}, \quad (97)$$

which leads to the known Lehmann inequality for the constant Z.

8. CONCLUSION

We have shown above that our procedure allows us to connect the two traditional methods of introducing the field renormalization constant: the Hamiltonian method and the pure axiomatic method going back to Lehmann. It was shown that the form of the spectral representations depends vitally on whether or not subtractions have been made. If subtractions have been made, one must carefully investigate the problem of a possible single-particle contribution to the spectral density and also the connected problem in which direction the Klein-Gordon operators act, which appear as a result of the subtractions. The final form of the spectral representation can only be established with the help of a special single-particle unitarity condition.

But this scrupulous procedure is fruitful, for it allows one not only to connect the quantities appearing in the Hamiltonian and axiomatic methods but also to show the self-consistency of the formulas for the connection of the operators in the two theories[3] with the formulas for their matrix elements. Finally, in the course of the investigation we have also solved the burning problem of the correspondence between the introduction of the field renormalization constant via the Green's function which is quadratic in the fields, and via the single-particle matrix element, which is linear in the field.

One may hope that further application of our method to more copious theories (for example, to theories with vector fields which are the object of study in current algebra) will make it possible to deal with their many difficulties.

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