RELAXATION OF A QUANTUM OSCILLATOR

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We use the kinetic equation (1) to consider the relaxation of a quantum harmonic oscillator to the thermodynamic equilibrium state. We use the method of generating functions to solve this equation. We consider a number of exact solutions for different boundary conditions. For large quantum numbers Eq. (1) is transformed into the Fokker-Planck equation and the Green function for that equation is found. We show how to apply the results to the problem of the statistics of photocounts. We generalize the kinetic equation (1) to the case when the oscillator is acted upon by an external classical force f(t) which is an arbitrary function of the time. We briefly consider ways to describe the evolution of the oscillator (characteristic functions, quasi-probability distributions in the α-plane) which are applicable when the force f(t) is present and we obtain the equation of motion for the relevant quantities. The partial solutions for the density matrix ρ obtained in the present paper are given in a table. We discuss the so-called "harmonic oscillator paradox."

1. INTRODUCTION

THERE is at the moment going on an intense discussion of the problems of the quantum theory of a laser[1-4] and of the photostatistics of laser light.[5-9] In most papers one uses a single-mode model of a laser leading to a consideration of a quantum oscillator interacting with an active medium. Owing to the complexity of the equations occurring in the theory their solution is found for a limited number of cases, mainly stationary cases.[1] It is therefore of interest to analyze the simple linear problem of the evolution of a quantum oscillator interacting with a dissipative medium. This problem is also of independent interest since it is the simplest model describing the statistical properties of coherent light propagating in a weakly absorbing medium (see[10-13] in this connection). The problem considered is important also because it belongs to the very small number of problems in non-equilibrium quantum statistical mechanics which can be solved exactly. Some results about the Brownian motion of a quantum oscillator were obtained by Schwinger.[14]

The present paper is devoted to describing the relaxation of a quantum oscillator using the following equation for its density matrix ρ:

$$d\rho / dt = -{}^1\!/_2\gamma[(\nu+1)(a^+a\rho - 2a\rho a^+ + \rho a^+a) + \nu(aa^+\rho - 2a^+\rho a + \rho aa^+)]. \tag{1}$$

Here γ is the damping constant of the oscillator (see (7) and (9) below), ν the average number of quanta for the oscillator when it is in a state of thermodynamic equilibrium:

$$\nu = \xi / (1-\xi) = (e^{\hbar\omega/kT}-1)^{-1}, \quad \xi = e^{-\hbar\omega/kT} \tag{2}$$

(in optics $\nu \ll 1$), and a and a^+ the usual annihilation and creation operators for the vibrational quanta. The kinetic equation (1) is essentially already contained in Landau's well known paper[15] (for the case $\nu = 0$) and can also be obtained from the general theory of the relaxation of quantum systems.[12,16] This equation was written down in its clearest form in the paper by Shen[13] for a field oscillator interacting with a two-level system of atoms. A simple derivation of Eq. (1) with the advantage of an obvious transition to the classical limit is given in[17]. It is important that the operators in (1) refer only to the oscillator; the interaction with the thermostat is taken into account phenomenologically using the constants γ and ν. We note also that Eq. (1) can describe not only damping but also the linear build-up of the vibrations of the oscillator (in a medium with a negative temperature). For this it is necessary to change the sign of γ and to assume that $\nu < -1$.

To solve the kinetic equation (1) we use in the present paper the method of generating functions (Sec. 2). Several concrete examples are discussed in Secs. 3, 4, 6, and 7. In particular, we give in Sec. 8 a solution of the well-known harmonic oscillator paradox.[18-22] We note also that the results obtained here have a direct relevance to problems of the statistics of photocounts (see Sec. 5).

2. METHOD OF GENERATING FUNCTIONS

Changing in (1) to the occupation number representation and introducing the variable $\tau = \gamma t$, we get

$$d\rho_{mn} / d\tau = (\nu+1)\sqrt{(m+1)(n+1)}\rho_{m+1,\,n+1} - [(m+n)(\nu+{}^1\!/_2)+\nu]\rho_{mn} + \nu\sqrt{mn}\rho_{m-1,\,n-1}. \tag{3}$$

We shall call the set of elements ρ_{mn} with a fixed value of the difference m – n = k the "k-th diagonal" of the matrix ρ (for k = 0 we get the level populations $w_n = \rho_{nn}$). It is clear from (3) that the elements of the different diagonals evolve independently without mixing. This property is characteristic for a harmonic oscillator when there is no external field f(t) and is connected with the fact that $\rho_{mn} \propto e^{i(m-n)\omega t}$. To solve Eqs. (3) we apply the method of generating functions.

Let, for instance, $k > 0$. We form the function

$$G_k(z,\tau) = \sum_{n=0}^{\infty}\left[\frac{(n+k)!}{n!k!}\right]^{1/2}\rho_{n+k,\,n}(\tau)z^n, \tag{4}$$

where z is an auxiliary variable. Multiplying (3) by z^n and summing over n from 0 to ∞ we get an equation for $G_k(z, \tau)$:

$$\frac{\partial G_k}{\partial \tau}=(1-z)\left\{(1+\nu(1-z))\frac{\partial G_k}{\partial z}-\nu(k+1)G_k\right\}-\frac{k}{2}G_k. \quad (5)$$

It is important that (5) is a first-order equation and we can thus solve it in the general form:

$$G_k(z,\tau)=p^{k/2}(1+q\nu\zeta)^{-k-1}f_k\left(1-\frac{p\zeta}{1+q\nu\zeta}\right). \quad (6)$$

Here $f_k(z) = G_k(z, 0)$ is a function determined by the boundary conditions; $\zeta = 1 - z$ and we have introduced the notation

$$p=e^{-\tau}, \quad q=1-e^{-\tau} \quad (7)$$

Finding the matrix elements $\rho_{mn}(\tau)$ leads in accordance with (4) to expanding (6) in a Taylor series. This will be illustrated in Sec. 3 by a number of examples. Here we consider some general properties of the evolution of $\rho_{mn}(\tau)$ following from (6).

1) When z = 1 we have from (6)

$$\langle (a^+)^r a^{r+k}\rangle_\tau=\mathrm{Sp}\,\{\rho(\tau)(a^+)^r a^{r+k}\}=\sqrt{k!}\left.\frac{\partial^r G_k(z,\tau)}{\partial z^r}\right|_{z=1} \quad (8)$$

In particular, we have for r = 0

$$\langle a^k\rangle_\tau=e^{-k\tau/2}\langle a^k\rangle_0 \quad (9)$$

(independent of the temperature of the thermostat).

2) We can express the average value $\bar{n}$ and the dispersion $\overline{\Delta n^2}$ in terms of the generating function $G_0(z, \tau)$:

$$\bar{n}(\tau)=G_0'|_{z=1}=\bar{n}_0 p+\nu q \quad (\bar{n}_0=\bar{n}(0)), \quad (10a)$$

$$\overline{\Delta n^2}(\tau)=G_0''-(G_0')^2+G_0'|_{z=1}=\overline{\Delta n_0^2}p^2+(2\nu+1)(\bar{n}-\nu)pq+\nu(\nu+1)(1-p^2). \quad (10b)$$

We emphasize that Eqs. (10) are true for any initial distribution $w_n(0)$. The peculiar point about an oscillator is that the evolution of $\overline{n(\tau)}$ and $\overline{\Delta n^2(\tau)}$ to the values corresponding to the thermal distribution (ν and $\nu^2 + \nu$, respectively) are completely determined by $\overline{n_0}$ and $\overline{\Delta n_0^2}$ and are independent of the higher-order moments of the initial distribution. Weber[23] had earlier obtained Eq. (10a) for $\overline{n(\tau)}$ (see also[12-14]).

3) Putting z = 0 in (6) we get the following simple formula for the occupation of the ground state level:

$$w_0(\tau)=\frac{1}{1+q\nu}f_0\left(\frac{(1+\nu)q}{1+q\nu}\right). \quad (11)$$

As $\tau \to \infty$ $w_0(\tau)$ approaches asymptotically the value $w_0(\infty) = (1 + \nu)^{-1}$ corresponding to the Planck distribution.

4) Expanding (6) when $p \to 0$ we get equations describing the approach of the occupations w_n to the thermal distribution:

$$w_n(\tau)=(1-\xi)\xi^n\{1+a_n p+b_n p^2+\ldots\}, \quad \tau\to\infty, \quad (12)$$

where

$$a_n=(n-\nu)(\bar{n}_0-\nu)/\nu(\nu+1),$$
$$b_n=[(n-\nu)(n-3\nu-1)-\nu(\nu+1)][\overline{\Delta n_0^2}-\nu(\nu+1)+(\bar{n}_0-\nu)(\bar{n}_0-3\nu-1)][4\nu^2(\nu+1)^2]^{-1}. \quad (13)$$

If $\overline{n_0} \neq \nu$ the difference $\delta_n = |w_n(\tau) - w_n(\infty)|$ decreases as $p = e^{-\tau}$; when $\overline{n_0} = \nu$, but $\overline{\Delta n_0^2} \neq \nu(\nu + 1)$ we will have $\delta_n \sim e^{-2\tau}$, and so on. The larger the number of the early moments of the initial distribution $w_n(0)$, which are the same as the corresponding moments of the Planck distribution, the faster the oscillator approaches equilibrium. It can be seen from (6) that the off-diagonal elements ρ_{mn} are damped like $\sim\exp\{-\frac{1}{2}|m-n|\tau\}$.

Equation (12) does not refer to the case of zero temperature of the medium ($\nu = 0$). In that case

$$w_k(\tau)\approx n_0^{[k]}p^k \quad \text{as} \quad \tau\to\infty, \quad (14)$$

where the $n_0^{[k]}$ are the so-called factorial moments[6] of the initial distribution:

$$n_0^{[k]}=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}w_n(0)=\langle a^{+k}a^k\rangle|_{\tau=0}. \quad (14a)$$

3. SOLUTIONS FOR PARTICULAR CASES

We now turn to concrete examples.

1. When $\tau = 0$ the oscillator is in a state of thermodynamic equilibrium with a temperature T_0 different from the temperature of the thermostat T:

$$\rho_{mn}(0)=(1-\xi_0)\xi_0^n\delta_{mn}, \quad \xi_0=\exp(-\hbar\omega/kT_0). \quad (15)$$

We note that for the Planck distribution we have

$$f_0(z)=\sum_{n=0}^{\infty}w_n z^n=\frac{1}{1+\nu\zeta}\quad\left(\nu=\frac{\xi}{1-\xi},\quad \zeta=1-z\right). \quad (16)$$

Using (6) we find $G_0(z, \tau) = (1 + (p\nu_0 + q\nu)\zeta)^{-1}$ and hence it follows that the distribution $w_n(\tau)$ at any time remains a Planck distribution but the temperature of the oscillator changes from T_0 to T in accordance with Eq. (10a): $\overline{n}(\tau) = p\nu_0 + q\nu$. This result was obtained by Schwinger.[14]

2. The initial state is the coherent state $|\alpha\rangle$:

$$\rho(0)=|\alpha\rangle\langle\alpha|,\quad |\alpha\rangle=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad (17)$$

where α is some complex number (see[5-7] for details of the properties of the state $|\alpha\rangle$). We first consider the evolution of the populations $w_n(\tau)$. When k = 0 Eq. (6) gives

$$G_0(z,\tau)=\frac{1}{1+q\nu\zeta}\exp\left(-\frac{p|\alpha|^2\zeta}{1+q\nu\zeta}\right). \quad (18)$$

Using the expression for the generating function for the Laguerre polynomials (see Eq. (8.975.1) in[24]) we find

$$w_n(\tau)=\frac{1}{1+q\nu}\left(\frac{q\nu}{1+q\nu}\right)^n\exp\left(-\frac{p|\alpha|^2}{1+q\nu}\right)L_n\left(-\frac{p|\alpha|^2}{q\nu(1+q\nu)}\right). \quad (19)$$

This equation simplifies for $\nu = 0$:

$$w_n(\tau;\nu=0)=\exp(-p|\alpha|^2)\frac{(p|\alpha|^2)^n}{n!}. \quad (20)$$

When $\nu > 0$ the distribution of the populations w_n is no longer a Poisson one (for $t > 0$). We note that Eq. (19) refers also to the case when the initial state is a superposition of coherent states $|\alpha\rangle$ with random phases:

$$\rho(0)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,||\alpha|e^{i\varphi}\rangle\langle|\alpha|e^{i\varphi}|,\quad \rho_{mn}(0)=e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\delta_{mn}. \quad (21)$$

3. Let the oscillator at $\tau = 0$ be in an N-quantum state: $\rho_{mn}(0) = \delta_{mn}\delta_{Nn}$. We have then

$$f_0(z)=z^N,\quad G_0(z,\tau)=\frac{[1+(q\nu-p)\zeta]^N}{(1+q\nu\zeta)^{N+1}},\quad \zeta=1-z. \quad (22)$$

Hence we find

$$\rho_{mn}=w_n(\tau)\delta_{mn},\quad w_n(\tau)=\frac{[q(1+\nu)]^N}{(1+q\nu)^{N+1}}\left(\frac{q\nu}{1+q\nu}\right)^n P_{N,n}\left(\frac{p}{q^2\nu(1+\nu)}\right), \tag{23}$$

where $P_{\lambda,n}$ is a polynomial defined from the following equation:

$$\frac{(1-t+zt)^\lambda}{(1-t)^{\lambda+1}}=\sum_{n=0}^{\infty}P_{\lambda,n}(z)t^n. \tag{24}$$

One can prove that[17]

$$P_{N,n}(z)=F(-N,-n,1;z)=\sum_{k=0}^{\min(N,n)}C_N{}^kC_n{}^kz^k, \tag{24a}$$

where $C_n^k = n!/k!(n-k)!$ are the binomial coefficients. We can appreciably simplify Eq. (23) for the w_n at the time τ_0 when $p = \nu/(\nu+1)$ i.e., $\gamma t_0 = \hbar\omega/kT$ (this is valid also when $\gamma < 0$, $T < 0$):

$$w_n(\tau_0)=C_{N+n}^n\left(\frac{\nu+1}{2\nu+1}\right)^{N+1}\left(\frac{\nu}{2\nu+1}\right)^n. \tag{25}$$

This is the so-called Pascal distribution.[25] At that instant ($\gamma t_0 = \hbar\omega/kT$) the distribution of the quasi-probabilities $P(\alpha)$ on the α-plane becomes a purely positive one from an alternating one (see Sec. 7).

We first consider the case $\nu = 0$. From (22) we find

$$G_0(z,\tau)=(pz+q)^N,\quad w_n(\tau)=C_N{}^np^nq^{N-n}, \tag{26}$$

i.e., w_n is a binomial distribution. This distribution arises also in the problem of the radioactive decay of N atoms each of which decays independently of the others with a lifetime γ^{-1}. The role of the atoms is here played by the different excitation quanta $\hbar\omega$ and the independence of their decay is connected with the linearity of the damping.

In the most interesting case $N \gg 1$, Eq. (26) becomes inconvenient. We can easily obtain for that case approximate formulae describing the change in $w_n(\tau)$ over the whole interval $(0 \le \tau < \infty)$:

$$w_n(\tau)=e^{-N\tau}(N\tau)^k/k!\quad\text{when } 0\leqslant\tau\ll 1,\ k=N-n=0,1,2,\ldots; \tag{27a}$$

$$w_n(\tau)=\frac{1}{\sqrt{2\pi Npq}}\exp\left\{-\frac{(n-Np)^2}{2Npq}\right\}\quad\text{when}\quad\frac{1}{N}\ll\tau\ll\ln N; \tag{27b}$$

$$w_n(\tau)=e^{-Np}(Np)^n/n!\quad\text{when } Np\lesssim 1. \tag{27c}$$

We note that (27b) leads to the Poisson distribution characteristic for a coherent state only when[1)] $p = e^{-\gamma t} \ll 1$; this approach is thus not uniform. In particular, the correlations Δ_m observed in experiments about photon counts (cf. Sec. 5) are not at all time-dependent (we are considering here a field oscillator of the light propagated in a linearly absorbing medium). If the initial state of the oscillator is an N-quantum one, we have (for $\nu = 0$)

$$\Delta_m=\frac{\langle(a^+)^ma^m\rangle-\langle a^+a\rangle^m}{\langle a^+a\rangle^m}=-1+\frac{N(N-1)\ldots(N-m+1)}{N^m} \tag{28}$$

(for all t). We note that in this case the sign of Δ_m is the opposite of what is obtained in the classical theory.

4. Similarly we can study the more complicated case when the matrix $\rho_{mn}(0)$ is not diagonal. Without giving the details of the calculations we present a few final formulae.

By virtue of the linearity of Eq. (1) we can write the solution for an arbitrary initial condition in the form

$$\rho_{lm}(\tau)=\sum_{l',m'=0}^{\infty}G(l,m;l',m'|\tau)\rho_{l',m'}(0), \tag{29}$$

where G is the analogue of the Green function for the discrete case:

$$G(l,m;l',m'|\tau)=\delta_{l-m,\,l'-m'}\frac{1}{k!}\left(\frac{N!N'!}{n!n'!}\right)^{1/2}\frac{p^{k/2}}{(1+q\nu)^{k+1}}\cdot\left(\frac{q\nu}{1+q\nu}\right)^n\left(\frac{q(1+\nu)}{1+q\nu}\right)^{n'}P^{(k)}_{n,n'}\left(\frac{p}{q^2\nu(\nu+1)}\right). \tag{30}$$

We have used here the notation

$$k=|l-m|=|l'-m'|\geqslant 0;\qquad n=\min(l,m),$$
$$N=\max(l,m),\quad n'=\min(l,m'),\quad N'=\max(l',m'). \tag{31}$$

The polynomial $P^{(k)}_{n,n'}(z)$ is defined by the formula

$$P^{(k)}_{n,n'}(z)=F(-n,-n',k+1;z)=\sum_{l=0}^{\min(n,n')}\frac{n!\,n'!\,k!}{l!(n-l)!(n'-l)!(k+l)!}z^l. \tag{32}$$

(for k = 0 these formulae go over into (23) and (24)). In principle, Eqs. (29) and (30) give us the possibility to find the time evolution of any density matrix $\rho_{mn}(0)$. We note that Eq. (30) for the Green function is appreciably simplified in the case $\nu = 0$:

$$G(m,n;m+s,n+s|\tau)=\frac{1}{s!}\left[\frac{(m+s)!(n+s)!}{m!\,n!}\right]^{1/2}p^{(m+n)/2}q^s, \tag{30a}$$

where $s \ge 0$.

As a particular case we consider the evolution of the coherent state $|\alpha\rangle$:

$$\rho_{mn}(0)=e^{-|\alpha|^2}\alpha^m\alpha^{*n}/\sqrt{m!\,n!}, \tag{33}$$

$$\rho_{n+k,\,n}(\tau)=R_{n+k,\,n}(\alpha\sqrt{p},\,q\nu)\quad(k\geqslant 0), \tag{34}$$

where we have denoted the function

$$R_{n+k,\,n}(\beta,\mu)=\sqrt{\frac{n!}{(n+k)!}}\frac{\beta^k\mu^n}{(\mu+1)^{n+k+1}}\exp\left(-\frac{|\beta|^2}{\mu+1}\right)L_n{}^k\left(-\frac{|\beta|^2}{\mu(\mu+1)}\right). \tag{34a}$$

by $R_{n+k,n}$. These formulae take an especially simple form in the zero-temperature case ($\nu = 0$):

$$\rho_{mn}(\tau)=e^{-|\beta|^2}\frac{\beta^m\beta^{*n}}{\sqrt{m!\,n!}},\qquad\beta=\beta(\tau)=\alpha e^{-\tau/2}, \tag{35}$$

i.e., the oscillator relaxes, remaining in a coherent state (cf.[13]). One can show (see the Appendix) that this is the only case when the state of the oscillator remains all the time a pure one during the relaxation process.

When $\nu \ne 0$ the interaction with the thermostat leads to the fact that the oscillator state ceases to be coherent for $t > 0$. For the case $|\beta|^2 \gg 1$ we can obtain from (34a) the approximate formula (with $\alpha\sqrt{p} = Re^{i\varphi} = \beta$, $\mu = q\nu \ll R^2$):

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1)The dispersion for the distribution (27b) is equal to $\sqrt{(Npq)}$ rather than to $\sqrt{(Np)}$.

$$\rho_{mn}(\tau)=\frac{1}{\sqrt{2\pi N(1+2\mu)}}\exp\left\{i(m-n)\varphi-\frac{[(m+n)/2-R^2]^2}{2N(1+2\mu)}-\frac{(m-n)^2(1+2\mu)}{2N}\right\}. \quad (36)$$

4. TRANSITION TO THE FOKKER-PLANCK EQUATION

The exact formulae for $\rho_{mn}(\tau)$ obtained earlier become very unclear when $m, n \gg 1$. Moreover, this is just the case of particular interest when the initial excitation of the oscillator is large. We can then go over from the discrete set of equations to the differential Fokker-Planck equation. When $m = n$, (3) becomes[2)]

$$dw_n/d\tau=(\nu+1)[(n+1)w_{n+1}-nw_n]-\nu[(n+1)w_n-nw_{n-1}]. \quad (37)$$

We shall assume that $n \gg 1$ and that w_n is a smooth function of n. We then get from (37)

$$\frac{\partial w}{\partial\tau}=\frac{1}{2}\frac{\partial^2}{\partial n^2}(Bw)-\frac{\partial}{\partial n}(Aw);\ A=\nu-n,\quad B=\nu+(2\nu+1)n. \quad (38)$$

This equation has the form of a diffusion equation and describes the relaxation process with good accuracy, as can be seen from the following: evaluating in the usual way[26] $\bar{n}(\tau)$ and $\overline{\Delta n^2}(\tau)$, we find for them from (38) expressions which are the same as the exact Eqs. (10). However, the n-dependence of the coefficients A and B makes it difficult to find its solution.[3)] We shall therefore simplify further.

Changing variables

$$x=\sqrt{n},\qquad V(x)=2\sqrt{n}w_n\quad (V(x)dx=w(n)dn), \quad (39)$$

we change from the discrete Eq. (37) to the equation

$$\frac{\partial V}{\partial\tau}=\frac{1+2\nu}{8}\frac{\partial^2 V}{\partial x^2}+\frac{1}{2}\frac{\partial}{\partial x}(xV), \quad (40)$$

which is the same as the Fokker-Planck equation for the Brownian motion of a classical oscillator.[26] The Green function for (40) has the form

$$G(x,x_0;\tau)=\frac{1}{\sqrt{\pi(\nu+1/2)q}}\exp\left\{-\frac{(x-p^{1/2}x_0)^2}{(\nu+1/2)q}\right\}. \quad (41)$$

We note that (41) is a Gaussian distribution in $\sqrt{n}$ rather than in n. The expressions for $\bar{n}$ and $\overline{\Delta n^2}$ following from (41) are the same as (10) only when $\bar{n}_0 p \gg 1 + 2\nu$. We shall also restrict ourselves to that case.

From (39) and (41) we are led to a formula for $w_n(\tau)$, which describes the relaxation of an N-quantum state (cf. (23)):

$$w(n,N;\tau)=\frac{1}{\sqrt{2\pi nq(1+2\nu)}}\exp\left\{-\frac{2(\sqrt{n}-\sqrt{Np})^2}{q(1+2\nu)}\right\},$$
$$\lim_{\tau\to 0} w(n,N;\tau)=\delta(n-N). \quad (42)$$

The populations $w_n(\tau)$ have a maximum at $n = Np$, near which Eq. (42) can be simplified:

$$w(n,N;\tau)=\frac{1}{\sqrt{2\pi Npq(1+2\nu)}}\exp\left\{-\frac{(n-Np)^2}{2Npq(1+2\nu)}\right\}. \quad (43)$$

Let now

$$w_n(0)=\frac{1}{\sqrt{2\pi N\sigma_0^2}}\cdot\exp\left\{-\frac{(n-N)^2}{2N\sigma_0^2}\right\}, \quad (44)$$

where $N \gg 1 + 2\nu$ and $\sigma_0 \sim 1$. From (44) and (43) it follows that the distribution $w_n(\tau)$ remains Gaussian also for $\tau > 0$:

$$w_n(\tau)=\frac{1}{\sqrt{2\pi N\sigma_t^2}}\cdot\exp\left\{-\frac{(n-Np)^2}{2N\sigma_t^2}\right\}\quad \sigma_t^2=p^2\sigma_0^2+pq(1+2\nu). \quad (44a)$$

Figure 1 shows the time-dependence of σ_t. This curve has a maximum when $\sigma_0 < \sqrt{(\nu + 1/2)}$. When $\sigma_0 = 0$, Eq. (44a) changes to (43). The coherent state $|\alpha\rangle$ (or the state (21)) corresponds to $N = |\alpha|^2$, $\sigma_0 = 1$; then $\sigma_t^2 = p(1 + 2q\nu)$.

These formulae are valid under the conditions $\bar{n}(\tau) = Np \gg N^{1/2}\sigma_t \gg 1$, which lead to the inequalities

$$Np \gg 1+2\nu,\qquad N[\sigma_0^2+(1+2\nu)\tau]\gg 1.$$

For an N-quantum state $\sigma_0 = 0$ and this gives $t \gg t_1 = [N\gamma(1 + 2\nu)]^{-1}$. The quantity t_1 is the "mixing time" necessary to change the initial δ-function distribution into a wide distribution to which the Fokker-Planck equation can be applied. The mixing time decreases with increasing excitation N and increasing temperature of the medium.

5. APPLICATION TO THE STATISTICS OF PHOTOCOUNTS

The results obtained above can be used in problems in the statistics of photocounts in optical receivers. We consider the process of the detection of light that is coherent in first order.[4)] The state of the field can then be considered to be one excited mode.[27] The relaxation of this mode when light passes through an absorbing medium situated in front of the detector or in it, is described by Eq. (1). The probability for the emission of k photoelectrons is equal to (see[8], Sec. 6.2):

$$w_k=\sum_{n=k}^{\infty}C_n^k\eta^k(1-\eta)^{n-k}\rho_{nn}, \quad (45)$$

where η is the quantum efficiency $(0 < \eta < 1)$ and ρ_{nn}

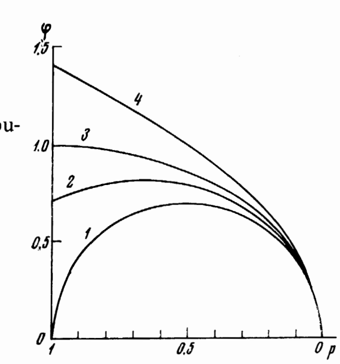

FIG. 1. Change in the quantity $\sigma_t = \sqrt{(\nu + 1/2)}\,\varphi(\tau, s)$ for the distribution (44) during relaxation. The dependence of $\varphi(\tau, s)$ on $p = e^{-\tau}$ is given for the following values of the parameter $s = \sigma_0/\sqrt{(\nu + 1/2)}$: Curve 1: s = 0, curve 2: s = 1/2, curve 3: s = 1, curve 4: s = 2.

2) One can consider the general case $m \neq n$ by the same method.

3) One can solve Eq. (58) for $\nu = 0$ but even in that case the solution has a rather complicated form.[17]

4) Coherence of first order means (according to Glauber[6]) that the correlation function $\langle E^{(-)}(r_1, t_1)E^{(+)}(r_2, t_2)\rangle$, which corresponds to the usual setup of photostatistical experiments (for instance, when one registers photoelectrons emitted over a time smaller than the coherence time of the field and with a cross-section less than the coherence cross-section; for details see [9]), can be factorized.

is the diagonal element of the density matrix for the photon oscillator. This formula connects the quantities ρ_{nn} with the probabilities w_n which are directly measured experimentally.

The transformation (45) can formally be considered to be an additional relaxation of the oscillator in a zero-temperature medium. It follows from Eqs. (45) and (26) that $w_k = \rho_{kk}(t + t_0)$ where t is the moment the field arrives at the photocathode, t_0 follows from the condition $\eta = e^{-\gamma t_0}$ and ν changes discontinuously

$$\nu(t') = \begin{cases} \nu, & 0 \leqslant t' \leqslant t \\ 0, & t \leqslant t' \leqslant t + t_0 \end{cases}. \tag{46}$$

We note in this connection that all results of Secs. 3 and 4 remain valid also in the case where γ and ν vary in time, provided we make the change

$$p \to p(\tau) = e^{-\tau}, \qquad \tau = \int_0^t \gamma(t')\,dt',$$

$$q(\nu + c) \to \int_0^\tau e^{-(\tau-\tau')}\nu(\tau')\,d\tau' + c(1-p) \tag{47}$$

(c is an arbitrary constant). This enables us to evaluate the probabilities w_k directly, using the technique developed in Sec. 3. In particular, the passage of light through a linear absorbing medium at zero temperature does not change its coherence properties. If we form the factorial moments of the photocounts[6]

$$n^{[k]} = \langle n(n-1)\ldots(n-k+1)\rangle = \langle (a^+)^k a^k\rangle \tag{48}$$

the quantities Δ_k which are independent of the light intensity:

$$\Delta_k = \frac{n^{[k]} - (n^{[1]})^k}{(n^{[1]})^k}, \tag{49}$$

then the absorption of the light will not influence the quantities Δ_k, as follows from the solution of the equations for the $n^{[k]}$ (cf. Eq. (28)).

6. RELAXATION OF THE F-DIMENSIONAL OSCILLATOR

The Hamiltonian of an f-dimensional oscillator possesses a "latent" symmetry (group SU(f)), which manifests itself in the strong degeneracy of the excited levels. The energy and degree of degeneracy of the n-th level are equal to

$$E_n = \left(n + \frac{f}{2}\right)\hbar\omega, \qquad D_n^{(f)} = \frac{(n+f-1)!}{n!(f-1)!}. \tag{50}$$

Equation (1) for the evolution of the density matrix can be generalized in an obvious way:

$$\frac{d\rho}{dt} = -\frac{\gamma}{2}\left\{(\nu+1)\sum_{i=1}^{f}(a_i^+a_i\rho - 2a_i\rho a_i^+ + \rho a_i^+a_i)\right.$$
$$\left. + \nu\sum_{i=1}^{f}(a_ia_i^+\rho - 2a_i^+\rho a_i + \rho a_ia_i^+)\right\}. \tag{51}$$

We can define generating functions in analogy with (4). Restricting ourselves for the sake of simplicity to diagonal elements, we have

$$G(z_1,\ldots,z_f,t) = \sum_{n_1,\ldots,n_f=0}^{\infty} \rho_{n_1\ldots n_f,\,n_1\ldots n_f}(t)\, z_1^{n_1}\ldots z_f^{n_f}. \tag{52}$$

The equation for $G(z_1, \ldots, z_f, t)$ is analogous to (5). Its solution has the form

$$G(z_1,\ldots,z_f;t) = \left\{\prod_{i=1}^{f}(1+q\nu\zeta_i)^{-1}\right\} f(\zeta_1',\ldots,\zeta_f'), \tag{53}$$

where

$$\zeta_i' = p\zeta_i/(1+q\nu\zeta_i) \quad (\zeta_i = 1 - z_i),$$
$$f(\zeta_1,\ldots,\zeta_f) = G(1-\zeta_1,\ldots,1-\zeta_f;0).$$

The population w_n of the n-th level is equal to

$$w_n = \sum_{n_1+\ldots+n_f=n} \rho_{n_1\ldots n_f\, n_1\ldots n_f}. \tag{54}$$

Writing down the generating function G_f for the populations, we have from (52) and (54):

$$G_f(z,t) = \sum_{n=0}^{\infty} w_n(t) z^n = G(z,\ldots z;t), \tag{55}$$

and by virtue of this we get from (53)

$$G_f(z,\tau) = (1+q\nu\zeta)^{-f} g\left(\frac{p\zeta}{1+q\nu\zeta}\right), \tag{56}$$

where

$$g(\zeta) = G_f(1-\zeta,0) = \sum_{n=0}^{\infty} w_n(0)(1-\zeta)^n.$$

To determine the evolution of the populations it is thus sufficient to know merely their initial distribution; one requires no more detailed information about the diagonal elements $\rho_{n_1\ldots n_f,\,n_1\ldots n_f}$.

For the particular case when $w_n(0) = \delta_{nN}$ we get

$$G_f(z,\tau) = \frac{[1+(q\nu-p)\zeta]^N}{[1+q\nu\zeta]^{N+f}},$$

whence

$$w_n(\tau) = D_n^{(f)}\frac{1}{(1+q\nu)^f}\left(\frac{q\nu}{1+q\nu}\right)^n\left(\frac{q+q\nu}{1+q\nu}\right)^N P_{n,N}^{(f-1)}\left(\frac{p}{q^2\nu(1+\nu)}\right) \tag{57}$$

(the polynomials $P_{n,N}^{(f-1)}$ are defined in (32)). As $\tau \to \infty$, Eq. (57) changes to the Planck distribution for the f-dimensional oscillator:

$$w_n = D_n^{(f)}(1-\xi)^f\xi^n \equiv \frac{(n+f-1)!}{n!\,(f-1)!}(1-\xi)^f\xi^n \quad (n = 0,1,2,\ldots). \tag{58}$$

Equations (10) for $\bar{n}(\tau)$ and $\overline{\Delta n^2}(\tau)$ change as follows:

$$\bar{n}(\tau) = \bar{n}_0 p + fq\nu,$$

$$\overline{\Delta n^2}(\tau) = \overline{\Delta n_0^2}p^2 + (2\nu+1)(\bar{n}_0 - f\nu)pq + f\nu(\nu+1)(1-p^2). \tag{59}$$

7. OTHER WAYS OF DESCRIBING THE RELAXATION

The interaction of the oscillator with the thermostat and radiation friction, which leads to spontaneous emission, has the nature of a random force. One easily generalizes (1) to the case when a classical (well-determined) force f(t), which arbitrarily depends on the time, is also acting upon the oscillator. The equation for ρ then takes the form

$$\frac{\partial\rho}{\partial t} = -i[V,\rho] - \frac{\gamma}{2}\{(\nu+1)(a^+a\rho - 2a\rho a^+ + \rho a^+a)$$
$$+ \nu(aa^+\rho - 2a^+\rho a + \rho aa^+)\}, \tag{60}$$

where

$$V = -f(t)x = -(2\omega)^{-1/2}\{f^*(t)e^{-i\omega t}a + f(t)e^{i\omega t}a^+\}.$$

Introducing into the discussion the quantity $u(t) = \langle a\rangle$ we find from (60)

$$du/dt = -{}^1/_2\gamma u + i(2\omega)^{-1/2}f(t)e^{i\omega t}; \quad (61a)$$

$$u(t) = u(0)e^{-\gamma t/2} + v(t), \quad (61b)$$

$$v(t) = \frac{i}{\sqrt{2\omega}}\int_0^t f(t')\exp\left[-\frac{\gamma}{2}(t-t') + i\omega t'\right]dt'.$$

For the average values of the operators $\hat{x}$ and $\hat{p}$ we have hence the same expressions as for a classical oscillator with damping:

$$\langle\hat{x}(t)\rangle = \sqrt{2/\omega}\,\mathrm{Re}\,(u(t)e^{-i\omega t}), \quad \langle\hat{p}(t)\rangle = \sqrt{2\omega}\,\mathrm{Im}\,(u(t)e^{-i\omega t}),$$

which is a generalization of the Ehrenfest theorem for the case of a quantum system with dissipation.

The term $-i[V, \rho]$ in (60) leads to a mixing of the different diagonals in the matrix ρ_{mn} and as a result the generating function method is no longer applicable. We turn therefore to other representations for ρ.

The density matrix ρ can be given not only through the matrix elements ρ_{mn} (occupation number representation) but also through characteristic functions and quasi-probability distributions. These representations for ρ turn out to be convenient for a number of problems.[4-6,28] We give some formulae referring to this case.

1. The characteristic functions (normal χ_N, ordinary χ_0, and antinormal χ_A) are determined as follows:[28,29]

$$\chi_N(\eta) = \mathrm{Sp}(\rho e^{\eta a^+}e^{-\eta^* a}), \quad \chi_O(\eta) = \mathrm{Sp}(\rho e^{\eta a^+ - \eta^* a}), \quad \chi_A(\eta) = \mathrm{Sp}(\rho e^{-\eta^* a}e^{\eta a^+}). \quad (62)$$

Then

$$\chi_K(\eta) = \exp(-\sigma_K|\eta|^2)\chi_N(\eta), \qquad \sigma_K = \begin{cases}0 \text{ for } K=N\\ {}^1/_2 \text{ for } K=O\\ 1 \text{ for } K=A.\end{cases} \quad (63)$$

The equation (60) for the functions χ_K has the same form, differing only in the values of the parameter σ_K:

$$\frac{\partial\chi_K}{\partial t} = -\gamma\left[\frac{1}{2}\eta_j\frac{\partial\chi_K}{\partial\eta_j} + (\nu+\sigma_K)|\eta|^2\chi_K\right] - \frac{2i}{\sqrt{2\omega}}\chi_K\,\mathrm{Re}\,[f(t)e^{i\omega t}\eta^*] \quad (64)$$

($j = 1,2$; $\eta = \eta_1 + i\eta_2$). Its general solution has the form

$$\chi_K(\eta,\tau) = \exp\{\eta v^* - \eta^* v - (\nu+\sigma_K)|\eta|^2 q\}\chi_K(p^{1/2}\eta, 0) \quad (65)$$

($v = v(t)$ is defined in (61b)). Louisell[3] already obtained Eq. (65) for χ_0 but not from Eqs. (60) and (64), using instead a model of an oscillator with damping[10] through an approximate integration of the Heisenberg equations of motion for $\hat{a}(t)$. From this it follows that this model is equivalent to the kinetic Eq. (60).

2. The quasi-probability distributions $W_K(\alpha)$ are connected with the characteristic functions $\chi_K(\eta)$ through a Fourier transformation:

$$W_K(\alpha) = \frac{1}{\pi^2}\int\chi_K(\eta)\exp\{\eta^*\alpha - \eta\alpha^*\}\,d^2\eta. \quad (66)$$

Then $W_N(\alpha)$ is the same as the weight functions in the P-representation for the density matrix:

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha, \quad W_N(\alpha) \equiv P(\alpha). \quad (67)$$

Furthermore, $W_0(\alpha) \equiv W(\alpha)$ is the Wigner function[30] ("density" in phase space) and, finally, $W_A(\alpha) \equiv \pi^{-1}Q(\alpha)$ where $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$ (see[6] for a similar discussion of the physical meaning of the functions W, P, and Q).

The quasi-probability distributions $W_K(\alpha)$ satisfy the Fokker-Planck equation:

$$\frac{\partial W_K}{\partial t} = -\frac{\partial}{\partial\alpha_i}(A_i W_K) + \frac{1}{2}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}(B_{ij}W_K); \quad (68)$$

$$A_i = -{}^1/_2\gamma\alpha_i + h_i, \qquad B_{ij} = {}^1/_2\gamma(\nu+\sigma_K)\delta_{ij},$$

$$\alpha = \alpha_1 + i\alpha_2, \quad h = h_1 + ih_2 = i(2\omega)^{-1/2}f(t)e^{i\omega t}. \quad (68a)$$

The solution of Eq. (68) has the form (cf.[4])

$$W_K(\alpha,\tau) = \int d^2\alpha'\,W_K(\alpha',0)\frac{1}{\pi q(\nu+\sigma_K)}\exp\left\{-\frac{|\alpha - \alpha' p^{1/2} - v(t)|^2}{q(\nu+\sigma_K)}\right\} \quad (69)$$

We have given the solutions of Eqs. (60), (64), and (68) for various initial conditions.

We make some remarks about the time evolution of the quasi-probabilities $W_K(\alpha, \tau)$.

1) The complex amplitude in the α-plane undergoes damping proportional to $p^{1/2}$ and is shifted under the action of the external field f(t) by the vector v(t) defined in (61b). This shift is equivalent to a unitary transformation on the density matrix ρ (see[17]). Moreover, when $\nu \neq 0$ fluctuations increase when one takes the interaction with the thermostat into account.

2) As $\tau \to \infty$ we have independently of the initial distribution

$$W_K(\alpha,\tau) = \frac{1}{\pi(\nu+\sigma_K)}\exp\left\{-\frac{|\alpha - v(\tau)|^2}{\nu+\sigma_K}\right\}. \quad (70)$$

3) For any distribution $W_K(\alpha, 0)$ the average $\langle\alpha_i\rangle$ and the dispersion $D_{ij} = \langle(\alpha_i - \langle\alpha_i\rangle)(\alpha_j - \langle\alpha_j\rangle)\rangle$ change as follows:

$$\langle\alpha_i(\tau)\rangle = p^{1/2}\langle\alpha_i(0)\rangle + v(\tau), \quad D_{ij}(\tau) = pD_{ij}(0) + {}^1/_2 q(\nu+\sigma_K)\delta_{ij}. \quad (71)$$

Here $\langle\alpha_1 + i\alpha_2\rangle$ is the same for all K and equal to $\mathrm{Tr}(\rho\hat{a})$ while the dispersions D_{ij} are connected for different K through the relation

$$D_{ij}^{(K)} = D_{ij}^{(N)} + {}^1/_2\sigma_K\delta_{ij}.$$

If the initial distribution $W_K(\alpha, 0)$ is Gaussian it remains so during the relaxation and it is completely determined by Eqs. (71). For an isotropic Gaussian distribution ($D_{ij} \propto \delta_{ij}$) the matrix elements ρ_{mn} are determined by Eq. (34a).

4) The N-quantum state is one of the most anticlassical ones: in the P- and W- planes we have corresponding to it an alternating distribution (the function $Q(\alpha)$ is, of course, positive). The argument of the Laguerre polynomials $L_N(x)$ describing the relaxation of the N-quantum state is equal to (see under A in the table)

$$x = |\alpha - v(\tau)|^2 F_K(\nu,\tau), \qquad F_K(\nu,\tau) = \frac{p}{(q\nu+\sigma_K)(p - q\nu - \sigma_K)} \quad (72)$$

(the form of the function $F_K(\nu, \tau)$ is given in Fig. 2). The function $F_N(\tau)$ has a minimum for $p = \sqrt{\xi}$ ($\tau = {}^1/_2\tau_0$) and becomes infinite for $p = \xi$ ($\tau = \tau_0 = \hbar\omega/kT$) (cf. with (25)). It is well known that all roots of the Laguerre polynomials $L_N(x)$ lie in the region $0 < x \lesssim N$, and for

$\rho_{n+k,n}(t),\ k \geqslant 0$	$W_K(\alpha, t)$
	A.N.-quantum state
Cf (23)	$\dfrac{(q\nu - p + \sigma_K)^N}{\pi(q\nu + \sigma_K)^{N+1}} \exp\left\{-\dfrac{\lvert\alpha\rvert^2}{q\nu + \sigma_K}\right\} L_N(x),$ $x = \lvert\alpha\rvert^2 F_K(\nu, t), \quad F_K$ см. (72)
	B. Superposition of Planck and coherent states
$R_{n+k,n}(p^{1/2}\alpha_0'\ \mu)$	$\dfrac{1}{\pi(\mu + \sigma_K)} \exp\left\{-\dfrac{\lvert\alpha - p^{1/2}\alpha_0\rvert^2}{\mu + \sigma_K}\right\};$
$R_{n+k,n}$ Cf (34 a)	$\mu(t) = p\nu_0 + q\nu$
	C. Superposition of a Planck distribution with a phase-averaged coherent one.
$\delta_{k,0} R_{n+k,n}(p^{1/2}\alpha_0, \mu)$	$\dfrac{I_0\left(\dfrac{2p^{1/2}\lvert\alpha\alpha_0\rvert}{\mu + \sigma_K}\right)}{\pi(\mu + \sigma_K)} \exp\left\{-\dfrac{\lvert\alpha\rvert^2 + p\lvert\alpha_0\rvert^2}{\mu + \sigma_K}\right\}$

Note: The superposition of two states is determined following Glauber (see [6], p. 181). The expressions given here refer to the case $f(t) = 0$. It is clear from (69) that the functions $W_K(\alpha, t)$ can be obtained when there is a force $f(t)$ present from those given here by changing the argument α to $\alpha - \nu(t)$ and the value of $\sigma_{n+k,n}(t)$ under B is obtained by replacing $p^{1/2}\alpha_0$ by $p^{1/2}\alpha_0 + \nu(t)$. We note that the equations under B describe at $\nu_0 = 0$ the relaxation of the coherent state $|\alpha_0\rangle$ and at $\alpha_0 = 0$ the relaxation of the Planck distribution (15).

$-\infty < x < 0$ the polynomials $L_N(x) > 0$. From this it follows that for $0 < \tau < \frac{1}{2}\tau_0$ the circles on which $P(\alpha) = 0$ are widened; when $\frac{1}{2}\tau_0 < \tau < \tau_0$ they contract to a zero radius while for $\tau > \tau_0$ the functions $P(\alpha)$ remain always positive. The circles for the function $W_0(\alpha, \tau)$ on which $W_0(\alpha) = 0$ contract monotonically to a point and when $\tau > \tau_0'$ the distribution $W_0(\alpha)$ becomes positive,

$$\tau_0' = \ln\left(1 + \frac{1}{2\nu + 1}\right) < \frac{\tau_0}{2} = \frac{1}{2}\ln\left(1 + \frac{1}{\nu}\right).$$

We note that as $\tau \to 0$ the function $P(\alpha)$ for the N-quantum state is strongly singular:

$$P(\alpha, \tau = 0) = \frac{N!}{(2N)!}\frac{e^{r^2}}{2\pi r}\left(\frac{\partial}{\partial r}\right)^{2N}\delta(r), \quad r = |\alpha| \tag{73}$$

(see [7]) and the quasi-probabilities $W_0(\alpha)$ and $Q(\alpha)$ have no singularities whatever.

8. THE HARMONIC OSCILLATOR PARADOX

In conclusion we dwell upon the so-called harmonic oscillator paradox.

According to the quantum theory of radiation the line width Γ_{ab} arising when there is a transition from the level a to the level b is under normal conditions equal to $\Gamma_{ab} = \gamma_a + \gamma_b$ where γ_a and γ_b are the widths of

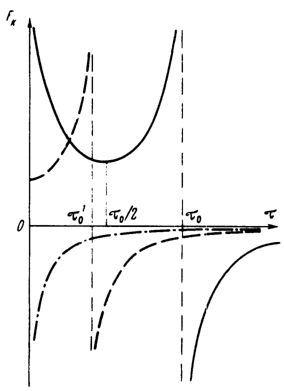

FIG. 2. The functions $F_K(\nu, \tau)$ for K = N (solid curve), K = 0 (dashed curve), and K = A (dash-dotted curve).

these levels.[18,19] For the harmonic oscillator $(x)_{n,n-1} \sim \sqrt{n}$ and therefore in the dipole approximation $\gamma_n = n\gamma$, $\Gamma_{n,n-1} = (2n - 1)\gamma$, where γ is the line width according to classical electrodynamics which is equal to $\gamma = 2e^2\omega_0^2/3mc^3$. In the region $n \gg 1$ where the transition to the classical theory must take place the disagreement with it apparently increases. This is the well-known harmonic oscillator paradox discussed in[20,21] (see also[19], p. 70, and[22] p. 112).

Weisskopf and Wigner[20] have shown from the example of n = 2 that taking the fact that the oscillator levels are equidistant into account leads to the fact that the factor $(2n - 1)$ in $\Gamma_{n,n-1}$ disappears and this leads to agreement with the classical theory. However, Weisskopf and Wigner's method requires the consideration of the whole wave function of the system (oscillator plus radiation field) and is therefore extremely unwieldy for $n \gg 1$. By virtue of this Weisskopf and Wigner[20] restricted themselves to the simplest case n = 2. We show how this paradox can be resolved on the basis of the kinetic Eq. (1) in the general case for any initial state. The spectrum of the quanta emitted by the oscillator during its relaxation is given by the formula

$$\frac{dE}{d\omega} \sim \int_0^\infty dt_1 \int_0^\infty dt_2 \langle a^+(t_2)a(t_1)\rangle e^{i\omega(t_1 - t_2)}$$
$$= 2\,\mathrm{Re}\int_0^\infty dt \int_0^\infty d\tau\, e^{i\omega\tau}\langle a^+(t+\tau)a(t)\rangle. \tag{74}$$

The correlation function $\langle a^+(t + \tau)a(t)\rangle$ is determined by the kinetic Eq. (1). Following the usual method (see, e.g., Sec. 10 in[12]), we get

$$\langle a^+(t+\tau)a(t)\rangle = e^{i\omega_0\tau}\sum_{k=0}^\infty \rho_{kk}(0)\sum_{m=1}^\infty m^{1/2}G(m, m; k, k\,|\,t)$$
$$\times \sum_{n=1}^m n^{1/2}G(n-1, n; m-1, m\,|\,\tau) = \bar{n}(0)\exp\left\{-\left(i\omega_0 + \frac{\gamma}{2}\right)t\right\}$$
$$\times \exp\left\{\left(i\omega_0 - \frac{\gamma}{2}\right)(t + \tau)\right\} \tag{75}$$

(for calculations we used the explicit Eq. (30a) for the

Green function G for $\nu = 0$). Substituting (75) into (74) gives the Lorentz form of the line with the classical width γ which is independent of the initial excitation:

$$\frac{dE}{d\omega} = \bar{n}(0)\frac{\gamma}{2\pi[(\omega-\omega_0)^2+\gamma^2/4]}. \qquad (76)$$

The fact that the spectrum of the oscillator is equidistant leads to the fact that the off-diagonal elements ρ_{mn} are not simply damped as $\sim\exp(-(m+n)\gamma t/2)$ but also change into one another along the diagonal $m - n = \text{const}$. The correlation time is therefore of the order γ^{-1} rather than $(N\gamma)^{-1}$, i.e., the narrowing of the line to the classical limit (76) occurs thanks to the interference of quanta emitted during transitions between different levels.

In conclusion the authors wish to express their deep gratitude to Ya. B. Zel'dovich for drawing our attention to the harmonic oscillator paradox, for his interest in this work, and for discussions of the results, and also to A. L. Golger who took part in the initial stages of this work.

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APPENDIX

We elucidate under what conditions the oscillator remains during relaxation all the time in a pure state. For a pure state $\rho^2 = \rho$, $\mathrm{Tr}\,\rho^2 = 1$; conversely, the condition $\mathrm{Tr}\,\rho^2$ is a criterion for the purity of a state. The difference of $\mathrm{Tr}\,\rho^2$ from unity can serve as a measure for the deviation of a given state from a pure one. Denoting $\mathrm{Tr}\,\rho^2$ by P_2 we have from (1):

$$\frac{dP_2}{dt} = -2\gamma\{\mathrm{Sp}\,A + \nu\,\mathrm{Sp}(BB^+)\} = -2\gamma\{\nu\,\mathrm{Sp}\,\rho^2 + (2\nu+1)\mathrm{Sp}\,A\}, \quad (A.1)$$

where $A = a^+a\rho^2 - a\rho a^+\rho$, $B = [\rho, A]$. Let the oscillator initially be in the pure state $\rho(0) = |\psi\rangle\langle\psi|$. From (A.1) we find

$$\frac{dP_2}{dt}(0) = -2c\gamma, \quad c = \nu + (2\nu+1)\{\langle\varphi|\varphi\rangle - |\langle\psi|\varphi\rangle|^2\} \geqslant \nu, \quad (A.2)$$

where $|\varphi\rangle = a|\psi\rangle$, $\langle\psi|\psi\rangle = 1$. For sufficiently small Δt the quantity $P_2(\Delta t) = 1 - 2c\gamma\Delta t + \ldots < 1$, i.e., the state changes from a pure one to a mixed one.

Examples show (see below) that the change in time of $P_2(t)$ is not always monotonic. However, the oscillator can not return to a pure state (in a finite time interval). Indeed, if $P_2(t_0) = 1$, $P_2(t)$ will have a maximum in that point and $dP_2/dt\,(t_0) = 0$ which contradicts (A.2) (provided $c > 0$). As $t \to \infty$ the state of the oscillator becomes pure, if $\nu = 0$.

We must still consider the case $c = 0$. The equation $c = 0$ is attained only when the condition

$$\nu = 0, \quad |\varphi\rangle \equiv \hat{a}|\psi\rangle = \alpha|\psi\rangle \qquad (A.3)$$

is satisfied (α is an arbitrary complex number), i.e., the initial state $|\psi\rangle$ must be coherent. On the other hand, it was shown in Sec. 3 that a coherent state during relaxation remains all the time coherent when $\nu = 0$. Thus, the relaxation of a coherent state at zero temperature of the thermostat is the only case when the state of a quantum system remains pure, dissipation notwithstanding. In all other cases the oscillator changes to a mixed state at once.

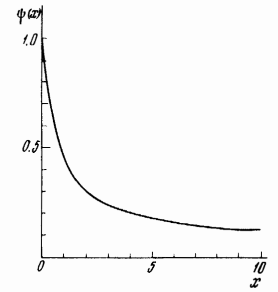

FIG. 3. The function $\psi(x) = e^{-x}I_0(x)$.

We consider a few examples. The quantity P_2 is for the distribution (21) equal to

$$P_2 = e^{-2|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{4n}}{(n!)^2} = \psi(2|\alpha|^2), \qquad (A.4)$$

where $\psi(x) = e^{-x}I_0(x)$ (see Fig. 3). The time-dependence of P_2 for $\nu = 0$ has the form

$$P_2(\tau) = \psi(2p|\alpha|^2), \qquad (A.5)$$

i.e., $P_2(\tau)$ increases monotonically with time. One can show[17] that for the binomial distribution (26) when $N \gg 1$

$$P_2(\tau) \approx \psi(2Npq), \qquad (A.6)$$

i.e., $P_2(\tau)$ initially decreases from unity to a magnitude of the order of $(\pi N)^{-1/2}$ when $\tau = \ln 2$, and then again increases to unity.

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