REFINEMENT OF THE RESULTS OF THE THEORY OF SUSPENSION VISCOSITY

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A systematic error occurring in the calculation of the viscosity of a low-concentration suspension is indicated. It is shown that the correct expression for the effective viscosity of a suspension of small spheres is given by (15), where φ is the small volume concentration of the solid phase.

IN 1906–1911, Finstein[1] (see also [2], Sec. 22), in studying the flow of a suspension of spheres with small volume concentration φ, found an expression for the effective viscosity of the system:

$$\eta = \eta_0(1 + 2{,}5\varphi), \tag{1}$$

where η_0—viscosity of the liquid in which the spheres are suspended. Subsequently the theory was extended to include suspensions of ellipsoidal particles[3,4]. In the present note we indicate, using a suspension of spheres as an example, that the calculations contain a systematic error. The question of the correctness of the obtained results deserves a special discussion, in view of the fact that the results of the theory are widely used to determine the dimensions and shapes of particles suspended in a viscous liquid[5].

Assume, just as in[1,2], that the flow of the incompressible liquid undisturbed by the spheres is described by a velocity distribution $v_i^{(0)} = \nu_{ik}x_k$, where ν_{ik} is a constant symmetrical tensor of the velocity gradients. The perturbation due to the sphere was calculated in the Stokes approximation by Einstein[1,2] under the condition that at large distances from the particle the velocity field approaches asymptotically the unperturbed value. The stresses that cause the asymptotically uniform motion of the liquid, both without the particle and with it, are then different. This is what is meant by the change of the effective viscosity of the system when the spheres are introduced. Thus, the problem reduces to a calculation of the stresses that maintain an asymptotically specified motion of the liquid with the particle. The viscosity of the system must then be defined as the ratio of the calculated stresses to the asymptotically specified velocity gradients.

The expressions for the corrections to the velocity and the pressure, in a coordinate system with origin at the center of the sphere, have for the region $r > R$ the form[1,2]

$$v_i' = \frac{5}{2}\left(\frac{R^5}{r^4} - \frac{R^3}{r^2}\right)\nu_{kl}n_i n_k n_l - \frac{R^5}{r^4}\nu_{ik}n_k, \tag{2}$$

$$p' = -5\eta_0\frac{R^5}{r^3}\nu_{ik}n_i n_k, \tag{3}$$

where R—radius of the sphere, n_i—unit vector in the direction of the radius vector r_i, and $r_k = x_k - x_{\alpha k}$, where $x_{\alpha k}$ is the coordinate of the center of the sphere numbered α. For the region $r < R$, the velocity is $v_i = 0$. From (2) we get the true gradients of the velocity. For the region outside the sphere we have

$$\frac{\partial v_i'}{\partial x_k} = \frac{5}{2}\left(5\frac{R^3}{r^3} - 7\frac{R^5}{r^5}\right)\nu_{jl}n_k n_i n_j n_l + \frac{5}{2}\left(\frac{R^5}{r^5} - \frac{R^3}{r^3}\right) \times (\nu_{jl}\delta_{ik}n_j n_l + \nu_{kl}n_i n_l + \nu_{jk}n_i n_j) + 5\frac{R^5}{r^5}\nu_{ij}n_k n_j - \frac{R^5}{r^5}\nu_{ik}. \tag{4}$$

for the region inside the sphere $\partial v_i / \partial x_k = 0$.

Let us find the average values. Taking into account the smallness of the concentration, we write an expression for the velocity

$$v_i = \nu_{ik}x_k + \sum_\alpha v_i'(x_k - x_{\alpha k}),$$

which is valid everywhere except the regions inside the spheres, where

$$v_i = \nu_{ik}x_{\alpha k}. \tag{6}$$

In calculating the average, we can, bearing in mind the large distance between the spheres, average first over the angles around each sphere, after which the last term in (5) vanishes. Taking into account the smallness of the spheres, we get from (5) and (6)

$$\bar{v}_i = \nu_{ik}x_k. \tag{7}$$

From this we get the tensor of the observed velocity gradients

$$\frac{\partial \bar{v}_i}{\partial x_k} = \nu_{ik}, \tag{8}$$

which differs from the average value of the tensor of the true gradients, defined in analogous fashion with the aid of expression (4):

$$\overline{\frac{\partial v_i}{\partial x_k}} = (1 - \varphi)\nu_{ik}. \tag{9}$$

Following[2], we define the stress tensor in terms of the average value of the momentum-flux tensor, which coincides in the approximation linear in the velocity with the stress tensor

$$\bar{\sigma}_{ik} = \frac{1}{V}\int \sigma_{ik}\,dV. \tag{10}$$

We break up the integration region into two regions (liquid and particle), and for the liquid we have

$$\sigma_{ik} = -p\delta_{ik} + \eta_0\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \tag{11}$$

The integration of the volume of the particle is replaced by integration over the surface:

$$\int_{\Omega} \sigma_{ik}\,dV = \frac{1}{2} \oint (P_i r_k + P_k r_i)\,df. \tag{12}$$

The force acting on the surface of the particle is calculated from the formula

$$P_i = -pn_i + \eta_0 \left(\frac{\partial v_i}{\partial r} + \frac{\partial v_k}{\partial r} \right)_{r=R} n_k. \tag{13}$$

From the last relations, using (3) and (4), we get the stress tensor of the system

$$\overline{\sigma}_{ik} = -p_0 \delta_{ik} + 2\eta_0 (1 + 1.5\varphi) \nu_{ik}. \tag{14}$$

Since ν_{ik} is the tensor of the asymptotically observed velocity gradients, we get from (14) the following expression for the effective viscosity of the suspension:

$$\eta = \eta_0 (1 + 1.5\varphi). \tag{15}$$

If the viscosity is determined from the ratio of the true gradients of the velocity (9) to the average value, as was done by Einstein[1], we get expression (1). However, since the observed velocity gradient is ν_{ik}, it must be recognized that formula (15) is valid. The foregoing calculation of the stress tensor is perfectly equivalent to the calculation of Landau and Lifshitz[2], but they, while assuming ν_{ik} as the observed gradient, did not obtain the expression (16) only because they assumed without justification, at the end of their calculations, that $\overline{\partial v_i / \partial x_k} = \nu_{ik}$, in lieu of the correct expression (9).

The results of the theory of the viscosity of a suspension of ellipsoidal particles[3,4] are likewise not accurate: the initial effective relative viscosity of the system is overestimated by a factor φ. However, unlike Einstein, Jeffrey[3] and Kuhn[4] assumed the unperturbed velocity gradient as the observed one. In this case the discrepancy is due to the careless regularization of the divergent expressions for the energy dissipation.

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