STRONG FLUCTUATIONS OF A PLANE LIGHT WAVE MOVING IN A MEDIUM WITH WEAK RANDOM INHOMOGENEITIES

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The problem of the strong fluctuations of the amplitude of a plane light wave moving in a randomly inhomogeneous medium is considered. It is assumed that fluctuations of the directions of propagation of the wave are small. Two methods of solution of the problem are developed: perturbation theory and integration of the law of conservation of energy along the random ray trajectories. Comparison of these methods makes it possible to get a series of exact relations for the averaged quantities. It is shown, in particular, that the mean value for the amplitude level can be expressed in terms of the mean values of quantities along the trajectories and in terms of the mean values of some functions of the trajectories. In the approximation of geometric optics, for propagation of a wave in a turbulent medium, the asymptotic value of the mean value of the amplitude level in the large fluctuation region is obtained. The behavior of the probability density of the amplitude level in the large fluctuations region is discussed.

IN a study of the propagation of light in a turbulent medium,[1,2] a region of strong fluctuations was discovered experimentally. In this region, with increase in the distance traveled by the light wave in the inhomogeneous medium, the mean square of the fluctuations of the logarithm of the amplitude ceases to increase and remains approximately constant (of the order of unity). In the region of strong fluctuations, it is inconvenient to use calculations based on some form of perturbation theory. Experiments were undertaken[3-6], going beyond the framework of perturbation theory, and results were obtained which are valid even in the case of strong fluctuations, by means of the use of mathematical methods developed in quantum field theory. However, being applicable to the wave equation for the field of the light wave, these methods were not fully successful. The comparison given in[7] of experimentally measured values of the mean square of the value of the fluctuations of the angle of arrival of the light wave in a turbulent medium with values calculated on the basis of perturbation theory has shown that, at large distances between the source and the point of observation (and precisely then the fluctuations of the amplitude are no longer described by perturbation theory), the dispersion of the angle of arrival of the light wave is satisfactorily described by the expression computed from perturbation theory. In view of this fact, it is necessary to investigate separately the fluctuations of amplitude and phase of the light.

A method has been set forth[8,9] for the calculation of strong fluctuations of the amplitude, based on the approximation of geometric optics, using the integration of the law of conservation of energy along random ray trajectories. A Gaussian approximation was used in[8] for the trajectory, and the result was obtained that in the case of very strong fluctuations, the mean value of the logarithm of the amplitude (amplitude level) saturates. In the present paper, a comparison is given of the method of integration over the random ray trajectories and perturbation theory, which allows us to obtain some exact relations for the averaged values and to improve considerably the results obtained in[8].

We note that all the results obtained in the study of light propagation in random media are easily extended to the case of sound propagation.[10]

1. FUNDAMENTAL EQUATIONS

Let us consider the problem of the propagation of a plane light wave with amplitude A_0 and wave vector k in a homogeneous and isotropic medium, the dielectric constant of which fluctuates weakly relative to its mean value. The electromagnetic field in the case in which the wavelength is short in comparison with the scale of the inhomogeneity can be described in the scalar approximation by the function $\varphi(\mathbf{r}) = \chi + iS$, where $\chi = \ln(A/A_0)$ is the amplitude level, S the departure of the phase from the phase of the plane wave (which is equal to kx) and the distance x is measured in the direction of the initial motion of the plane wave. If we neglect effects connected with large angle light scattering, then, as is well known,[10] the function $\varphi(\mathbf{r})$ satisfies the equation

$$\Delta_\perp\varphi + 2ik\partial\varphi/\partial x + (\nabla_\perp\varphi)^2 + k^2\varepsilon = 0, \tag{1.1}$$

which must be solved with the boundary condition $\varphi(0, \boldsymbol{\rho}) = 0$. Here $\boldsymbol{\rho}$ are the transverse coordinates, the index $\perp$ denotes derivatives along the transverse coordinates, and ϵ denotes the departure of the values of the dielectric constant from their mean value; this departure we shall assume to be homogeneous and isotropic.

Separating out the imaginary part in (1.1), we get the equation for $\chi(\mathbf{r})$:

$$2k\frac{\partial\chi}{\partial x} + 2\nabla_\perp S\cdot\nabla_\perp\chi + \Delta_\perp S = 0. \tag{1.2}$$

For small fluctuations in the direction of propagation of the wave, we can use an approximation which consists in the replacement of the total phase of the wave $S(x, \rho)$ by $S^0(x, \rho)$, which is the solution of the linearized equation (1.1) in the method of smooth perturbations (MSP). Thus the equation

$$2k\partial\chi / \partial x + 2\nabla_\perp\chi \cdot \nabla_\perp S^0 + \Delta_\perp S^0 = 0 \tag{1.3}$$

is basic in the problem under consideration.

It is convenient to transform to Fourier components all quantities in the transverse coordinates; namely, we set

$$\varphi(x, \rho) = \int \varphi_{\mathbf{p}}(x) e^{i\mathbf{p}\rho} d\mathbf{p}, \qquad \varphi_{\mathbf{p}}(x) = \frac{1}{(2\pi)^2} \int \varphi(x, \rho) e^{-i\mathbf{p}\rho} d\rho. \tag{1.4}$$

Then Eq. (1.3) can be rewritten in the form

$$\chi_{\mathbf{p}}(x) = \frac{p^2}{2k} \int_0 d\xi S_{\mathbf{p}}{}^0(\xi)$$
$$+ \frac{1}{k} \int_0^x d\xi \int d\mathbf{q}_1 d\mathbf{q}_2 \, \mathbf{q}_1\mathbf{q}_2 \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}) S_{\mathbf{q}_1}{}^0(\xi) \chi_{\mathbf{q}_2}(\xi), \tag{1.5}$$

where the phase $S^0_{\mathbf{p}}(x)$ in the MSP is given by the expression

$$S_{\mathbf{p}}{}^0(x) = \frac{k}{2} \int_0^x d\xi \, \varepsilon_{\mathbf{p}}(\xi) \cos\left[\frac{p^2}{2k}(\xi - x)\right], \tag{1.6}$$

which can be rewritten in the form

$$S_{\mathbf{p}}{}^0(x) = \frac{k}{2} \int_0^x d\xi \, \varepsilon_{\mathbf{p}}(\xi). \tag{1.6'}$$

if the geometric approximation is valid. Equations (1.3) or (1.5) express the law of energy conservation.

Equation (1.3) can be rewritten in the form of an integral over some trajectory. We introduce the vector $\mathbf{a} = \{k, \nabla_\perp S^0\}$. We then obtain

$$\mathbf{a}\nabla\chi = -{}^1\!/_2\Delta_\perp S^0(x, \rho). \tag{1.7}$$

Introducing the unit vector $\mathbf{l} = \mathbf{a}/a$, directed along the ray arriving at the point (x, ρ), we rewrite Eq. (1.3) in the following form:

$$\frac{d}{dl}\chi(x, \rho) = -\frac{1}{2a}\Delta_\perp S^0(x, \rho). \tag{1.3'}$$

The differential equation of the ray has the form

$$d\mathbf{r} / dl = \mathbf{a} / a. \tag{1.8}$$

Taking the x-component of Eq. (1.8), we get $dx/dl = k/a$, or $dl = (a/k)dx$.

Similarly, for the transverse components, we have $d\mathbf{R}/dl = \nabla_\perp S^0/a$ or

$$d\mathbf{R} / dx = k^{-1}\nabla_\perp S^0. \tag{1.9}$$

Let $\mathbf{R} = \mathbf{R}(x, \xi; \rho)$ be the equation of the ray (running coordinate ξ), satisfying the condition $\mathbf{R}(x, x; \rho) = \rho$. Integrating (1.3′) and (1.9) with the conditions $\chi(0, \rho) = 0$, $\mathbf{R}(x, x; \rho) = \rho$, we get

$$\chi(x, \rho) = -\frac{1}{2k} \int_0 d\xi \Delta_\perp S^0(\xi, \mathbf{R}(x, \xi; \rho)),$$
$$\mathbf{R}(x, \xi; \rho) = \rho - \frac{1}{k} \int_\xi^x d\eta \nabla_\perp S^0(\eta, \mathbf{R}(x, \eta; \rho)). \tag{1.10}$$

Equation (1.3) can also be written in the form

$$\operatorname{div}(A^2\mathbf{a}) = 0, \tag{1.11}$$

where A is the amplitude of the light wave. Assuming that we are interested only in the mean value and that $\langle A^2\mathbf{a}\rangle = f(x)$ by virtue of the inhomogeneity along the transverse coordinates, we get the relation $\partial\langle A^2\rangle/\partial x = 0$ or

$$\langle A^2\rangle = A_0{}^2, \tag{1.12}$$

which expresses the law of energy conservation in the approximation considered. Turning from (1.12) to the amplitude level, we find

$$\langle\exp\{2\chi(x)\}\rangle = 1. \tag{1.13}$$

We note that if χ is a Gaussian random variable, it then follows from (1.13) that

$$\langle\chi(x)\rangle = -\sigma_\chi{}^2, \tag{1.14}$$

where σ^2_χ is the variance of fluctuations of the amplitude level and $\chi(x) = \chi(x, 0)$. The ray defined by Eq. (1.1), in the presence of diffraction, differs from the ray constructed in the approximation of geometric optics, since $S^0(r)$ differs from the phase in the geometric approximation. However, as follows from (1.11), the energy is propagated along the rays constructed by means of Eq. (1.10), since it follows from (1.11) that the energy flux density is conserved along the ray tube defined by the relation (1.10). In what follows, we shall always mean by the word ray the ray in the sense of (1.10).

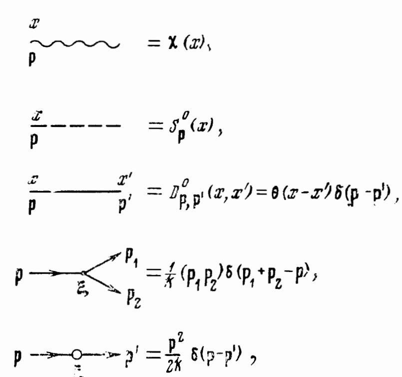
In the approximation at hand, when the phase S^0 is assumed to be given and the probability of infinite values of $\nabla_\perp S^0$ is equal to zero (we shall assume the distribution of $\nabla_\perp S^0$ to be Gaussian; see below), the equation of the ray arriving at a given point has a unique solution. It then follows that there is the relation

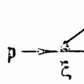
$$\mathbf{R}(\xi_1, \xi; \mathbf{R}(x, \xi_1; \rho)) \equiv \mathbf{R}(x, \xi; \rho), \tag{1.15}$$

the meaning of which is that we must take as a boundary condition for the ray the condition $\mathbf{R} = \mathbf{R}(x, \xi_1; \rho)$ for $\xi = \xi_1$, in addition to the condition $\mathbf{R} = \rho$ for $\xi = x$.

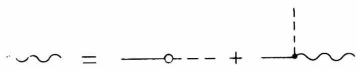
2. PERTURBATION THEORY

We consider Eq. (1.5). The solution of this equation can be represented in the form of a series obtained by its iteration. It is more convenient to develop the perturbation theory, and also to represent the resultant series graphically, placing some analytic expression in correspondence with each diagram, just as is done in quantum field theory. We introduce the notation shown below:



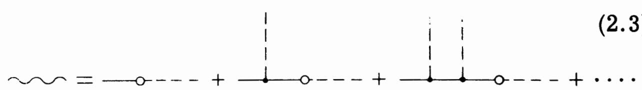
 $=\Lambda^{p}_{p_1,p_2}=\frac{1}{2k}[p_1(i\lambda p_1+2p_2)]\delta(p_1+p_2-p)$ (2.1)

Then we can describe Eq. (1.5) graphically in the form

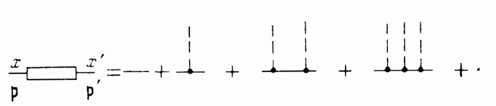
 (2.2)

where each element of the graph corresponds to an analytic expression from (2.1), at each vertex the law of conservation of wave numbers is satisfied along all wave numbers converging on the vertex, integration is carried out over the entire space while integration over the running coordinate ξ is carried out in the limits from 0 to ∞.

Carrying out the successive iterations, we get the solution of the equation in the form of a series:

 (2.3)

If we introduce, in place of the line D^0, its generalization D, namely as the sum of graphs having two free ends:

 (2.4)

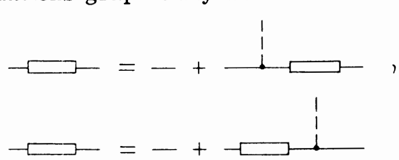
then (2.3) can be represented in the form

 (2.5)

According to the general rules, one can write down the analytic expression corresponding to the graph (2.5):

$$\chi_{\mathbf{p}}(x)=\frac{1}{2k^2}\int_0^\infty d\xi\int d\mathbf{q}\, q^2 S_{\mathbf{q}}{}^0(\xi)D_{\mathbf{p},\mathbf{q}}(x,\xi). \tag{2.5'}$$

It is easy to see from the graph (2.4) that equations are satisfied by the function $D_{\mathbf{p},\mathbf{p}'}(x, x')$, namely, representing these equations graphically we obtain

 (2.6)

The corresponding equations in analytic form are

$$D_{\mathbf{p},\mathbf{p}'}(x,x')=\overset{0}{D}_{\mathbf{p},\mathbf{p}'}(x,x')$$
$$+\frac{1}{k}\int_0^x d\xi\int d\mathbf{q}_1\, d\mathbf{q}_2\, \mathbf{q}_1\mathbf{q}_2\delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{p})S_{\mathbf{q}_1}{}^0(\xi)D_{\mathbf{q}_2,\mathbf{p}'}(\xi,x'),$$
$$D_{\mathbf{p},\mathbf{p}'}(x,x')=\overset{0}{D}_{\mathbf{p},\mathbf{p}'}(x,x')$$
$$+\frac{1}{k}\int_{x'}^\infty d\xi\int d\mathbf{q}_1\, d\mathbf{q}_2\, \mathbf{q}_2\mathbf{p}'\delta(\mathbf{q}_2+\mathbf{p}'-\mathbf{q}_1)S_{\mathbf{q}_2}{}^0(\xi)D_{\mathbf{p},\mathbf{q}_1}(x,\xi). \tag{2.6'}$$

Comparing (2.5′) and (1.10), we get the connection of the function D with the trajectories of the ray:

$$D_{\rho,\rho'}(x,x')=4\pi^2\theta(x-x')\delta(\rho'+\mathbf{R}(x,x';\rho)). \tag{2.7}$$

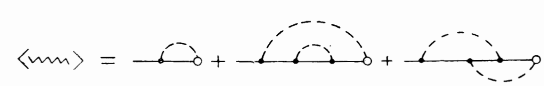
Taking it into account that **R** is a random trajectory, we see that ⟨D⟩ is the probability density for R. Completing the Fourier transformation with respect to ρ and ρ', we have

$$D_{\mathbf{p},\mathbf{p}'}(x,x')=\frac{1}{4\pi^2}\theta(x-x')\int\exp\{i[\mathbf{p}'\mathbf{R}(x,x';\rho)-\mathbf{p}\rho]\}\,d\rho. \tag{2.8}$$

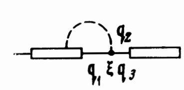
3. CALCULATION OF ⟨χ⟩

We now consider the diagram (2.3) for $\chi_p(x)$. We shall assume that the field of the random variable $\epsilon_p(x)$—the dielectric constant—is Gaussian and hence the field of $S_p^0(x)$ is also Gaussian (generally speaking, only the Gaussian nature of the field $S_p^0(x)$ is necessary which, by virtue of the central limit theorem, can take place even in the case in which the field $\epsilon_p(x)$ is not Gaussian).

We consider $\langle\chi_p(x)\rangle$. For averaging, all diagrams which have an odd number of vertices (odd number of lines S^0) vanish, while diagrams having an even number of vertices (lines S^0) decomposte into all possible sums of diagrams in which S^0 is pairwise connected. As a result, we have the following series:

 (3.1)

The rules of correspondence for each diagram of an analytic expression are obvious. Because of the homogeneity in the transverse coordinates, it is evident that $\langle S_p^0(x_1)S_q^0(x_2)\rangle = \Phi_p^{S^0}(x_1, x_2)\delta(\mathbf{p}+\mathbf{q})$ and $\langle\chi_p(x)\rangle = \langle\chi(x)\rangle\delta(\mathbf{p})$, where $\langle\chi(x)\rangle = \langle\chi(x, 0)\rangle$. Diagrams are omitted from the series (3.1) consisting of several parts connected only by the lines D^0 (weakly connected diagrams). All diagrams of such a type are equal to zero. Actually, we consider the diagram of the form

 (3.2)

where by the rectangle is indicated some closed part of the diagram. The factor $\mathbf{q}_2\mathbf{q}_3\delta(\mathbf{q}_2+\mathbf{q}_3-\mathbf{q}_1)$ will stand in the vertex of ξ, but, because of the spatial homogeneity in the transverse coordinates, the right rectangle will be equal to $f(\xi)\delta(\mathbf{q}_3)$ and consequently, the entire diagram (3.2) will be identically equal to zero.

One can show that in the case of the applicability of the geometric optics approximation, we can use the expression

$$\Phi_{\mathbf{p}}{}^{S^0}(x_1,x_2)={}^1\!/_2k^2\pi\Phi_{\mathbf{p}}{}^{\epsilon}\min\{x_1,x_2\}, \tag{3.3}$$

for $\Phi_p^{S^0}(x_1, x_2)$, where Φ_p^{ϵ} is a three-dimensional spectrum of the field ϵ of the two-dimensional vector **p**. We are interested in the passage of light through a turbulent medium. In this case, we can assume that[10]

$$\Phi_{\mathbf{p}}{}^{\epsilon}=AC_\epsilon{}^2p^{-11/3}\exp(-p^2/\varkappa_m{}^2), \tag{3.4}$$

where A and C_ϵ^2 are constants and κ_m characterizes the internal scale of the turbulence. An expression of the type (3.3) can easily be obtained even in the presence of diffraction effects, but it appears to be very cumbersome in that case.

Let us consider the first diagram on the right side of

(3.1). We get for it

$$\langle\chi_1(x)\rangle = -\frac{1}{2k^2}\int_0^x d\xi_1\int_0^{\xi_1} d\xi_2\int d\mathbf{p}\, p^4\Phi_{\mathbf{p}}{}^{S^0}(\xi_1,\xi_2). \qquad (3.5)$$

Substituting the value of $\Phi_{\mathbf{p}}^{S^0}$ from (3.3) and (3.4) in (3.5), we get, after integration,

$$\langle\chi_1(x)\rangle = -\sigma_0^2, \qquad (3.6)$$

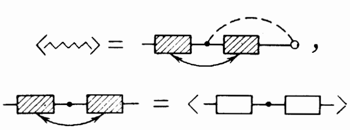
where

$$\sigma_0^2 = {}^1/_{24}\pi^2\Gamma({}^7/_6)AC_\varepsilon^2 x^3\varkappa_m{}^{7/3} \qquad (3.7)$$

is the variance of the amplitude level in the MSP in the approximation of geometric optics. It is clear from dimensional considerations that the diagrams which contain four vertices will be proportional to σ_0^4, those which contain six vertices will be proportional to σ_0^6 and so on. Consequently, the parameter by which it can be decided whether the level fluctuations are strong or weak is precisely σ_0^2. In the case $\sigma_0^2 \ll 1$, the fluctuations are small and the MSP is valid. In the case $\sigma_0^2 \gtrsim 1$, the fluctuations are not small and we must use, say for $\langle\chi\rangle$, the complete series (3.1). A similar picture is also valid for any moments of the function χ. Consequently, we can conclude that

$$\langle\chi^k(x)\rangle = f_k(\sigma_0^2).$$

It is seen from (3.1) that there is a special vertex which describes the correlation of the transferable quantity $\Delta_\perp S^0$ with the trajectory of the ray defined by $\nabla_\perp S^0$. Separating out these means, we get the diagram



(3.8)

for $\langle\chi\rangle$. To it corresponds the analytic expression

$$\langle\chi_{\mathbf{p}}(x)\rangle = -\frac{1}{2k^2}\iint_0^\infty d\xi_1\, d\xi_2\int d\boldsymbol{\varkappa}\, d\mathbf{q}_1\, d\mathbf{q}_2\varkappa^2(\boldsymbol{\varkappa}\mathbf{q}_2)\,\Phi_{\varkappa}^{S^0}(\xi_1,\xi_2)\cdot$$
$$\cdot\,\delta(\mathbf{q}_2-\boldsymbol{\varkappa}-\mathbf{q}_1)\,\langle D_{\mathbf{p},\mathbf{q}_1}(x,\xi_1)\,D_{\mathbf{q}_2,\boldsymbol{\varkappa}}(\xi_1,\xi_2)\rangle. \qquad (3.8')$$

Using for D expressions in terms of R from (2.8), carrying out the integrations, and using (1.15) we can obtain the formula

$$\langle\chi(x)\rangle = -\frac{1}{2k^2}\int_0^x d\xi_1\int_0^{\xi_1} d\xi_2\int d\boldsymbol{\varkappa}\,\varkappa^4\Phi_{\varkappa}^{S^0}(\xi_1,\xi_2)\cdot\langle\exp\{-i\boldsymbol{\varkappa}[\mathbf{R}(x,\xi_1;0)$$
$$-\mathbf{R}(x,\xi_2;0)]\}\rangle + \frac{1}{2k^2}\int_0^x d\xi_1\int_0^{\xi_1} d\xi_2\int d\boldsymbol{\varkappa}\,\varkappa^2\Phi_{\varkappa}^{S^0}(\xi_1,\xi_2)\int d\boldsymbol{\rho}\,\langle\delta[\boldsymbol{\rho}-\mathbf{R}(x,\xi_1;0)$$
$$\times\,\boldsymbol{\varkappa}\frac{\partial}{\partial\boldsymbol{\rho}}[\boldsymbol{\varkappa}(\boldsymbol{\rho}-\mathbf{R}(\xi_1,\xi_2;\boldsymbol{\rho}))]\exp\{-i\boldsymbol{\varkappa}[\mathbf{R}(x,\xi_1;0)-\mathbf{R}(x,\xi_2;0)]\}\rangle. \qquad (3.9)$$

We note that the mean value in the first integral in (3.9) is the characteristic function for the increase in the transverse displacements of the ray. If we now return to the expression for $\chi(x,\rho)$ as an integral over the random trajectories, then we see that $\langle\chi\rangle$ is expressed in terms of the mean values of the same transferable quantities and in terms of the mean values of some functions of the trajectories. In the region of strong fluctuations, as simple estimates show, the mean square of the displacement of the ray becomes large in comparison with the dimensions of the inhomogeneities which make the principal contribution to the fluctuations. This means that for $\xi_1 \neq \xi_2$ the difference $\mathbf{R}(c, \xi_1; 0) - \mathbf{R}(x, \xi_2; 0)$ in the exponent takes on large values and after averaging, the quantity

$$\langle\exp\{-i\boldsymbol{\varkappa}[\mathbf{R}(x,\xi_1;0)-\mathbf{R}(x,\xi_2;0)]\}\rangle$$

is seen to be small—the smaller the larger the value of σ_0. It then follows that the principal contribution to the integral is made by the region $\xi_1 \approx \xi_2$. If we consider the exponent for a small fixed value of $\xi_1 - \xi_2$, it should be maximum for $\xi_1 \approx \xi_2 \approx x$, since the ray, because of the boundary conditions, reaches the point of observation and near this point the difference $\mathbf{R}(x, \xi_1; 0) - \mathbf{R}(x, \xi_2; 0)$ is minimal.

On the basis of the considerations given, it is clear that the principal contribution to the integral is made by the region $\xi_1 \approx \xi_2 \approx x$. Therefore, we can expand $\mathbf{R}(x, \xi; \rho)$ in a series in ρ in the vicinity of the point of observation and limit ourselves to the first two terms (linear approximation), i.e., represent $\mathbf{R}$ in the form

$$\mathbf{R}(x,\xi;\boldsymbol{\rho}) = \boldsymbol{\rho} - \frac{i}{k}\int_\xi^x d\eta\int d\mathbf{q}\,\mathbf{q}S_{\mathbf{q}}{}^0(\eta)\,e^{i\mathbf{q}\boldsymbol{\rho}}. \qquad (3.10)$$

In this approximation the trajectory of the ray is a Gaussian random function. Substituting (3.10) in (3.9), we can complete the averaging with the help of the formula

$$\langle e^z\rangle = \exp(\langle z^2\rangle/2), \qquad \langle z_1e^z\rangle = \langle z_1z\rangle\exp(\langle z^2\rangle/2),$$

where z and z_1 are random Gaussian quantities with mean values equal to zero. By carrying over the averaging and integration, after transition to dimensionless quantities, we obtain, in the geometric approximation,

$$-\langle\chi\rangle = I_1 - I_2 - I_3,$$

$$I_1 = \frac{12\sigma_0^2}{\Gamma({}^7/_6)}\int_0^1 d\xi_1\int_0^{\xi_1} d\xi_2\,\xi_2\int d\boldsymbol{\varkappa}\,\varkappa^5\Phi_\varkappa{}^\varepsilon\exp\{-6\sigma_0^2\varkappa^2(\xi_1-\xi_2)^2(\xi_1+2\xi_2)\},$$

$$I_2 = \frac{48\sigma_0^4}{\pi[\Gamma({}^7/_6)]^2}\int_0^1 d\xi_1\int_0^{\xi_1} d\xi_2\,\xi_2(\xi_1-\xi_2)^2(\xi_1+2\xi_2)\int_0^\infty d\varkappa\,\varkappa^3\Phi_\varkappa{}^\varepsilon$$
$$\times\int d\mathbf{p}\,(\boldsymbol{\varkappa}\mathbf{p})^3\Phi_{\mathbf{p}}{}^\varepsilon\psi_{\mathbf{p}}(\xi_1,\xi_2),$$

$$I_3 = \frac{72\sigma_0^4}{\pi[\Gamma({}^7/_6)]^2}\int_0^1 d\xi_1\int_0^{\xi_1} d\xi_2\,\xi_2(1-\xi_1)(\xi_1{}^2-\xi_2{}^2)\int_0^\infty d\varkappa$$
$$\times\,{}^3\Phi_\varkappa{}^\varepsilon\int d\mathbf{p}\,(\boldsymbol{\varkappa}\mathbf{p})^2p^2\Phi_{\mathbf{p}}{}^\varepsilon\psi_{\mathbf{p}}(\xi_1,\xi_2),$$

$$\psi_{\mathbf{p}}(\xi_1,\xi_2) = \exp\{-6\sigma_0^2\varkappa^2(\xi_1-\xi_2)^2(\xi_1+2\xi_2) -$$
$$-6\sigma_0^2p^2(1-\xi_1)^2(1+2\xi_1) - 18\sigma_0^2\boldsymbol{\varkappa}\boldsymbol{\rho}(1-\xi_1)(\xi_1{}^2-\xi_2{}^2)\}, \qquad (3.11)$$

where σ_0^2 is the dispersion of the amplitude level in the MSP. The expression (3.11) actually shows that as $\sigma \to \infty$ only the region $1 - \xi_1 \approx \xi_1 - \xi_2 \approx 1/\sigma_0$ is important, in which the expansion (3.10) is valid. In the case of weak fluctuations $\sigma_0 \ll 1$ and $-\langle\chi\rangle \approx \sigma_0^2$. In the case of strong fluctuations, $\sigma_0 \gg 1$ and

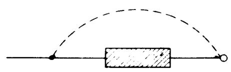
$$I_1 \approx 2\sqrt{2\pi}\,\frac{\Gamma({}^2/_3)}{\Gamma({}^1/_6)}\,\sigma_0 = 1.22\sigma_0, \qquad I_2 + I_3 \approx 1. \qquad (3.12)$$

Thus, in the case of strong fluctuations, the asymptote of $\langle\chi\rangle$ is described by the formula (3.12) and we see that saturation does not take place.

We note that in the linear approximation for **R**

$$\mathbf{R}(x, \xi_1; 0) - \mathbf{R}(x, \xi_2; 0) = \mathbf{R}(\xi_1, \xi_2; 0) \tag{3.13}$$

and the asymptote of $\langle\chi\rangle$ is described by the asymptotic diagram



(3.14)

where we should assume that the wave number of the line $\langle S^0 S^0\rangle$, which spans the entire diagram, is much larger than all the internal wave numbers. The linear approximation for **R** in this case is seen to be equivalent to the situation that for the diagram (2.14) the wave numbers **q** of $S^0_{q_i}$ are much smaller than the wave numbers of the free ends. In this case, the approximation of (3.10) for **R** gives the following expression for D:

$$D_{\mathbf{p},\mathbf{p}'}(x, x') = \frac{1}{4\pi^2}\theta(x-x')\int d\rho \exp\Big\{-i(\mathbf{p}-\mathbf{p}')\rho + \frac{1}{k}\int_{x'}^{x} d\xi \int d\varkappa\,\mathbf{p}'\varkappa S_\varkappa^0(\xi)\,e^{i\varkappa\rho}\Big\}. \tag{3.15}$$

The diagram (3.14) with the approximation (3.15) used for D indicates the summation of several diagrams of the exact solution for $\langle\chi\rangle$. Direct comparison of the expansion of $\langle\chi\rangle$ in a series in σ_0^2 with the terms considered in (3.14) shows that in terms of order σ_0^4 we make an error of 12%, in terms of order σ_0^6, an error of 21%; beyond that, the error decreases, approaching zero as the power of σ_0 goes to infinity. A linear approximation was also used in[8] for the rays but, as a preliminary, the correlation of the transferable quantities was not separated. In view of this, the linear approximation for rays means there assigning an expression obtained from (3.14) to all diagrams (including the weakly coupled, which are equal to zero). It is therefore clear that errors in each order in σ_0 amount to hundreds and thousands of percent; they are the greater the higher the power of σ_0, as a consequence of which the asymptote obtained in[8] for $\langle\chi\rangle$ is not correct.

4. VARIANCE OF THE AMPLITUDE LEVEL

Let us consider the expression for $\langle\chi^2\rangle$. Here we must introduce the series (2.5), square it and take the average. In averaging, diagrams of two types appear—connected and non-connected—which decompose into two parts:

$$\langle\chi^2(x)\rangle = \langle\chi^2(x)\rangle_{\text{cn}} + \langle\chi^2(x)\rangle_{\text{unc}}$$

It is obvious that $\langle\chi^2(x)\rangle_{\text{unc}} = \langle\chi(x)\rangle^2$ and consequently,

$$\sigma_\chi^2 = \langle\chi^2(x)\rangle_{\text{con}} \tag{4.1}$$

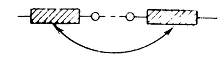
In the case $\sigma_0 \ll 1$ the variance of the amplitude level will be described by the following diagram

(4.2)

calculation of which gives $\sigma_\chi^2 \approx \sigma_0^2$. For arbitrary values of σ_0, we must carry out an analysis similar to that made for $\langle\chi\rangle$. However, we shall not do this, in view of the extremely cumbersome nature of the results obtained. We note that one can represent σ_χ^2 in the form

$$\sigma_\chi^2 = I_1 + I_4,$$

where I_1 is pictured by the diagram



(4.3)

and is numerically equal to the first integral in (3.9). The term I_4 will be determined by expressions of the type of the first integral in (3.9). Evidently, the asymptote of σ_χ^2 will also be determined by the asymptote of I_1, although verification of this is made difficult by the extreme tediousness of the computations. However, the situation in which the asymptote of σ_χ^2 will be determined by the vicinity of the point of observation and the linear approximation for the ray **R** can be used in the case $\sigma_0 \gg 1$ remains in force, since one can, in this case, repeat the same discussions which were made in the derivation of the formula (3.8). Moreover, one can expect that the correlation of the quantity $\Delta_\perp S^0$, taken at one point, with the functionals of the trajectory, determined by the distribution of S^0 in a large region of space, will be small, so that the term I_4 corresponding to account of this correlation will be small in comparison with I_1, and therefore the relation

$$\sigma_\chi^2 = -\langle\chi\rangle. \tag{4.4}$$

will be satisfied approximately. The same relation is also valid for $\sigma_0 \ll 1$. However, it is clearly impossible to regard (4.4) as an interpolation formula for all values of σ_0.

5. THE CUMULANTS OF THE AMPLITUDE LEVEL

We now proceed to consideration of the higher moments of the amplitude level. The probability distribution of the random variable χ is described by the characteristic function

$$\varphi_\lambda(x) = \langle\exp\{i\lambda\chi(x)\}\rangle.$$

The moments $m_n = \langle\chi^n\rangle$ and the cumulants K_n of the field χ are connected with $\varphi_\lambda(x)$ by the relations

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty}\frac{(i\lambda)^n}{n!}m_n = \exp\Big\{\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{n!}K_n\Big\}. \tag{5.1}$$

The equation (1.13) imposes definite conditions on the cumulant $K_n(x)$ of the field χ; that is, (1.13) is equivalent to the relation

$$\sum_{n=1}^{\infty}\frac{2^n}{n!}K_n(x) = 0, \tag{5.2}$$

which follows from (1.13) and (5.1) for $\lambda = 2/i$.

It was shown above that in the case of the propagation of a wave in a turbulent medium, all the one-point moments of the field χ are functions of the parameter σ_0^2 in the geometric approximation. Consequently, in this case, $K_n(x) \equiv K_n(\sigma_0^2)$ and Eq. (5.2) should be satisfied for any order in σ_0^2. We consider the mean $\langle\chi^n(x)\rangle$. To calculate this mean value, we must multiply n of the series (2.5) and average over all realizations of the field ϵ. In the averaging, we obtain a series consisting of diagrams which must be separated into two classes—connected and non-connected, which decompose into several connected pieces. It is easy to see that

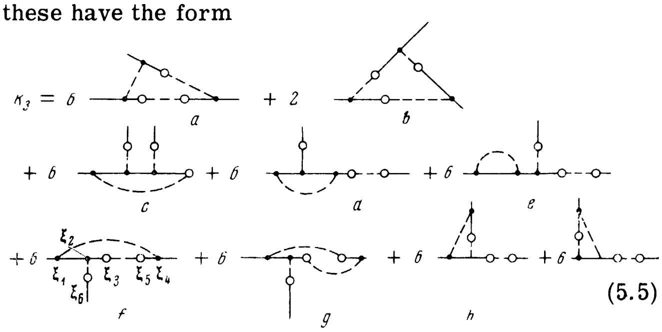
$$\langle\chi^n(x)\rangle_{\text{cn}} \equiv K_n(x). \tag{5.3}$$

Equation (5.3) is a consequence of the relation between the moments m_n and the cumulants K_n, which follows from (5.1):

$$m_l = \sum_{j=1}^{l} \frac{l!}{j!} \sum_{n_1+\cdots+n_j=l} \frac{K_{n_1}K_{n_2}\ldots K_{n_j}}{n_1!\,n_2!\ldots n_j!}. \tag{5.4}$$

Here j is the set of cofactors K_α, among which there can be identities. For example, $m_1 = K_1$, $m_2 = K_2 + K_1^2$, $m_3 = K_3 + 3K_2K_1 + K_1^3$ and so on. It follows from Eq. (5.4) that the moment of the n-th order is expressed in terms of the sum of products of cumulants of order no higher than n. Since products of cumulants are non-connected graphs, then the only random graph in the expansion of m_n will be equal to K_n. This discussion is, of course, not a proof of Eq. (5.3). However, such a proof can be given under very broad assumptions. We note that Eq. (5.3) is the analog of the well-known relation for the expansion of the thermodynamic potential and the proof can be transferred to the case under discussion without change (see, for example, [11]).

Let us consider the case $\sigma_0^2 \ll 1$, i.e., weak fluctuations of the field χ. In this case, the first terms of the expansion of the cumulants in a series in σ_0^2 are given by the first connected diagrams that are different from zero. The first terms of the expansion for $K_1 = \langle\chi\rangle$ and $K_2 = \sigma_\chi^2$ were discussed above. We consider $K_3 = \langle(\chi - \langle\chi\rangle)^3\rangle$. In this case, the diagrams proportional to σ_0^4 vanish and the expansion begins with terms proportional to σ_0^6. In all there are nine topologically different diagrams that are non-zero and are proportions to σ_0^6; these have the form



(5.5)

By comparing each diagram according to the general rules of the analytic expression, and after integration over all wave numbers, including the wave numbers of the free ends, we obtain the expression

$$K_3 = -378\sigma_0^6[6A_a + 6A_d + 12A_e - 6A_f - 6A_h - 6A_i] - 54\sigma_0^6[2A_b + 6A_c - 6A_g],$$

where A_i are the numerical coefficients which are determined by integrals over the longitudinal coordinates. Thus, we have, as an example, for A_f

$$A_f = \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \int_0^1 d\xi_4 \int_0^{\xi_4} d\xi_5 \int_0^1 d\xi_6 \min\{\xi_1, \xi_4\} \min\{\xi_2, \xi_6\} \min\{\xi_3, \xi_5\}.$$

Calculating these coefficients, we obtain

$$K_3 = -{}^{111}/_{70}\sigma_0^6$$

and consequently, we have for the coefficient of asymmetry $S = K_3/(K_2)^{3/2}$

$$S \approx -1.59\,\sigma_0^3. \tag{5.6}$$

The expansion for the cumulants of higher order begins with a term of higher order than σ_0^6 and consequently, we have from (5.2)

$$\sigma_\chi^2 + \langle\chi\rangle = {}^{37}/_{35}\sigma_0^6. \tag{5.7}$$

6. PROBABILITY DISTRIBUTION OF THE AMPLITUDE LEVEL IN THE REGION OF STRONG FLUCTUATIONS

For arbitrary values of σ_0 one can, in principle, carry out an analysis for K_n that is analogous to that given above for $K_1 = \langle\chi\rangle$. However, even in the case $K_2 = \sigma_\chi^2$, the calculations become too involved. The qualitative behavior of the cumulants for $\sigma_0^2 \gg 1$ and consequently the probability density of the amplitude level can be obtained by considering the characteristic function of the field χ. We consider the function $\Phi(x, \rho) = \exp\{i\lambda\chi(x, \rho)\}$. The function Φ satisfies the expression following from (1.3):

$$\frac{\partial\Phi}{\partial x} = -\frac{i\lambda}{2k}\Phi\Delta_\perp S^0 - \frac{1}{k}\nabla_\perp S^0 \nabla_\perp \Phi, \tag{6.1}$$

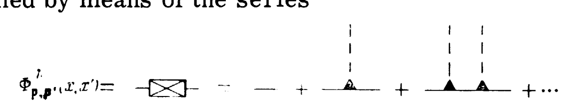
which must be solved with the boundary condition $\Phi(0, \rho) = 1$. Carrying over a Fourier transformation in Eq. (6.1) over the transverse coordinates, we obtain

$$\Phi_{\mathbf{p}}(x) = \delta(\mathbf{p}) + \frac{1}{2k}\int_0^x d\xi \int d\mathbf{q}_1 d\mathbf{q}_2 [\mathbf{q}_2(i\lambda\mathbf{q}_2 + 2\mathbf{q}_1)] \times \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}) S_{\mathbf{q}_2}{}^0(\xi)\Phi_{\mathbf{q}_1}(\xi). \tag{6.2}$$

In view of the homogeneity in the transverse coordinates, the characteristic function of the amplitude level satisfies the relation

$$\langle\Phi_{\mathbf{p}}(x)\rangle = \varphi_\lambda(x)\delta(\mathbf{p}). \tag{6.3}$$

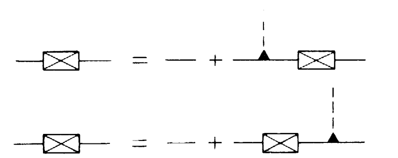
We introduce the function $\Phi^\lambda_{\mathbf{p},\mathbf{p}'}(x, x')$, which is determined by means of the series



6.4)

which goes over into the function D for $\lambda = 0$.

It is clear from (6.4) what equations are satisfied by the function Φ^λ, namely, the equations



(6.5)

to which corresponds the analytic description

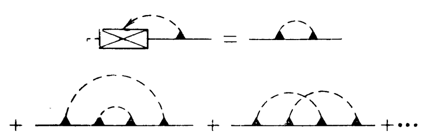
$$\Phi^\lambda_{\mathbf{p},\mathbf{p}'}(x,x') = D^0_{\mathbf{p},\mathbf{p}'}(x,x') + \int_0^x d\xi \int d\mathbf{q}_1 d\mathbf{q}_2 \Lambda^{\mathbf{p}}_{\mathbf{q}_1,\mathbf{q}_2} S_{\mathbf{q}_1}{}^0(\xi)\Phi^\lambda_{\mathbf{q}_2,\mathbf{p}'}(\xi, x'),$$

$$\Phi^\lambda_{\mathbf{p},\mathbf{p}'}(x,x') = D^0_{\mathbf{p},\mathbf{p}'}(x,x') + \int_{x'}^\infty d\xi \int d\mathbf{q}_1 d\mathbf{q}_2 \Phi^\lambda_{\mathbf{p},\mathbf{q}_1}(x,\xi)\Lambda^{\mathbf{q}_1}_{\mathbf{q}_2,\mathbf{p}'} S_{\mathbf{q}_2}{}^0(\xi). \tag{6.6}$$

Evidently, $\Phi_{\mathbf{p}}(x) = \Phi^\lambda_{\mathbf{p},0}(x, 0)$.

Let us consider $\langle\Phi_{\mathbf{p},\mathbf{p}'}(x, x')\rangle$. For averaging, we obtain diagrams which can be divided into two types—

weakly connected and strongly connected. We classify as weakly connected diagrams consisting of at least two pieces, connected only by the lines D^0. All the remaining diagrams we classify as strongly connected. Introducing the function $K^{\lambda}_{p,p'}(x, x')$, which represents the sum of strongly connected diagrams, the first diagrams of which have the form

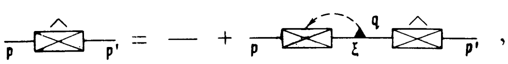


(6.7)

we can represent $\langle\Phi^{\lambda}\rangle$ in the form of the following series:

(6.8)

The diagram (6.8) represents an iteration series of the following equation:



(6.9)

which can be written analytically in the form

$$\langle\Phi^{\lambda}_{p,p'}(x,x')\rangle = \delta(p-p')\theta(x-x') + \int_0^{\infty} d\xi \int dq\, K^{\lambda}_{p,q}(x,\xi)\langle\Phi^{\lambda}_{q,p'}(\xi,x')\rangle. \tag{6.10}$$

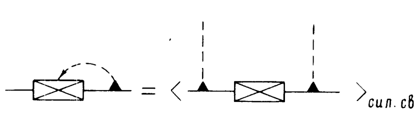
Setting p′ = 0, c′ = 0 in (6.10) and integrating with respect to p, we obtain the following equation for the characteristic function of the amplitude level

$$\varphi_{\lambda}(x) = 1 + \int_0^{\infty} d\xi \int dp\, K^{\lambda}_{p,0}(x,\xi)\varphi_{\lambda}(\xi). \tag{6.11}$$

We note that by virtue of the homogeneity over the transverse coordinates, we can represent $K^{\lambda}_{p,p'}(x, \xi)$ in the form

$$K^{\lambda}_{p,p'}(x,\xi) = K^{\lambda}(x,\xi)\theta(x-\xi)\delta(p-p'),$$

and, using the first equation of $\Phi^{\lambda}_{p,p'}$ (6.5) for $K^{\lambda}_{p,p'}$, we can write $K^{\lambda}_{p,p'}(x, x')$ in the following fashion:



(6.12)

This means that we can write $K^{\lambda}(x, \xi)$ in the form

$$K^{\lambda}(x,\xi) = \int_{\xi}^{x} d\eta\, L(\eta,\xi),$$

$$L(\eta,\xi) = \int dq_1\, dq_2\, d\varkappa_1\, d\varkappa_2 \Lambda^0_{q_1,\varkappa_1}\Lambda^{\varkappa_2}_{q_2,0}$$

$$\times \langle S_{q_1}{}^0(\eta) S^0_{q_2}(\xi)\Phi^{\lambda}_{\varkappa_1,\varkappa_2}(\eta,\xi)\rangle_{\text{сил. св.}}$$

and rewrite Eq. (6.1) as follows:

$$\varphi_{\lambda}(x) = 1 + \int_0^x d\eta \int_0 d\xi\, L(\eta,\xi)\varphi_{\lambda}(\xi). \tag{6.13}$$

It was shown above that the asymptote of the various one-point moments at $\sigma_0 \gg 1$ is determined by the immediate vicinity of the line $\eta \sim \xi$ while this vicinity is the narrower the larger σ_0. Therefore, as $\sigma_0 \to \infty$, we can rewrite (6.11) in the form of the equation

$$\varphi_{\lambda}(x) = 1 + \int_0^x d\eta\, \varphi_{\lambda}(\eta) \int_0^{\eta} d\xi\, L(\eta,\xi), \tag{6.14}$$

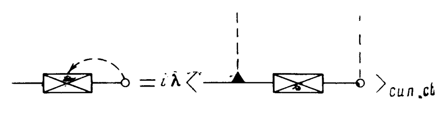
the solution of which is

$$\varphi_{\lambda}(x) = \exp\left\{\int_0^x d\eta \int_0^{\eta} d\xi\, L(\eta,\xi)\right\}. \tag{6.15}$$

Consequently, we can conclude that the asymptote of the logarithm of the characteristic function for $\sigma_0 \gg 1$ will be determined by the expression

$$K_{\lambda}(x) = \int_0^x d\xi\, K^{\lambda}(x,\xi)$$

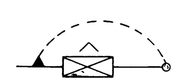
or by diagrams of the form



(6.16)

which must be integrated along the wave number of the free end.

Thus we have obtained the result that the asymptote of the logarithm of the characteristic function for $\sigma_0 \gg 1$ is determined by the asymptote of the strongly connected diagram. This allows us to find the asymptote of the probability density of the amplitude level χ as $|\chi| \to \infty$ in the region of very strong fluctuations. Actually, the asymptote of the probability density as $|\chi| \to \infty$ is determined by the asymptote $K_{\lambda}(x)$ as $\lambda \to 0$. Taking it into account that $K_{\lambda}(x) \approx i\lambda(i\lambda - 2)|\langle\chi\rangle/2$ as $\lambda \to 0$, we see that the "tails" of the probability density distribution of the amplitude level correspond to a Gaussian distribution with mean value $\langle\chi(x)\rangle$ and dispersion $\sigma^2_{\chi} = -\langle\chi\rangle$. The qualitative picture of the behavior of the cumulants, and consequently, of the probability density of the amplitude level for $\sigma_0^2 \gg 1$ can be obtained by choosing for $K_{\lambda}(x)$ a diagram similar to that which yields the asymptote $K_1 = \langle\chi(x)\rangle$, i.e., the following diagram:



(6.17)

and in the calculation of $\langle\Phi_{\lambda}\rangle$ to use the fact that the asymptote of the cumulants is determined by the neighborhood of the point of observation. Therefore, we can assume that the wave number of the line spanning the entire diagram is much greater than the internal wave numbers, and therefore,

$$\int dq\, q(i\lambda q + 2q_1)\delta(q + q_1 - \varkappa)S_q{}^0(\xi)$$

$$\approx \delta(q_1 - \varkappa)\int dq\, q(i\lambda q + 2q_1)S_q{}^0(\xi).$$

In this approximation,

$$\Phi^{\lambda}_{\varkappa_1,\varkappa_2}(\xi_1,\xi_2) = D^0_{\varkappa_1,\varkappa_2}(\xi_1,\xi_2)\exp\left\{\frac{1}{2k}\int_{\xi_2}^{\xi_1} d\eta \int d\varkappa\, \varkappa(i\lambda\varkappa + 2\varkappa_1)S_{\varkappa}{}^0(\eta)\right\}, \tag{6.18}$$

and consequently,

$$K_{\lambda}(x) = \frac{i\lambda(i\lambda-2)}{(2k)^2}\int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \int dq\, q^4\Phi_q{}^{S^0}(\xi_1,\xi_2)$$

$$\times\exp\left\{-\frac{1}{8k^2}\int_{\xi_2}^{\xi_1}\int d\eta_1 d\eta_2 \int d\varkappa(\lambda^2\varkappa^4 + 4(\varkappa q)^2)\Phi_{\varkappa}{}^{S^0}(\eta_1,\eta_2)\right\}. \tag{6.19}$$

Using the expression from (3.3) for $\Phi_q^{S^0}$, completing the integration in (6.19) in the wave numbers, and also taking into consideration that for $\sigma_0^2 \gg 1$ the principal region of integration is $\xi_1 \approx \xi_2$, we put (6.19) in the form

$$K_\lambda(x) = \frac{\sigma_0 i\lambda(i\lambda - 2)}{3\sqrt{2}} \int_0^{18\sigma_0^2} dz \left(1 - \sqrt{\frac{z}{18\sigma_0^2}}\right) \frac{\exp\{-\lambda^2 z/12\}}{\sqrt{z}(1+z)^{7/6}}. \tag{6.20}$$

It follows from (6.20) that the cumulants satisfy the relations $K_{2(n+1)}(x) + (n+1)K_{2n+1}(x) = 0$,

$$K_{2k+1}(x) = -\frac{3^{k-7/3}}{2^{k-5/6}} \frac{2k+1}{(2k-4/3)(2k-1/3)} \sigma_0^{2k-1/3} \quad (k = 1, 2, \ldots),$$

$$K_2(x) = -K_1(x) = 2\sqrt{2\pi} \frac{\Gamma(2/3)}{\Gamma(1/6)} \sigma_0, \quad K_3(x) = -\frac{3^{5/3}}{10 \cdot 2^{1/6}} \sigma_0^{5/3}. \tag{6.21}$$

It follows from (6.21) that for $\sigma_0 \gg 1$ the sums of each two successive terms of the series in Eq. (5.2) are equal to zero.

We see that for $\sigma_0 \gg 1$ the cumulants of the amplitude level ought to increase with increase in σ_0; thus, in particular, for the asymmetry coefficient, we get

$$S \approx -0.46 \sigma_0^{1/6}. \tag{6.22}$$

The qualitative behavior of the probability density of the amplitude level for $\sigma_0 \gg 1$ can be obtained by completing the inverse Fourier transformation of the characteristic function, i.e., for the probability density of the field χ we have

$$\varphi_\chi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{K_\lambda(x) - i\lambda\chi(x)\}\, d\lambda, \tag{6.23}$$

where $K_\lambda(x)$ is given by Eq. (6.20). It follows from (6.23) that for $|\chi - \langle\chi\rangle|/|\langle\chi\rangle| \gg 1$, the function $\varphi_\chi(x)$ corresponds to a Gaussian probability distribution with mean value equal to $\langle\chi\rangle$ and dispersion $\sigma_\chi^2 = -\langle\chi\rangle$, in accord with what was pointed out above. In the region $|\chi - \langle\chi\rangle| \lesssim \sigma_0$, the distribution of probabilities ought to have a more sloping portion than in the case of the Gaussian distribution, since the growth of the cumulants with increase in σ_0 shows that just this portion plays the important role. The length of this piece increases with increase in σ_0 and the role of the Gaussian "tails" is diminished.

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