PROPAGATION OF PLANE ELECTROMAGNETIC WAVES IN A CONSTANT FIELD

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We calculate the polarization operator in a constant crossed field, and also the radiative corrections to the Green's function of the photon in this field. It is shown that two waves with different dispersion laws can propagate in a constant crossed field. The refractive indices are determined and the question of the direction of propagation and polarization of the two waves is studied.

1. INTRODUCTION

A number of recent papers[1-4] are devoted to an investigation of the influence of the effects of polarization of vacuum on the propagation of photons in a constant and homogeneous electromagnetic field. In particular, Klein and Nigam[1] and Baier and Breitenlohner[2] considered the phenomenon of birefringence in vacuum. The propagation of electromagnetic waves in an external field was described with the aid of the dielectric and magnetic-permeability tensors of vacuum. It turned out here that the refractive index depends on the polarization of the wave. However, the vacuum was regarded in[1,2] as a medium without dispersion or absorption, making it possible to obtain only a limited number of results. In addition, the results of[1,2] are valid only in the weak field approximation, $B \ll B_0$, where $B_0 = m^2/e = 1.3 \times 10^{13}$ absolute Heaviside units.

In this paper we calculate the polarization operator in a constant and homogeneous electromagnetic field with $\mathbf{E} \perp \mathbf{H}$, $E = H = B$ (crossed field) and the radiative corrections to the photon Green's function in this field.

We furthermore investigate the propagation of plane electromagnetic waves in a crossed field with the aid of Maxwell's classical equations. The external field is regarded as a homogeneous anisotropic medium with dispersion and absorption, whose dielectric tensor is determined in terms of the polarization operator. It is shown that in a crossed field there can propagate two waves with different dispersion laws and with different refractive indices. The sign of the correction to the real part of the refractive indices depends on the magnitude of the invariance $\kappa = e\sqrt{(k_\mu F_{\mu\nu})^2/m^3}$, where $F_{\mu\nu}$ is the tensor of the external electromagnetic field, $k_\mu = \{\mathbf{k}, i\omega_0\}$, $\mathbf{k}$—wave vector of the wave, and $\omega_0 = |\mathbf{k}|$. $\mathrm{Re}\, n_{1,2} > 1$ when $\kappa \ll 1$ and $\mathrm{Re}\, n_{1,2} < 1$ when $\kappa \gg 1$. The imaginary part of the refractive indices determines the probability of pair production by a photon in the crossed field. We calculate the magnitude and direction of the group-velocity vectors of the two waves, and consider the question of the wave polarization. It is shown that perturbation theory is applicable to the calculation of the radiative corrections to the quantities under consideration in the case when $\kappa \gg 1$, provided the following conditions are simultaneously satisfied

$$1 \ll \varkappa \ll \left(\frac{\omega_0}{m}\right)^3 \frac{1}{\alpha^{3/2}}, \quad \alpha = \frac{e^2}{4\pi} = \frac{1}{137}.$$

The presented calculations are valid if the field dimensions $L \gg m/eB$, where m/eB is the distance over which the photon interacts effectively with the field, i.e., the distance in which the field changes the momentum of the virtual pair by an amount $\sim m$. The results are applicable also for weakly-inhomogeneous and slowly-varying fields, if the characteristic dimension of the inhomogeneity is $l \gg m/eB$, $l \gg \lambda$, where λ is the photon wavelength and the external-field frequency is $\omega \ll eB/m$, $\omega \ll \omega_0$.

The results obtained a good approximation for constant electric and magnetic fields if the following conditions are satisfied

$$|f| \ll 1, \quad |f| \ll \varphi(\varkappa),$$

where $f = e^2F_{\mu\nu}^2/2m^4$, and $\varphi(\kappa)$ equals κ^3 when $\kappa \ll 1$ and $\kappa^{4/3}$ when $\kappa \gg 1$[5]. The first condition is satisfied for all real fields, and satisfaction of the second condition can be obtained by making the photon energy high. Thus, for example, at a magnetic field intensity $B \sim 10^7$ absolute Heaviside units and a photon energy $\omega_0 \sim 10^6 m$, we get for the invariant $\kappa \sim B\omega_0/B_0 m \sim 1$, whereas the invariant $f = B^2/B_0^2 \sim 10^{-12}$. We note however, that under the foregoing conditions the effects under consideration are small (for example, the angle between the directions of propagation of the two waves $\theta \sim \alpha(m/\omega_0)^2 \sim 10^{-12}\alpha$). The optimal magnitude of the effect is reached when the conditions $B \gtrsim B_0$, $\omega_0 \lesssim m$ are satisfied (for example, $\theta \sim \alpha$).

2. POLARIZATION OPERATOR AND GREEN'S FUNCTION OF A PHOTON IN A CROSSED FIELD

The polarization operator in an external field, in the lowest order of perturbation theory, is given by

$$\Pi_{\mu\nu}(x, x') = e^2\, \mathrm{Sp}\, \{\gamma_\mu G^e(x, x')\gamma_\nu G^e(x', x)\}, \tag{1}$$

where $G^e(x, x')$—Green's function of the electron in the external field. In a constant and homogeneous field[6]

$$G^e(x, x') = \Phi(x, x')S^e(x - x'), \tag{2}$$

where $\Phi(x, x') = \Phi^{-1}(x, x')$ and $S^e(x - x')$, which depends only on the difference of the coordinates, has in a crossed field the form

$$S^e(x) = \frac{1}{2(4\pi)^2}\int_0^\infty \frac{ds}{s^3}\exp\left\{-im^2s + i\frac{x^2}{4s} - i\frac{s}{12}e^2(F_{\sigma\lambda}x_\lambda)^2\right\}$$
$$\times\left\{-i\gamma_\mu\left(\delta_{\mu\lambda} + esF_{\mu\nu} + \frac{e^2s^2}{3}F_{\mu\sigma}F_{\sigma\lambda}\right)x_\lambda + 2ms\right\}\left\{1 - i\frac{es}{2}\sigma_{\rho\tau}F_{\rho\tau}\right\}, \tag{3}$$

where $\sigma_{\rho\tau} = (-1/2)i[\gamma_\rho, \gamma_\tau]$. After substituting (2) in

(1) we see that the polarization operator in the homogeneous field depends, as it should, only on the coordinate difference

$$\Pi_{\mu\nu}(x, x') = \Pi_{\mu\nu}(x - x') = e^2 \operatorname{Sp}\{\gamma_\mu S^e(x - x')\gamma_\nu S^e(x' - x)\}. \quad (1')$$

Calculating the trace and going over to a momentum space, we obtain the tensor $\Pi_{\mu\nu}(k)$, which contains diverging terms. To eliminate these divergences, we separate from $\Pi_{\mu\nu}(k)$ the zeroth term $\Pi^{(0)}_{\mu\nu}(k)$ of the expansion in the field:

$$\Pi_{\mu\nu}(k) = \Pi^{(0)}_{\mu\nu}(k) + \Pi_{\mu\nu}'(k),$$

which coincides with the self-energy photon part of second order in the absence of the field. The divergences in $\Pi^{(0)}_{\mu\nu}(k)$ are eliminated in the usual manner, while the tensor $\Pi'_{\mu\nu}(k)$ is free of divergences. We finally obtain

$$\Pi_{\mu\nu}(k) = F_1(k^2,\varkappa)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + F_2(k^2,\varkappa)e^2(k_\sigma F_{\sigma\mu})(k_\lambda F_{\lambda\nu})$$
$$+ F_3(k^2,\varkappa)\left(\delta_{\mu\sigma} - \frac{k_\mu k_\sigma}{k^2}\right)e^2 F_{\sigma\rho}F_{\rho\tau}\left(\delta_{\tau\nu} - \frac{k_\tau k_\nu}{k^2}\right), \quad (4)$$

$$F_1(k^2,\varkappa) = i\frac{\alpha}{4\pi}\int_{-1}^{1} d\eta\,(1-\eta^2)\left\{k^2 \ln\left[1 + \frac{k^2}{4m^2}(1-\eta^2)\right]\right.$$
$$- \int_0^\infty d\tau \exp\left[-im^2\tau - i\frac{k^2}{4}(1-\eta^2)\tau\right]$$
$$\times\left[\frac{k^2}{\tau}\left(\exp\left\{-i\frac{\varkappa^2 m^6}{48}(1-\eta^2)^2\tau^3\right\} - 1\right)\right.$$
$$\left.\left. + \frac{\varkappa^2 m^6}{6}\tau[2 + (1-\eta^2)]\exp\left\{-i\frac{\varkappa^2 m^6}{48}(1-\eta^2)^2\tau^3\right\}\right]\right\}, \quad (5)$$

$$F_2(k^2,\varkappa) = i\frac{\alpha}{16\pi}\int_{-1}^{1} d\eta\,(1-\eta^2)^2$$
$$\times \int_0^\infty d\tau\,\tau \exp\left[-im^2\tau - i\frac{k^2}{4}(1-\eta^2)\tau - i\frac{\varkappa^2 m^6}{48}(1-\eta^2)^2\tau^3\right], \quad (6)$$

$$F_3(k^2,\varkappa) = i\frac{\alpha}{24\pi}\int_{-1}^{1} d\eta\,(1-\eta^2)[2 + (1-\eta^2)]$$
$$\times \int_0^\infty d\tau\,\tau \exp\left[-im^2\tau - i\frac{k^2}{4}(1-\eta^2)\tau - i\frac{\varkappa^2 m^6}{48}(1-\eta^2)^2\tau^3\right] \quad (7)$$

In the limiting case of a weak field $\kappa \ll 1$, formulas (5)–(7) give a result that coincides with the corresponding formula of [2]. We note that the series in powers of κ, which represents the polarization operator, is asymptotic. In the limiting case of a strong field $\kappa \gg 1$, the functions F_1, F_2, and F_3 are given

$$F_1(k^2,\varkappa) \approx -\frac{\alpha m^2}{\pi^2}(3\varkappa)^{2/3}\frac{3\sqrt{3}}{7}\Gamma^4\left(\frac{2}{3}\right)e^{i\pi/6},$$
$$F_2(k^2,\varkappa) = -\frac{1}{3m^6\varkappa^2}F_1(k^2,\varkappa); \quad F_3(k^2,\varkappa) = -\frac{k^2}{m^6\varkappa^2}F_1(k^2,\varkappa). \quad (8)$$

Knowing the polarization operator in the external field, we can solve the Dyson equation

$$G^\gamma_{\mu\nu}(k) - D^c_{\mu\lambda}(k)\Pi_{\lambda\sigma}(k)G^\gamma_{\sigma\nu}(k) = D^c_{\mu\nu}(k),$$
$$D^c_{\mu\nu}(k) = \frac{1}{ik^2}\left[\delta_{\mu\nu} + (d_l - 1)\frac{k_\mu k_\nu}{k^2}\right],$$

and find the photon Green's function in this field. In our case this solution is given by

$$G^\gamma_{\mu\nu}(k) = \frac{1}{ik^2 - F_1(k^2,\varkappa)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$
$$+ \frac{F_2(k^2,\varkappa)e^2(k_\sigma F_{\sigma\mu})(k_\lambda F_{\lambda\nu})}{[ik^2 - F_1(k^2,\varkappa)][ik^2 - F_1(k^2,\varkappa) - \varkappa^2 m^6 F_2(k^2,\varkappa)]}$$
$$+ \frac{F_3(k^2,\varkappa)}{[ik^2 - F_1(k^2,\varkappa)][ik^2 - F_1(k^2,\varkappa) - (\varkappa^2 m^6/k^2)F_3(k^2,\varkappa)]}$$
$$\times\left(\delta_{\mu\sigma} - \frac{k_\mu k_\sigma}{k^2}\right)e^2 F_{\sigma\rho}F_{\rho\tau}\left(\delta_{\tau\nu} - \frac{k_\tau k_\nu}{k^2}\right) + \frac{d_l}{ik^2}\frac{k_\mu k_\nu}{k^2} \quad (9)$$

We note that formula (4) gives the most general form of the gauge-invariant tensor of second rank, which can be constructed from $F_{\mu\nu}$, $\delta_{\mu\nu}$, and k_μ. Therefore formula (9) also gives the general form of the exact Green's function in the crossed field with all the radiative corrections. On the other hand, the functions F_1, F_2, and F_3 are calculated by perturbation theory.

It is seen from (9) that the interaction of the photons with the external electron magnetic field leads to the appearance of two different poles in the Green's function $G^\gamma_{\mu\nu}(k)$, which do not coincide with the pole of the free Green's functions. (The quantity $ik^2 - F_1(k^2, \kappa) - \kappa^2 m^6 k^{-2} F_3(k^2, \kappa)$, which enters in the denominator of the third term in (9), contains a factor k^2 which cancels out a light factor in $F_3(k^2, \kappa)$. The remaining factor never vanishes and thus does not lead to the appearance of a third pole in $G^\gamma_{\mu\nu}(k)$. This corresponds to two possible states of the photon with different polarizations, which interact in different manners with the external field. The effects to which this leads will be considered in the next section. We note that, just as in the free case, the radiative corrections make no contribution to the longitudinal part of the Green's function (9).

3. PROPAGATION OF PLANE ELECTROMAGNETIC WAVES IN A CROSSED FIELD

The propagation of plane electromagnetic waves in an external field will be described with the aid of the classical Maxwell equations

$$\mathbf{kh} = 0, \quad [\mathbf{kh}] = -\omega\mathbf{e} - i4\pi\mathbf{j},$$
$$\mathbf{ke} = -i4\pi\rho, \quad [\mathbf{ke}] = \omega\mathbf{h}, \quad (10)^*$$

where **e** and **h** are respectively the vectors of the electric and magnetic field of the wave, **k** is the wave vector, and $j_\mu = \{\mathbf{j}, i\rho\}$ is the Fourier component of the four-vector of the current in vacuum, induced by the joint action of the external field and tye type of the wave, which can be written in the form

$$j_\mu = -\frac{i}{4\pi}\Pi_{\mu\nu}(k)a_\nu(k), \quad (11)$$

where $a_\nu(k)$ is the Fourier component of the four-potential of the wave field and $\Pi_{\mu\nu}(k)$ is the polarization operator, which for the case of a crossed field is given by formula (4). Using the simple relations

$$k_i\Pi_{ij}(k) = -i\omega\Pi_{4j}(k), \quad k_i\Pi_{i4}(k) = -i\omega\Pi_{44}(k),$$

which follow directly from the condition of gauge invariance of the polarization operator, we can rewrite (11) in the form

$$j_\mu = -\frac{1}{4\pi\omega}\Pi_{\mu j}(k)e_j. \quad (11')$$

Substituting (11′) in (10), we rewrite the system of Maxwell's equations in the form

$$\mathbf{kh} = 0, \quad [\mathbf{kh}] = -\omega\mathbf{d}, \quad \mathbf{kd} = 0, \quad [\mathbf{ke}] = \omega\mathbf{h}, \quad (12)$$

where

$$d_i = \varepsilon_{ij}(\mathbf{k},\omega)e_j,$$
$$\varepsilon_{ij}(\mathbf{k},\omega) = \delta_{ij} - \frac{i}{\omega^2}\Pi_{ij}(k). \quad (13)$$

Thus, the propagation of plane electromagnetic waves in

*$[\mathbf{kh}] = \mathbf{k} \times \mathbf{h}$.

an external field is described by a system of equations (12), which coincides with Maxwell's equations for plane waves in a homogeneous anisotropic medium with dispersion and absorption, whose dielectric tensor is determined by formulas (13) and (4).

The connection between ω and $\mathbf{k}$ is established by the equation

$$|\omega^2\varepsilon_{ij}(\mathbf{k},\omega)-\mathbf{k}^2\delta_{ij}+k_ik_j|=0, \qquad (14)$$

and after substituting in it formulas (13) and (4) we get

$$[ik^2-F_1(k^2,\varkappa)][ik^2-F_1(k^2,\varkappa)-\varkappa^2m^6F_2(k^2,\varkappa)]=0. \qquad (15)$$

As expected, Eqs. (15) are obtained by equating to zero the expressions in the denominator of the Green's function (9).

Equations (14) and (15) have two solutions corresponding to two different waves:

$$\omega_1(\mathbf{k})\approx\omega_0+\frac{i}{2\omega_0}[F_1(0,\varkappa)+\varkappa^2m^6F_2(0,\varkappa)], \qquad (16)$$

$$\omega_2(\mathbf{k})\approx\omega_0+\frac{i}{2\omega_0}F_1(0,\varkappa), \qquad (17)$$

in formulas (16) and (17) and further, $\omega_0 = |\mathbf{k}|$ and in the invariant κ the vector $k_\mu = \{\mathbf{k}, i\omega_0\}$.

The imaginary parts of the frequencies $\omega_{1,2}(\mathbf{k}) = \omega'_{1,2}(\mathbf{k}) + i\omega''_{1,2}(\mathbf{k})$ can be represented in the form

$$\omega''_{1,2}(\mathbf{k})=\frac{e^2m^2\varkappa}{128\pi^{3/2}\omega_0}\int\limits_{(4/\varkappa)^{2/3}}^{\infty}dy\,\frac{8\rho+1\mp3}{\sqrt{y}\,\rho\sqrt{\rho(\rho-1)}}\,v'(y)=-\frac{1}{2}F_{\parallel,\perp}(\varkappa), \qquad (18)$$

where $\rho = (1/4)\kappa y^{3/2}$, v(y)—Airy function, and $\mathbf{F}_{\parallel,\perp}$—probabilities of pair production in the crossed field by a photon polarized respectively parallel and perpendicular to the electric-field vector[7].

The wave propagation direction is determined by the direction of the group-velocity vector, which is defined, by virtue of satisfaction of the condition $|\omega''_{1,2}| \ll \omega'_{1,2}$, by the relation

$$\mathbf{v}^{(1,2)\,\mathrm{gr}}=\partial\omega_{12}'(\mathbf{k})\,/\,\partial\mathbf{k}.$$

Formulas (16) and (17) take in the limiting case $\kappa \ll 1$ the form

$$\omega_1\approx\omega_0-2\frac{\alpha}{45\pi}\frac{m^2\varkappa^2}{\omega_0},\qquad \omega_2\approx\omega_0-7\frac{\alpha}{90\pi}\frac{m^2\varkappa^2}{\omega_0}. \qquad (19)$$

In a system in which the electric-field intensity vector $\mathbf{E}$ is directed along the axis 1 and the magnetic-field intensity vector H is directed along the axis 2 (henceforth, the "special system"), we have $\kappa = B\gamma/B_0m$, where $\gamma = \omega_0 - k_3$. If the wave vector has components $k_1 = k_2 = 0$ and $k_3 = -\omega_0$, the direction of the group-velocity vectors for both waves coincides with the direction of the wave vector, and their moduli are equal to

$$v^{(1)\mathrm{gr}}\approx1-8\lambda,\quad v^{(2)\mathrm{gr}}\approx1-14\lambda, \qquad (20)$$

where $\lambda = (\alpha/45\pi)(B/B_0)^2$. At any other direction of the wave vector, the group-velocity vectors are directed at a certain angle to the vector $\mathbf{k}$. For example, in the case $k_1 = \omega_0$ and $k_2 = k_3 = 0$ we have

$$\begin{aligned} &v_1^{(1)\mathrm{gr}}\approx1-4\lambda,\quad v_2^{(1)\mathrm{gr}}=0,\quad v_3^{(1)\mathrm{gr}}\approx4\lambda,\\ &v_1^{(2)\mathrm{gr}}\approx1-7\lambda,\quad v_2^{(2)\mathrm{gr}}=0,\quad v_3^{(2)\mathrm{gr}}\approx7\lambda. \end{aligned} \qquad (21)$$

Thus, the angles between the vectors $\mathbf{v}^{(1)\mathrm{gr}}$ and $\mathbf{v}^{(2)\mathrm{gr}}$ and the axis 1 are respectively

$$\theta_1\approx4\lambda,\quad \theta_2\approx7\lambda$$

and the angle between $\mathbf{v}^{(1)\mathrm{gr}}$ and $\mathbf{v}^{(2)\mathrm{gr}}$ is

$$\theta=\theta_1-\theta_2\approx3\lambda.$$

In the opposite limiting case $\kappa \gg 1$ we have

$$\begin{aligned} \omega_1(\mathbf{k})&\approx\omega_0+\frac{am^2}{14\pi^2\omega_0}(3\varkappa)^{2/3}\sqrt{3}\,\Gamma^4\left(\frac{2}{3}\right)(1-i\sqrt{3}),\\ \omega_2(\mathbf{k})&\approx\omega_0+\frac{am^2}{28\pi^2\omega_0}(3\varkappa)^{2/3}3\sqrt{3}\,\Gamma^4\left(\frac{2}{3}\right)(1-i\sqrt{3}). \end{aligned} \qquad (22)$$

We note that the second terms in (22) are radiative corrections to ω_0, and consequently the formulas in (22) are valid only if the following conditions are simultaneously satisfied:

$$1\ll\varkappa\ll\left(\frac{\omega_0}{m}\right)^3\frac{1}{\alpha^{3/2}}. \qquad (23)$$

In the opposite case, perturbation theory cannot be used to calculate the radiative corrections to ω_0 or $\epsilon_{ij}(\mathbf{k}, \omega)$.

In the case $k_1 = k_2 = 0$ and $k_3 = -\omega_0$ in the special system, the directions of the vectors $\mathbf{v}^{(1)\mathrm{gr}}$ and $\mathbf{v}^{(2)\mathrm{gr}}$ coincide with the direction of the wave vector $\mathbf{k}$, and their moduli are equal to

$$\begin{aligned} v^{(1)\mathrm{gr}}&\approx1-\frac{\alpha}{14\pi^2}\left(6\frac{B}{B_0}\right)^{2/3}\left(\frac{m}{\omega_0}\right)^{4/3}\frac{1}{\sqrt{3}}\Gamma^4\left(\frac{2}{3}\right),\\ v^{(2)\mathrm{gr}}&\approx1-\frac{\alpha}{28\pi^2}\left(6\frac{B}{B_0}\right)^{2/3}\left(\frac{m}{\omega_0}\right)^{4/3}\sqrt{3}\,\Gamma^4\left(\frac{2}{3}\right). \end{aligned} \qquad (24)$$

In the case $k_1 = \omega_0$ and $k_2 = k_3 = 0$, we have

$$\begin{aligned} v_1^{(1)\mathrm{gr}}&\approx1-\frac{\alpha}{14\pi^2}\left(3\frac{B}{B_0}\right)^{2/3}\left(\frac{m}{\omega_0}\right)^{4/3}\frac{1}{\sqrt{3}}\Gamma^4\left(\frac{2}{3}\right)\\ v_2^{(1)\mathrm{gr}}&=0,\\ v_3^{(1)\mathrm{gr}}&\approx-\frac{\alpha}{7\pi^2}\left(3\frac{B}{B_0}\right)^{2/3}\left(\frac{m}{\omega_0}\right)^{4/3}\frac{1}{\sqrt{3}}\Gamma^4\left(\frac{2}{3}\right). \end{aligned} \qquad (25)$$

The corrections for the group velocity of the second wave differ from the corrections in formula (25) by the factor 3/2. The angles between the vectors $\mathbf{v}^{(1)\mathrm{gr}}$ and $\mathbf{v}^{(2)\mathrm{gr}}$ and the axis 1 are respectively

$$\theta_1\approx v_3^{(1)\mathrm{gr}},\qquad \theta_2\approx v_3^{(2)\mathrm{gr}}={}^3/_2\,v_3^{(1)\mathrm{gr}}$$

and the angle between $\mathbf{v}^{(1)\mathrm{gr}}$ and $\mathbf{v}^{(2)\mathrm{gr}}$ is

$$\theta=\theta_2-\theta_1\approx{}^1/_2\,v_3^{(1)\mathrm{rp}}.$$

The refractive index is determined by the relation

$$n_{1,2}\left(\omega_0,\frac{\mathbf{k}}{|\mathbf{k}|}\right)=\frac{|\mathbf{k}|}{\omega_{1,2}(\mathbf{k})}. \qquad (26)$$

Hence, using formulas (16) and (17), we get

$$\begin{aligned} n_1\left(\omega_0,\frac{\mathbf{k}}{|\mathbf{k}|}\right)&\approx1-\frac{i}{2\omega_0^2}[F_1(0,\varkappa)+\varkappa^2\mathrm{m}^6F_2(0,\varkappa)],\\ n_2\left(\omega_0,\frac{\mathbf{k}}{|\mathbf{k}|}\right)&\approx1-\frac{i}{2\omega_0^2}F_1(0,\varkappa). \end{aligned} \qquad (27)$$

In the limiting case $\kappa \ll 1$

$$\begin{aligned} n_1\left(\omega_0,\frac{\mathbf{k}}{|\mathbf{k}|}\right)&\approx1+2\frac{\alpha}{45\pi}\frac{m^2\varkappa^2}{\omega_0^2},\\ n_2\left(\omega_0,\frac{\mathbf{k}}{|\mathbf{k}|}\right)&\approx1+7\frac{\alpha}{90\pi}\frac{m^2\varkappa^2}{\omega_0^2}. \end{aligned} \qquad (28)$$

In the special system at $k_1 = k_2 = 0$ and $k_3 = -\omega_0$ we have

$$n_1\approx1+8\lambda,\quad n_2=1+14\lambda,$$

which coincides with the result of[2]. In the case $\kappa \ll 1$

$$n_1\left(\omega_0, \frac{\mathbf{k}}{|\mathbf{k}|}\right) \approx 1 - \frac{\alpha m^2}{14\pi^2\omega_0^2}(3\varkappa)^{2/3}\sqrt{3}\,\Gamma^4\left(\frac{2}{3}\right)(1 - i\sqrt{3}),$$

$$n_2\left(\omega_0, \frac{\mathbf{k}}{|\mathbf{k}|}\right) \approx 1 - \frac{\alpha m^2}{28\pi^2\omega_0^2}(3\varkappa)^{2/3}3\sqrt{3}\,\Gamma^4\left(\frac{2}{3}\right)(1 - i\sqrt{3}). \quad (29)$$

From a comparison of formulas (28) and (29) we see that the sign of the correction to the real part of the refractive indices depends on the magnitude of the invariant κ. Re $n_{1,2} > 1$ when $\kappa \ll 1$ and Re $n_{1,2} < 1$ when $\kappa \gg 1$.

The question of the polarization of the two waves is resolved in the standard manner (see, for example,[8]). It turns out here that for a specified direction of the wave vector, the electric-induction vectors **d** of the two waves are polarized in two mutually perpendicular planes. The same can be said concerning the polarization of the vector **e** of two waves with identical direction of the group-velocity vector. There exists one direction of the wave vector (in the special system $k_1 = k_2 = 0$, $k_3 = -\omega_0$), for which the directions of the vectors **e** and **d** coincide. For the first wave in this case **e** ∥ **E**, and for the second wave **e** ⊥ **E**.

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