SOVIET PHYSICS JETP VOLUME 28, NUMBER 3 MARCH, 1969

LARGE ANGLE QUASICLASSICAL SCATTERING AT HIGH ENERGIES

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Submitted April 6, 1967; resubmitted April 26, 1968

Zh. Eksp. Teor. Fiz. 55, 854–868 (September, 1968)

Some analytic properties of the scattering matrix as a function of the complex angular momentum are studied for nonrelativistic scattering on a Gaussian potential $U(r) = U_0 \exp(-r^2/a^2)$. The asymptotic behavior of the scattering amplitude for large momentum transfers is found. The possibility of a universal phenomenological description of the experimental data on high energy elastic pp-scattering throughout the whole angle range by means of a Gaussian potential is considered.

1. INTRODUCTION

EXPERIMENTAL data on high-energy elastic pp scattering have the following characteristic features. Scattering in the c.m.s. through intermediate and large angles[1] ($\theta_{c.m.} > 30°$), at a total energy $\sqrt{s} \approx 5-8$ GeV/c, and at all measured squared momentum transfers $-t = 2-25\ (\text{GeV/c})^2$, is well described by Orear's empirical formula[2], which can be written in the form

$$d\sigma / d\Omega = (9.7)^2 \exp(-6.6\, k_\perp), \tag{1}$$

where

$$k_\perp = k \sin\theta_{c.m} = \sqrt{(1+t/s)(-t)}$$

and k is the c.m.s. momentum. In (1), σ is expressed in $(\text{GeV}/\hbar c)^{-2}$, and t and s in $(\text{GeV/c})^2$. We shall use these units throughout. At large angles ($\theta_{c.m.} = 70-90°$), the Orear formula, under suitable limitation of the parameters for this narrower region, goes over into the Cocconi empirical formula[3] which, in the same units, can be written in the form

$$d\sigma / d\Omega = (26.8)^2 \exp(-3.4\sqrt{s}).$$

For forward scattering ($\theta_{c.m.} < 20°$), the diffraction peak formula given by Foley et al.[4] is valid:

$$d\sigma / d\Omega = (5.6k)^2 \exp(10t). \tag{2}$$

We can attempt to interpret these results on the basis of the potential-scattering model. The basis for this is the quasipotential approach in field theory. Within the framework of the potential model, two possible explanations have been proposed for the exponential smallness of the differential large-angle scattering: either scattering by a singular potential with strong absorption and at small distances (an essentially complex potential)[5], or else scattering by a smooth potential in the angle region forbidden by classical mechanics[6].

In[6] they discussed the Gaussian potential $U(r) = U_0 \exp(-r^2/a^2)$, which is characterized by the fact that at small angles, in the Born approximation, it leads to a cross section in the form

$$d\sigma / d\Omega \sim \exp(\tfrac{1}{2}a^2 t), \tag{3}$$

i.e., it imitates the diffraction peak (2), and at large angles in the zeroth quasiclassical approximation it has the form (1), with the exponential containing a constant that depends weakly on s; if the choice $U_0 \sim k$ is made it leads to constancy of the total cross section at large k in the Born approximation[7].

The present paper is devoted to the study of the asymptotic properties of the amplitude of scattering by this potential at large k and θ. A very effective procedure of finding the asymptotic solutions of the Schrödinger equation at large complex angular momenta λ and of investigating the properties of the scattering matrix $S(\lambda, k)$ is to use the Zwaan method[8] (the WKB method using the complex plane of the radius vector). In[9] they investigated with the aid of the Zwaan method, in conjunction with the Regge method[10], scattering by an analytic even potential with poles at complex points of the radius vector plane. In the present paper we use a similar procedure to study scattering by a potential having an entire radius-vector function.

In Sec. 2 we find for the scattering matrix $S(\lambda)$ a formula which is asymptotic as $\lambda \to \infty$, after which we calculate the matrix explicitly by expansion in an asymptotic series in a parameter which is small at high energies. With the aid of the obtained approximation we find the region of holomorphy of $S(\lambda)$ and investigate its properties. In Sec. 3, the Watson integral for the scattering amplitude is calculated by the saddle-point method with the aid of expansion in an asymptotic series in the same small parameter, and we find the conditions for applicability of the asymptotic expansions. In Sec. 4 we discuss the general problems connected with the Zwaan method, the question of the relativistic generalization of the obtained results, and the agreement with experiment.

2. SCATTERING MATRIX

Thus, we solve the nonrelativistic Schrödinger equation for a potential $U(r) = U_0 \exp(-r^2/a^2)$, the WKB form of which in units $\hbar = 2m = 1$ is given by

$$\left[\frac{d^2}{d\rho^2} + 1 - \frac{\lambda^2}{\rho^2} - \alpha^2 \exp\left(-\frac{\rho^2}{h^2}\right)\right] \varphi(\rho) = 0, \tag{4}$$

where $\varphi(\rho)$—radial part of the l-the partial wave function multiplied by r, m—reduced mass, k—c.m.s. momentum, and

$$\rho = kr, \quad \lambda = l + 1/2, \quad \alpha = |\alpha| e^{-i\Delta} = U_0^{1/2}(k)/k, \quad h = a(k)k. \tag{5}$$

The spin effects are ignored here, and the inelastic

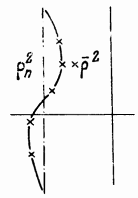

FIG. 1.

processes are taken into account by introducing a complex potential. The potential is assumed to decrease at $+\infty$ ($|\arg h| < \pi/4$), and its phase is assumed to be of the absorption type ($0 \le 2\Delta < \pi$). The quantities α and h have in general independent asymptotic behaviors with increasing k. We assume that $\alpha \to 0$ as $k \to \infty$.

A. The solution of (4) by the WKB method begins with finding the zeroes of the quasimomentum

$$q(\rho) = 1 - \frac{\lambda^2}{\rho^2} - \alpha^2 \exp\left(-\frac{\rho^2}{h^2}\right). \tag{6}$$

An investigation shows (see Appendix I) that at small values of λ there is a series of dynamic zeroes ρ_n ($-\infty < n < \infty$) due to the Gaussian part of the potential (Fig. 1), and two kinematic zeroes $\bar{\rho} \approx \pm\lambda$ due to the kinematic part of the potential λ^2/ρ^2. A situation of importance to the scattering is one in which the kinematic zero approaches the dynamic zero ρ_0 with changing λ ($|\operatorname{Im}\rho_0|$ is minimal in the ρ_n series). The λ corresponding to this zero is the saddle point in the Watson integral (see Sec. 3).

B. An exact solution of (4) in a certain region of the ρ plane adjacent to the root, containing the root itself, is represented asymptotically by a super-position of two linearly independent WKB solutions[8,9]:

$$\varphi_\pm(\rho) = q^{-1/4} \exp[\pm i\xi(\rho)], \quad \xi(\rho) = \int_{\rho_s}^{\rho} q^{1/2}\, d\rho,$$

where ρ_s is one of the roots. To find the connection between the coefficients of the superposition in different regions, a system of Stokes lines is used, namely the lines of the level $\operatorname{Im}\xi(\rho) = 0$ and the conjugate lines of the levels $\operatorname{Re}\xi(\rho) = 0$.

The Stokes-line picture corresponding to the situation essential for the scattering, when $\bar{\rho}$ is close to ρ_0, is shown in Fig. 2 (situation 1). Figure 3 shows the successive change of the pictures on going from the pre-pole configuration of the level lines (dashed) through the pole configuration (solid line) (the S-matrix has a pole) to the post-pole configuration (dash-dot). For details see Appendix II.

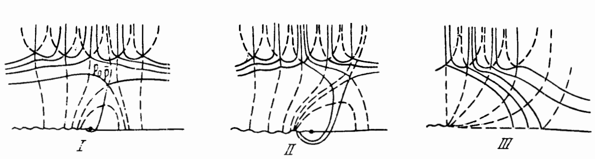

FIG. 2. The ρ plane. The Stokes lines for the general case, when the potential and the orbital momentum differ from zero: I – case when there are no dynamic zeroes in the influence zone ($|\rho|<|\lambda|$) of the kinematic zero $\rho \approx \lambda$; II – case when the kinematic zero ρ is close to the line of the dynamic zeroes; III – case when a large number of dynamic zeroes is located in the zone of influence of the kinematic zero.

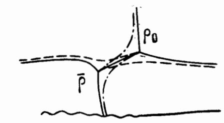

FIG. 3. The ρ plane. Pole configuration of the level lines – solid lines. Pre-pole – dash dash, post-pole – dash-dot.

C. If $j_\lambda(\rho)$ is the exact solution of (4) with asymptotic behavior $j_\lambda(\rho) \approx \text{const}\cdot\rho^\lambda$ at $\rho \to 0$, then, by determining its asymptotic form at $\rho \to +\infty$ with the aid of the Stokes structure, we find the value of the phase shift $\delta(\lambda)$ of the scattering matrix $\tilde{S}(\lambda)$]. For $= \exp[2i\tilde{\delta}(\lambda)]$. For the pre-pole configuration, the phase shift takes the usual form (see Appendix III):

$$\tilde{\delta}(\lambda) = \int_{\bar{\rho}}^{\infty} (\sqrt{q} - 1)\, d\rho - \bar{\rho} + \pi\lambda/2. \tag{7}$$

The guaranteed region of validity of (7) is shown in Fig. 4 (region $R_0(\lambda)$ with a cut along the negative semiaxis). The radius of the region $R_0(\lambda)$ is of the order of $|h\sqrt{\ln(\alpha^{-2})}|$. The saddle-point parameter of the Watson integral is located in the sector R_0'. The guaranteed region of holomorphy of $S(\lambda)$ is of the form of $R(\lambda)$ (Fig. 4). The boundary of this region is determined by those values of λ, at which there occurs either the pole configuration (left-side of the boundary $\operatorname{Re}\lambda < 0$), or where $\bar{\rho} \approx \lambda$ merges with one of the dynamic zeroes (right part of the boundary $\operatorname{Re}\lambda > 0$). It tends asymptotically to the value $|\arg\lambda| = T/4$. Detailed questions of this section are considered in Appendix III.

D. We present an asymptotic calculation of the WKB phase shift $\tilde{\delta}(\lambda, \alpha)$ as $\alpha \to 0$. To this end, we introduce a new independent variable $w(\lambda)$, putting

$$\lambda\rho_s^{-1}(\lambda) = \cos w,$$

where ρ_s is one of the zeroes of the quasimomentum $q(\rho)$. On the basis of formula (6) this yields

$$\lambda = ih\cos w\sqrt{\ln(\alpha^{-2}\sin^2 w)}, \quad \rho_s = ih\sqrt{\ln(\alpha^{-2}\sin^2 w)}. \tag{8}$$

The quantity $\lambda(w)$ is a multi-sheeted analytic function with an essential singularity at infinity and with a logarithmic singularity at zero. The sheet corresponding to $|\arg w| < \pi$ has two branch points of the root type, $w \approx \alpha$ and $w \approx \pi - \alpha$.

The quantity $w(\lambda)$ is a multi-sheeted function with an essential singularity at infinity. Each sheet of $w(\lambda)$ is an entire function of λ. The sheet $\bar{w}(\lambda)$ corresponding to the kinematic root $\bar{\rho}(\lambda)$ will be called the kinematic sheet, and the corresponding branch $\bar{S}(\lambda)$ will be called the kinematic branch. The calculations are given in Appendix IV (formula (IV.3)).

We now investigate the properties of $S(\lambda, \alpha)$ with

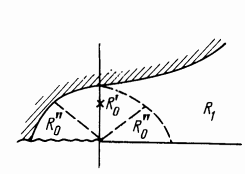

FIG. 4. The λ plane, R_0' – region of validity of the WKB asymptotic expansion of the phase shift; $R_0 = R_0' \cup R_0''$ – minimum region of accuracy of the Zwaan method; $R = R_0 \cup R_1$ – minimal region of holomorphy of $S(\lambda)$.

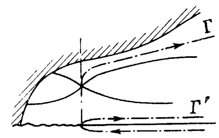

FIG. 5. The λ plane. Structure of relief of the integrand of the Watson integral. Level line passing through the saddle points – solid; path of steepest descent – dashed; integration contour Γ′ and deformed integration contour Γ = Γ₋ ∪ Γ₊ – dash-dot.

the aid of the principal term of the asymptotic expansion of the WKB phase shift $\tilde{\delta}(\lambda, \alpha)$:

$$\tilde{\delta}_0(\lambda, \alpha) = \lambda(w - \operatorname{tg} w). \quad (9)$$

In the region $R_1(\lambda)$ (Fig. 4), where $w \approx 0$, we get with allowance for (8)

$$\tilde{\delta}_0(\lambda, \alpha) \approx \frac{1}{3}\alpha^3\lambda \exp\left(-\frac{3}{2}\frac{\lambda^2}{h^2}\right) \quad (10)$$

and the following asymptotic property

$$\delta(\lambda, \alpha) \to 0 \text{ as } \lambda \to \infty \ (|\arg\lambda| < \pi/4). \quad (11)$$

Thus, the kinematic sheet $\tilde{S}(\lambda)$ is at the same time a physical sheet. The dynamic branch $S_n(\lambda)$ corresponding to the dynamic branch $w_n(\lambda)$ (such that $w_n(\lambda) \to \infty$ as $\lambda \to \infty$) does not possess this property In the region of $R_0'(\lambda)$ we have

$$\tilde{\delta}_0(\lambda, \alpha) \approx \lambda \arccos\frac{\lambda}{ih\sqrt{\ln\alpha^{-2}}} - i\sqrt{h^2\ln\alpha^{-2} + \lambda^2}, \quad (12)$$

so that the second asymptotic property is valid in $R_0'(\lambda)$:

$$\tilde{\delta}(\lambda, \alpha) \to \infty \text{ as } \alpha \to 0. \quad (13)$$

From the estimate (see [10])

$$\delta(\lambda, \alpha) = \tilde{\delta}(\lambda, \alpha)[1 + O(\lambda^{-1})]$$

it follows that in $R_0'(\lambda)$

$$\delta(\alpha, \lambda) / \tilde{\delta}(\lambda, \alpha) \to 1 \text{ as } \alpha \to 0, \quad (14)$$

inasmuch as $\lambda \to \infty$ when $\alpha \to 0$ at fixed w, in accordance with (8). In other words, in this region, the WKB form $\tilde{S}(\lambda, \alpha)$ gives an asymptotic representation of the exact $S(\lambda, \alpha)$.

3. SCATTERING AMPLITUDE

A. The scattering amplitude $f(\theta, \alpha)$ for a scattering angle $\theta \neq 0$ is given by the Watson integral[9] along the contour Γ′ (Fig. 5):

$$f(\theta, \alpha) = \frac{1}{2k}\int_{\Gamma'} S(\lambda, \alpha)\frac{P_{\lambda-1/2}(-\cos\theta)}{\cos\pi\lambda}\lambda\, d\lambda, \quad (15)$$

where P—Legendre functions. Within the limits of the holomorphy region of $S(\lambda, \alpha)$, we deform the integration contour Γ′ into the contour $\Gamma = \Gamma_- \cup \Gamma_+$. Rotation of the contour through a certain angle $|\arg\lambda| < \pi/4$ is possible by virtue of the exponential decrease of the integrand. Further, we have[11]

$$\frac{P_{\lambda-1/2}(-\cos\theta)}{\cos\pi\lambda} = \left(\frac{2}{\pi\lambda\sin\theta}\right)^{1/2} e^{\mp i(\theta+\pi/4)}[1 + O(\lambda^{-1})]. \quad (16)$$

Therefore, neglecting the contribution of the vicinity of zero,

$$f(\theta, \alpha) = \left(\frac{-i}{2\pi k^2\sin\theta}\right)^{1/2}\int_{\Gamma_-}\exp\left\{2i\left[\tilde{\delta}(\lambda) - \frac{\lambda\theta}{2}\right]\right\}\lambda^{1/2}[1 + O(\lambda^{-1})]\,d\lambda$$
$$+ \left(\frac{i}{2\pi k^2\sin\theta}\right)^{1/2}\int_{\Gamma_+}\exp\{2i[\tilde{\delta}(\lambda) + \lambda\theta/2]\}\lambda^{1/2}[1 + O(\lambda^{-1})]\,d\lambda, \quad (17)$$

where $\tilde{\delta}(\lambda)$—WKB phase shift (see (IV.3)). Introducing the phase functions

$$N(\mp\theta, \lambda) = 2\tilde{\delta}(\lambda) \mp \lambda\theta, \quad (18)$$

we obtain the values of the saddle-point parameters $w_\mp$ in first order in $\ln^{-1}\alpha$:

$$w_\mp = \pm\frac{\theta}{2} - \frac{\ln 2}{\operatorname{tg}(\theta/2)\ln(\alpha^{-2}\sin^2(\theta/2))}. \quad (19)$$

This property of 2w, in addition to the definition (8), makes it possible to treat it as the complex scattering angle connected with the given complex angular momentum (impact parameter) at the given potential. The integration path $\Gamma(\lambda)$ for $0 < 2\Delta < \pi$ is mapped into the contour $\Gamma(w)$ of the w plane (Fig. 6), from which we see that, first, there is one saddle point each in the upper and lower half planes and, second, $\arg(-\theta) = -\pi$. Thus, the saddle points in first order in $\ln^{-1}\alpha$ are

$$\lambda_\mp = \mp ih\cos(w_\mp)\sqrt{\ln[\alpha^{-2}\sin^2(\pm\theta/2)]}, \quad (20)$$

and the direction of the steepest descent in each of them coincides with the imaginary axis. For the saddle-point values of the phase functions this yields

$$iN(-\theta, \lambda_-) = -2h\sin\frac{\theta}{2}\sqrt{\ln\left(\alpha^{-2}\sin^2\frac{\theta}{2}\right)}\times\left[1 + \frac{\ln 2 - 1}{\ln(\alpha^{-2}\sin^2(\theta/2))} + o(\ln^{-2}\alpha)\right],$$
$$iN(\theta, \lambda_+) = -2h\sin\frac{\theta}{2}\sqrt{\ln\left(\alpha^{-2}\sin^2\frac{\theta}{2}\right)}\times\left[1 + \frac{\ln 2 - 1 - i\pi}{\ln(\alpha^{-2}\sin^2(\theta/2))} + o(\ln^{-2}\alpha)\right]. \quad (21)$$

The structure of the relief of the phase functions $iN(\mp\theta, \lambda)$ is shown in Fig. 5. For the level lines passing through the upper saddle point, by definition,

$$\operatorname{Re} iN(\theta, \lambda) = \operatorname{Re} iN(\theta, \lambda_+),$$

whereas in the region $|\lambda| \to \infty$, $0 < \arg\lambda < \pi/4$ we have

$$\operatorname{Re} iN(\theta, \lambda) \approx -\theta\operatorname{Im}\lambda,$$

from which it is clear that the level line going upward to the right behaves like a curve of the parabolic type.

Further, inasmuch as for the steepest-descent line, by definition, $\operatorname{Im} iN(\theta, \lambda) = \operatorname{Im} iN(\theta, \lambda_+) \approx 0$ and $\operatorname{Im} iN(\theta, \lambda) \approx \theta\operatorname{Re}\lambda$ increases with increasing $|\lambda|$ along the line $0 < \arg\lambda < \pi/4$, the steepest-descent line goes off into the forbidden region. But this is immaterial, since outside the influence of the saddle point the path of integration can be deformed arbitrarily within the limits of the valley. Inasmuch as we can choose the integration path along which

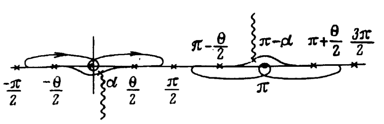

FIG. 6. The w plane. Integration contour Γ (w) for the Watson integral.

the form of the region of holomorphy of $S(\lambda)$, the position of its singularities, and the behavior at infinity. The region of holomorphy of $S(\lambda)$ turns out to coincide, in a certain sense, with the region where the potential is small. The main difficulty lies in constructing the pictures of the Stokes lines, which depend essentially on the parameter λ. Certain general criteria are given in the present paper[2)]. A simultaneous consideration of the level lines and the conjugate level lines, being more complete, turns out at the same time also more convenient. To calculate the phase integrals, it is convenient to replace the implicit dependence of the zeroes of $q(\rho, \lambda)$ on λ by their parametric dependence on a certain parameter w, which is possible in the general case. This makes it possible to transfer the dependence of the integration limits on the potential to under the integral sign in the phase integral and to carry out an asymptotic expansion, assuming the presence of a parameter which is small at high energies. The parameter w makes it possible to investigate also the properties of the scattering matrix. A certain canonical representative of the parametrization group realizes in the simplest manner the connection between the potential and the scattering amplitude. For scattering at classically forbidden angles, the saddle-point value of this canonical parameter w coincides with the scattering angle, making it possible to treat it as a complex scattering angle.

2. A few remarks of physical nature. The zeroth approximation of the amplitude of scattering by a Gaussian potential is given in the ordinary notation by

$$f(t) = a\sqrt{\ln (t/t_0)} \exp [-a\sqrt{(-t) \ln (t/t_0)}], \qquad (28)$$

where $t_0 = -4MU_0$ (M = proton mass), and consequently depends only on t. The dependence on s enters in the symmetrization term $f(u)$ and appears at scattering angles close to $\pi/2$. Comparing the expressions for the Born (4) and quasiclassical (28) approximations, we see that the former is an expansion in powers of a^2t, and the latter in powers of $(\ln t/t_0)^{-1}$. Accordingly, the regions of their applicability are given by $-t \ll a^{1/2}$ and $-t \gg -t_0$, and in order for them to overlap it is necessary to have $-t_0 a^2 \ll 1$.

As to the realization of the scheme described in the introduction, we note here two items. 1) the relativistic generalization of the nonrelativistic formula is generally speaking, ambiguous; it is sufficient, for example, to use the relativistic and nonrelativistic neutralizers in the forms $4M^2s^{-1}$ and $1 - 4M^2s^{-1}$. For uniqueness it is necessary to introduce additional assumptions. 2) The parameter a can depend on t and its value at $t = 0$ given by the diffraction-peak formula cannot, generally speaking, be transferred to the region of large t. This arbitrariness decreases the value of the nonrelativistic potential model, since the required properties can always be obtained by complicating the relativistic generalization. With respect to the very simple formula (28), we note that there is no system of parameters (a, U_0) lying in the region of applicability of (28) and reconciling it and (3) with the formulas of Orear (1) and Foley et al. (2).

In conclusion, I am grateful to S. P. Alliluev, S. S. Gershtein, and A. A. Logunov for suggesting the problem and for numerous useful discussions.

APPENDIX I

We present the results of an investigation of the equation $q(\rho) = 0$ in the form of a picture of the motion of its roots in the ρ^2 plane when λ^2 varies from $-\infty$ to $+\infty$ along the real axis.

Let $\arg h = 0$ and $\Delta = 0$. If $\lambda^2 \gg 1$, then in the right half-plane of ρ^2 there exists a root $\bar{\rho}^2 \approx \lambda^2$, which we shall call kinematic, since it is determined by the kinematic potential. In addition there exists a series of roots which are essentially connected with the potential and are therefore called dynamic. The central part of the series is described asymptotically, at large λ^2, by the formula

$$\rho_n{}^2 \approx -h^2(\ln \lambda^2 + 2\pi n i), \quad |n| \sim 1, \qquad (I.1)$$

and the zeroes with large imaginary part deviate slightly from the asymptote $\mathrm{Re}\, \rho^2 = -h^2 \ln \alpha^{-2}$, and satisfy the formula

$$\rho_n{}^2 \approx -h^2(\ln \alpha^{-2} + 2\pi n i), \quad |n| \gg 1 \qquad (I.2)$$

(Fig. 8, solid line). When λ^2 approaches zero, the kinematic group remains close to λ^2 and equals $\bar{\rho}^2 \simeq (1 + \alpha^2)\lambda^2$, whereas the series of dynamic roots approaches the asymptote. At $\lambda^2 = 0$, the kinematic root lands at zero, and the series of dynamic roots is described by formula (I.2) for all n (Fig. 8, dash-dot). If λ^2 becomes negative, the central part of the series and the kinematic zero move opposite each other (Fig. 8, dashed), with $\bar{\rho}^2 \approx \lambda^2$, and the dynamic zeroes are displaced little. Further, there is a real $\lambda d_0^2 \approx -h^2 \ln \alpha^{-2}$ such that the kinematic zero coalesces with the dynamic zero ρ_0^2 $(n = 0)$, forming a double zero $\lambda d_0^2 \approx -h^2 \ln \alpha^{-2}$. With further shift of λ^2 to the left, the kinematic zero is halted on the line of dynamic zeroes which, in turn, lagging λ^2, is drawn together with it to the left, crossing, in particular, the asymptote $\mathrm{Re}\, \rho^2 = -h^2 \ln \alpha^{-2}$. When $|\lambda^2| \gg h^2 \ln \alpha^{-2}$, the central part of the series is described as before by (I.1), with allowance for the fact that now $\arg \lambda^2 = \pi$, and the remote zeroes satisfy formula (I.2).

If λ^2 is complex, then the picture for $\bar{\rho}^2$ close to ρ_n^2 is shown in Fig. 1. The double zero ρd_n^2 (corresponding to the merging of ρ^2 and ρ_n^2) approaches the imaginary axis with increasing n. In all other respects the behavior of the roots is similar.

The general case $\arg h \neq 0$ and $\Delta \neq 0$ is considered analogously. In particular, for remote dynamic zeroes $(|n| \gg 1)$ formula (I.2) remains in force, from which

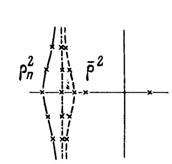

FIG. 8.

[2)]The technical details of the present work, and particularly of the appendices, are contained in the author's thesis submitted to the Moscow Physicotechnical Institute, 1966.

Re $iN(\mp\theta, \lambda)$ decreases monotonically with increasing distance from the saddle point, the accuracy estimate $O(\lambda_\pm^{-1})$ holds true for the calculations. We note that $\lambda_\pm \to \infty$ as $\alpha \to 0$.

Thus, taking into account the fact that the WKB accuracy, the accuracy of the asymptotic representation of the Legendre functions, and the accuracy of the saddle-point method are of the same order $O(\lambda_\mp^{-1})$, we get

$$f(\theta, \alpha) = \tilde{f}(\theta, \alpha)[1 + O(\lambda_\mp^{-1})], \tag{22}$$

where $\tilde{f}(\theta, \alpha)$—amplitude corresponding to the exact WKB phase shift (without expansion in $\ln^{-1}\alpha$). In other words, the WKB formula gives the asymptotic representation of the exact scattering amplitude as $\alpha \to 0$. Since it is sufficient to calculate the saddle-point pre-exponential factors in zeroth order in $\ln^{-1}\alpha$ in order to obtain the asymptotic representation of $f(\theta, \alpha)$, we obtain ultimately

$$\tilde{f}(\theta, \alpha) = \frac{a}{2}\left(\ln \alpha^{-2} \sin^2 \frac{\theta}{2}\right)^{1/2} [e^{iN(-\theta, \lambda_-)} + e^{iN(\theta, \lambda_+)}][1 + O(\ln^{-1}\alpha)], \tag{23}$$

where the $N(\mp\theta, \lambda_\mp)$, calculated in first order in $\ln^{-1}\alpha$, are given in (21).

B. As already noted, the parameters α and h have generally speaking independent asymptotic behaviors as $k \to \infty$, determined by the behavior of the force and the radius of the interaction with increasing energy. In order for the presented asymptotic calculations to be valid, it is necessary to have 1) $\alpha \to 0$, 2) $\lambda_\mp \to \infty$ as $k \to \infty$ and at fixed $\theta \neq 0$, i.e., in accordance with (20)

$$h^{-1} \ln^{-1/2} \alpha = 0(1) \text{ as } k \to \infty. \tag{24}$$

In other words, $h(k)$ should increase with increasing k, and if it does decrease, it should do so not faster than $\ln^{-1/2}\alpha$. In turn, the order of growth of $h(k)$ determines the number of terms of the expansion of the phase functions in (23) needed for the valid presence of the pre-exponential factor, since this factor, when transferred to the exponent, makes a contribution of the order of $\ln(\ln \alpha)$. Thus, if the radius of the potential a does not depend on k, so that $h(k) \sim k$, and with this $U_0 a^2$ is bounded from below with increasing k, then a correction of arbitrarily high order in the exponential makes a contribution larger than the contribution of the pre-exponential factor, since

$$\ln(\ln \alpha) = o[\alpha^{-1}(\ln \alpha)^{-m+1/2}] \text{ as } \alpha \to 0.$$

Consequently, at the given values of the parameters, the asymptotic expansion is valid only for $\ln f(\theta, \alpha)$ in the form[1)] (in the expanded notation)

$$\ln f(\theta, k) = -2ak \sqrt{\ln \frac{k^2 \sin^2(\theta/2)}{U_0}}\left[1 + O\left(\ln \frac{k^2 \sin^2(\theta/2)}{U_0}\right)\right] \sin \frac{\theta}{2}. \tag{25}$$

C. Let us discuss the question of the region of applicability of the obtained results. The trajectories of the saddle points $\lambda_\mp$ as functions of the scattering angle θ are shown in Fig. 7. When $0 < \theta < |\alpha|$ the saddle points are located near the real axis (although, strictly speaking, it was assumed in the calculation that $|\text{Im}\, \lambda_\pm \gg 1|$). The exponential decrease of $f(\theta, \alpha)$ with increasing k disappears now and the performed calculations are not suitable. If the quantum-mechanical uncertainty of the scattering angle is

$$\Delta\theta \approx 1/k|a| \ll \alpha,$$

then this region is the region of the classical scattering angles. When $\theta \approx |\alpha|$ both saddle-points corresponding to the two impact parameters merge in the vicinity of $\lambda = 0$ and then go off to the complex region. When $\theta = \pi$ both saddle points again vanish. Thus, the angle region of applicability corresponds to the loops on the trajectories of Fig. 7. Both the WKB and the asymptotic expansions are valid here. This region is the region of the classically inaccessible angles and, as follows, the amplitude is exponentially small. Finally, from the condition of the independence of the contributions of the saddle points we have the following limitations on the angles:

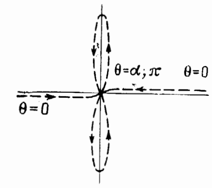

FIG. 7. λ plane. Trajectory of the saddle points of the Watson integral as a function of the scattering angle θ. The section close to the real axis corresponds to the classical scattering angles, and the loops of the trajectory correspond to the classical forbidden angles.

$$\theta \gg |\alpha|, \quad (\pi - \theta) \gg |h|^{-1/2}\left(\ln \frac{1}{|\alpha|}\right)^{-1/4}, \tag{26}$$

and from the condition of applicability of the asymptotic expansion we have a limitation on the energy:

$$\ln \frac{1}{|\alpha|} \gg 1. \tag{27}$$

When using the saddle-point method we have assumed that the parameter h is real. However, the results are valid in the region $|\arg h| < \pi/4$, where we have an exponentially decreasing oscillating potential, In this case the picture (Fig. 5) is rotated through an angle arg h, and the level lines passing through the saddle point become deformed in such a way that it is still possible to choose an integration path that lies in the valley within the limits of the holomorphy region.

4. CONCLUSION

1. Let us note certain general mathematical questions connected with the application of the Zwaan method to the study of the properties of scattering by potentials that are entire functions of r. First, these potentials lead to an infinite number of turning points, the determination of which is best performed by using, for example, a grapho-analytic method. The Zwaan method then turns out to be effective in cases when it is possible to confine oneself to one or at the most several zeroes. However, in the general case it is possible to draw several conclusions connected with

[1)] A similar result was obtained in [12] by summing the Born series. A discussion of questions connected with the indicated form of the exponential decrease of the scattering amplitude is contained in [13].

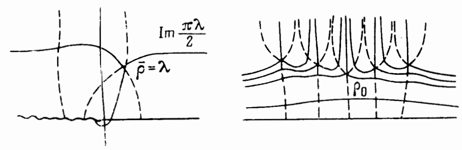

FIG. 9. The ρ plane. The Stokes lines of free motion ($\alpha = 0$) – left, central-symmetry motion ($\lambda = 0$) – right. Level lines – solid, conjugate level lines – dashed. All the pictures are symmetrical with respect to the orgin.

we see, for example, that the asymptote makes an angle $-2 \arg h$ with the imaginary axis.

APPENDIX II

The structure of the Stokes lines for the free motion ($\alpha = 0$) and for centrally-symmetrical moticn ($\lambda = 0$) is shown in Fig. 9. In the general case ($\alpha \neq 0$, $\lambda \neq 0$) the structure of the lines is as follows: the dynamic picture is superimposed on the free picture ($\alpha = 0$) corresponding to the given λ, and the two of them are smoothly inscribed. Then, the significant distortion takes place only within the limits of the tion $|\rho| \lesssim |\lambda|$ at large λ ($|\lambda \gtrsim h\sqrt{\ln \alpha^{-2}}$). Outside this region, a weak deviation from the free picture takes place in the lateral sectors, and from the dynamic picture in the upper and lower sectors. The joining of these pictures is effected with allowance for the continuous dependence of the pictures of the Stokes lines on the parameter λ. In particular, if two zeroes are on one level line at a certain λ, then: 1) there is an increment $\delta_1\lambda$ such that the zeroes are connected as before by the same level line; 2) there is an increment $\delta_2\lambda$ such that the common level line for the two zeroes (double line) is weakly split, and then, say, the upper level line is the one emerging from either the first or the second zero, depending on the sign of the increment $\delta_2\lambda$. It is then convenient to introduce a system of connecting lines, namely lines in the λ plane such that if λ lies on one of them, then any two zeroes in the ρ plane lie on one level line. With the aid of these criteria it is possible, for example, to consider the continuous deformation of the level-line picture when λ varies from zero to a given value along different paths in the λ plane (see footnote[2)]). We present here the pictures of three characteristic situations, which we shall need subsequently to investigate the properties of the scattering matrix $S(\lambda)$ (Fig. 2). It is assumed that cuts are made on the figure from the dynamic zeroes to $\mp i\infty$.

APPENDIX III

Let us find the WKB formula of the S matrix. To this end, we introduce in the vicinities of the singular points and the roots of the quasimomentum $q(\rho)$ the system of WKB solutions:

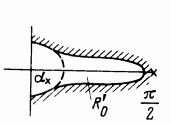

FIG. 10. The w plane. Region of holomorphy of $S(\lambda)$. Region of $R'_0(w)$ – two lateral sectors.

$$(0, \rho) = q^{-1/4} \exp i \left[\int_0 \left(\sqrt{q} + \frac{i\lambda}{\rho} \right) d\rho - i\lambda \ln \rho \right],$$

$$(\infty, \rho) = q^{-1/4} \exp - i \int_\rho^\rho (\sqrt{q} - 1) d\rho - \rho \quad ,$$

$$(\bar{\rho}, \rho) = q^{-1/4} \exp i \int_{\bar{\rho}}^{\rho} \sqrt{q}\, d\rho \qquad \text{(III.1)}$$

with asymptotic behavior

$$(0, \rho) \approx \text{const} \cdot \rho^\lambda \qquad \rho \to 0, \qquad \text{(III.2)}$$
$$(\infty, \rho) \approx \exp i\rho \qquad \rho \to +\infty$$

and the system of symbols[8]

$$[\rho_1, \rho_2] = q^{1/4}(\rho_1, \rho_2), \qquad \text{(III.3)}$$

where ρ_1 or ρ_2—any one of the points ρ, $\bar{\rho}$, p_0, 0, or $+\infty$. It is convenient to use this notation in order to connect different solutions, since the following simple properties hold:

$$[\rho_1, \rho_2] = [\rho_2, \rho_1]^{-1}, \quad (\rho_1, \rho) = [\rho_1, \rho_2](\rho_2, \rho). \qquad \text{(III.4)}$$

If $j_\lambda(\rho)$ is the exact solution of (4) with asymptotic behavior (III.2) when $\rho \to 0$ and

$$j_\lambda(\rho) \approx A(\lambda) e^{i\rho} + B(\lambda) e^{-i\rho} \qquad \rho \to +\infty,$$

then the S matrix is given by[9]

$$S(\lambda) = e^{i\pi(\lambda + 1/2)} A(\lambda) / B(\lambda).$$

Alternately, if $h_\lambda^{(1)}(\rho)$ is an exact solution with asymptotic behavior (III.2) when $\rho \to +\infty$ and $h_\lambda^{(1)}(\rho) \approx a(\lambda) e^{i\rho} + b(\lambda) e^{-i\rho}$ when $\rho \to -\infty$, then

$$S(\lambda) = a^{-1}(\lambda)[1 + e^{i\pi(\lambda + 1/2)} b(\lambda)].$$

The zeroes and poles of $S(\lambda)$, according to the first definition, are located at the zeroes of $A(\lambda)$ and $B(\lambda)$, for which it is necessary that at least two roots lie on one level line (Fig. 3, solid line). For the pole configuration and configurations close to it, we obtain the WKB formula for the S matrix in the form

$$\tilde{S}(\lambda) = e^{i\pi\lambda} \frac{[\bar{\rho}, \infty]^2}{1 + [\bar{\rho}, \rho_0]^2}. \qquad \text{(III.5)}$$

For the pre-pole situation (Fig. 3, dashed) $|[\bar{\rho}, \rho_0]| \ll 1$, so that $\tilde{S}(\lambda) = \exp(2i\tilde{\delta}(\lambda))$, where $\tilde{\delta}(\lambda)$—WKB phase shift:

$$\tilde{\delta}(\lambda) = \int_{\bar{\rho}}^{\infty} (\sqrt{q} - 1) d\rho - \bar{\rho} + \pi\lambda/2. \qquad \text{(III.6)}$$

For the pole configuration we get from (III.5) the Bohr phase-integral condition

$$\int_{\bar{\rho}}^{\rho_0} \sqrt{q}\, d\rho = \pi(k + 1/2) \qquad (k - \text{целое}).$$

The pole line in the λ plane extends from $\lambda_{d_0} \approx ih\sqrt{\ln \alpha^{-2}}$ to the left—downward to $\lambda \approx -|\lambda_{d_0}|$. The line symmetrical to it with respect to the real axis is the line of zeroes of $S(\lambda)$. For the post-pole configuration (Fig. 3, dash-dot) we have $|[\bar{\rho}, \rho_0]| \gg 1$, so that

$$\tilde{S}(\lambda) \approx e^{i\pi\lambda}[\rho_0, \infty]^2.$$

However, in order for this expression to be valid it is necessary that $\bar{\rho}$ be located far from the level line $\{0, \rho_0\}$, for which it is necessary to increase λ. It can be shown, however, that with increasing λ the level line $\{\rho_0, +\infty\}$ disappears, i.e., the direct connection between zero and $+\infty$ via of the level lines vanishes. And since the function $j_\lambda(\rho)$ decreases with increasing $|\rho|$ when $\mathrm{Re}\,\lambda < 0$, it is impossible to move away from the center across the level lines, and the method turns out to be ineffective (see Fig. 2 (III) when $\mathrm{Re}\,\lambda < 0$). The second definition of $S(\lambda)$ is likewise ineffective, since there is no direct connection between $-\infty$ and $+\infty$ via level lines.

Let us discuss the question of the applicability of the WKB method. When the solution is continued along a certain path into the ρ plane, it is necessary, generally speaking, to take into account the contributions made to the coefficient of the asymptotically smaller solution from the intersections of all the conjugate level lines. However, if the intersection occurs outside the region of influence of the zero, then these contributions can be neglected. Nothing can be said in the general case with respect to the dimensions of the zero-influence region. In situation I of Fig. 2, $\bar{\rho}$ is far from all the dynamic zeroes and there is a direct connection of the zero with $+\infty$, so that the WKB method is suitable. In the situation of type II, the WKB method is not suitable because of the passage of the level line $\{\bar{\rho}, +\infty\}$ near a number of remote zeroes. In addition, $\bar{\rho}$ is close to the line of the dynamic zeroes and consequently λ is close to the line of double zeroes in its plane. For these λ, the behavior of the level lines becomes critical: a small change of leads to an essential change in the picture; in particular, situations arise in which there are many bound zeroes. Near and beyond this line, the Zwaan method is ineffective, and this line is apparently the line of singularities of $S(\lambda)$. In situation III ($|\lambda| \gg 1$, $\mathrm{Re}\,\lambda > 0$), while $j_\lambda(\rho)$ does increase with increasing $|\rho|$, an intersection of the conjugate level lines emerging from a large number of close zeroes takes place on going from zero along the level line $\{\rho, i\infty\}$, so that the asymptotically small solution is continuously reconstructed with a certain coefficient (equal, generally speaking, to a part of the Stokes coefficient), and both solutions can have a commensurate amplitude on the level line $\{\rho, i\infty\}$. Therefore it is impossible to guarantee accuracy of the WKB method in this approach. However, knowing the general behavior of the roots as functions of λ it is possible to verify with the aid of an analytic consideration[10] the validity of (III.6) in the entire right-hand sector.

Thus, the guaranteed region of holomorphy of $S(\lambda)$ has the form of $R(\lambda)$ (Fig. 4). The region $R_0(\lambda)$, where the accuracy of the Zwaan method is guaranteed, has the form of the vicinity of the radius of order $|h\sqrt{\ln \alpha^{-2}}|$ with a cut along the negative semiaxis.

APPENDIX IV

Using the new pair of independent variables (w, α), we transform the expression (III.6) for the WKB phase shift $\tilde{\delta}(\lambda, \alpha)$ into

$$\tilde{\delta}(\lambda, \alpha) = \lambda\Bigg(\int_{\cos^{-s} w}^{\infty \cdot e^{i \arg \lambda}} \left\{ \left[1 - \frac{1}{t^2} - \sin^2 w \left(\frac{\alpha}{\sin w} \right)^{2(1-t^2\cos^2 w)} \right]^{1/2} - 1 \right\} dt - \frac{1}{\cos w} + \frac{\pi}{2} \Bigg). \tag{IV.1}$$

The region $R(w)$ corresponding to $R(\lambda)$ is shown in Fig. 10. We confine ourselves to that part of the region where $|\alpha^2 \sin^{-2} w| \ll 1$, i.e., to the two $R_0'(w)$ sections adjacent to the real axis. In the λ plane, the $R_0'(w)$ region corresponds to that part of $R_0(\lambda)$, where $|\arg\lambda \pm \pi/2| < \pi/4$ (Fig. 4). It is easy to see that the integral converges here. Further, for w $R_0'(w)$ and

$$t \in \Gamma(t) - [\Gamma \cap U_\varepsilon(1/\cos w)],$$

where $\Gamma(t)$ is the integration path and $\epsilon > 0$ is fixed, the sequence of functions $(n = 0, 1, 2, \ldots)$

$$\{\varphi_n(t)\} = (1 - t^{-2})^{-n} \left(\frac{\alpha}{\sin w} \right)^{2n(1-t^2\cos^2 w)} \tag{IV.2}$$

is asymptotic as $\alpha \to 0$ and uniform in the parameters t and w. Expanding the integrand in (IV.1) in an asymptotic series in $\{\varphi_n(t)\}$, integrating term by term, and separating the first terms of the asymptotic expansion by means of integration by parts, we obtain a series which is asymptotic as $\alpha \to 0$ in a certain vicinity of $\epsilon = 0$. Since the limits of the obtained series exist as $\epsilon \to 0$ and they themselves yield an asymptotic series as $\alpha \to 0$, we have completed the term-by-term transition to the limit as $\epsilon \to 0$. Ultimately this yields

$$\tilde{\delta}(\lambda, \alpha) \underset{\alpha \to 0}{=} \lambda \left\{ w - \left(1 + \frac{\ln 2 - 1}{\ln(\alpha^{-2}\sin^2 w)} \right) \mathrm{tg}\, w + O[\ln^{-2}(\alpha^{-2}\sin^2 w)] \right\}. \tag{IV.3}$$

Analogously

$$\frac{\partial\tilde{\delta}(\lambda, \alpha)}{\partial\lambda} \underset{\alpha \to 0}{=} w + \frac{\ln 2}{\ln(\alpha^{-2}\sin^2 w)} + O[\ln^{-2}(\alpha^{-2}\sin^2 w)]. \tag{IV.4}$$

We note that inasmuch as the derivative $\partial\tilde{\delta}(\lambda, \alpha)/\partial\lambda$ admits of an expansion of the same type as the function, and since all the terms of the asymptotic sequence (IV.3) are differentiable functions of the parameter w, then term-by-term differentiation is permissible. Taking into account the equality

$$\frac{\partial\lambda}{\partial w} = -\lambda \left[1 - \frac{1}{\ln(\alpha^{-2}\sin^2 w)\,\mathrm{tg}\, w} \right] \mathrm{tg}\, w \tag{IV.5}$$

this gives the same result as before. In exactly the same manner, in the first order in $\ln^{-1}\alpha$, we have

$$\frac{\partial^2\tilde{\delta}(\lambda, \alpha)}{\partial\lambda^2} = -(\lambda\, \mathrm{tg}\, w)^{-1}[1 + O(\ln^{-1}\alpha)]. \tag{IV.6}$$

We note that in the derivation of $\tilde{\delta}(\lambda, \alpha)$ (IV.3), a cut $\{0, i\infty\}$ was drawn through the plane of the radicand of the $\lambda(w)$ dependence (8), and the principal branch of the root was chosen. The choice of the second branch would yield

$$\tilde{\delta}_0(\lambda) = \lambda(\pi - w + \mathrm{tg}\, w).$$

Since $\lambda(\pi - w) = \lambda(w)$ for λ lying on different sheets of the roots, this is simply equivalent to two different definitions of $\delta_0(\lambda)$. The second case corresponds to the right-hand side of Fig. 6.

Further, since $\lambda(\pi - w) = -\lambda(w)$ for λ lying on the same sheet of the root, the following reflection property holds true:

$$\tilde{\delta}_0(-\lambda) = -\pi\lambda + \tilde{\delta}_0(\lambda),$$

which remains valid also for the exact WKB phase shift $\tilde{\delta}(\lambda)$. The exact $\delta(\lambda)$, of course, does not have this property. Finally, since

$$\lambda^*(w^*) = \lambda(w)[1 + O(h^{-1}\ln^{-1}\alpha)],$$

the expansion (IV.3) leads directly to unitarity in first order in $\ln^{-1}\alpha$.

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Translated by J. G. Adashko
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