CRITICAL FIELD OF SURFACE SUPERCONDUCTIVITY AND THE STRUCTURE OF A SUPERCONDUCTING LAYER IN AN INCLINED FIELD

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Two problems are considered. 1. The problem of calculating the critical field of surface superconductivity H_{c3} for pure superconductors is solved within the framework of the microscopic superconductivity theory. It is shown that $H_{c3}/H_{c2} \geq 1.99$ at $T = 0$ for specular reflection of the electrons at the metal surface and ≥ 2.09 for diffuse reflection (the numerical calculation was performed by a variational method). 2. The structure of a superconducting surface layer in an inclined magnetic field is investigated in the region of applicability of the Ginzburg–Landau equations ($T \to T_c$). It is shown that for an angle of inclination $\theta \neq 0$ the surface superconducting layer possesses a vortex structure similar to the Abrikosov vortex lattice. For small inclination angles the period of the structure is $a \sim \xi/\sqrt{\theta} \gg \xi(T)$, where $\xi(T)$ is the temperature-dependent coherence length.

INTRODUCTION

An investigation of the character of the onset of superconductivity near the surface of a metal in a strong magnetic field has been recently the subject of many papers, both theoretical and experimental. The task of the theory consists, first, of calculating the critical fidl field of the surface conductivity (H_{c3}) as a function of temperature, concentration of impurities introduced into the superconductors, and other parameters, and second, of an investigation of the structure of the superconducting surface layer in a field $H < H_{c3}$.

The problem of the theoretical calculation of the field H_{c3}, as well as the problem of calculating the critical field of the volume superconductivity H_{c2}[1,2] reduces to a solution of the linearized Gor'kov equation[3]

$$\Delta^*(\mathbf{r}) = |\lambda| T \sum_{\omega} \int d\mathbf{r}' \overline{G_\omega(\mathbf{r}, \mathbf{r}') G_{-\omega}(\mathbf{r}, \mathbf{r}')} \Delta^*(\mathbf{r}'), \qquad (1.1)$$

where $\Delta^*(\mathbf{r}')$ is the parameter of superconducting ordering, $G_\omega(\mathbf{r}, \mathbf{r}')$ are the thermodynamic Green's functions of the normal metal in a constant magnetic field $\mathbf{H}$. $\omega = (2n+1)\pi T$ are discrete frequencies, and λ is the Cooper interaction constant. In the calculation of the upper critical field H_{c2}, the quantity G_ω in (1.1) should be taken to mean the Green's function in an infinite metal, whereas in the case of calculation of the field H_{c3} of the surface superconductivity $G_\omega(\mathbf{r}, \mathbf{r}')$ should denote the Green's function for the half-space.

If the temperature is close to a critical temperature of the superconducting transition T_c, the integral equation (1.1) reduces to a differential Ginzburg–Landau equation[4,5] with corresponding boundary conditions on the surface.[6] In this case, as shown by Saint-James and de Gennes,[7] the critical field is $H_{c3} = 1.69\, H_{c2}$, and the superconducting surface layer, unlike the superconducting state below the field H_{c2}, is not vortical but is characterized by the existence of two superconducting currents of equal magnitude but opposite direction flowing along the surface of the metal. The described situation pertains to the case of a strictly parallel orientation of the magnetic field relative to the surface of the superconductor. In the case of inclined orientation, the critical field H_{c3} becomes a function of the angle of inclination θ: $H_{c3} = H_{c3}(\theta)$, the function $H_{c3}(\theta)$ changing from $1.69\, H_{c2}$ at $\theta = 0$ to a value $H_{c3} = H_{c2}$ at $\theta = \pi/2$.[8]

The task of the present investigation is, first, to study the temperature dependence of the critical field of the surface superconductivity H_{c3} in the case of parallel orientation ($\theta = 0$) outside the region of applicability of the Ginzburg–Landau equations (in particular, at $T = 0$), and second, a study of the structure of the superconducting surface layer in the case of inclined orientation of the magnetic field ($\theta \neq 0$).

As will be shown in Sec. 2, the ratio $\gamma(T) = H_{c3}(T)/H_{c2}(T)$, in the case of pure superconductors, reveals a noticeable temperature dependence. Thus, the magnitude of this ratio at $T = 0$ amounts to $\gamma(0) \geq 1.99$ in specular reflection of the electrons from the surface of the metal and $\gamma(0) \geq 2.09$ in diffuse reflection, which must be compared with the value $\gamma = 1.69$ at $T = T_c$ (which does not depend on the character of the reflection). Such a behavior of γ with changing temperature differs from the corresponding behavior of $\gamma(T)$ in alloys. As noted by de Gennes,[9] for extremely contaminated alloys ($l \ll \xi_0$) the ratio H_{c3}/H_{c2} does not depend on the temperature and equals 1.7 at all values of T. We note that attempts to calculate the temperature dependence of γ in pure superconductors near T_c were made by Ebneth and Tewordt[10] and by Luders[11] using modified Ginzburg–Landau equations containing spatial derivatives of the ordering parameters of order higher than the second. The method employed by us to calculate the field H_{c3} is similar to that employed by Abrikosov[6] and is based on a solution of an equation of the type (1.1), except that we are using a more exact expression for the Green's function G_ω in the presence of a magnetic field (in addition, we consider the case of not only specular but also diffuse reflection of the electrons from the surface of the metal). Unlike [6], we use in the calculation of (1.1) the method of quasiclassical trajectories of Shapoval and de Gennes[9, 12, 13] (the superior bar in formula (1.1) is introduced in the case of diffuse reflection of the electrons from the surface or in the presence of impurities in the volume, and denotes averaging over the statistical elements of the trajectory.

In Sec. 3 we investigate the structure of the superconducting surface layer in the case of a magnetic-field orientation that is inclined with respect to the surface of the superconductor ($\theta \neq 0$). For simplicity, we confine ourselves here to the case $T \to T_c$, but the qualitative picture of the investigated effect is not connected with this assumption. It is shown that in the case of inclined orientation of the magnetic field a superconducting surface layer has a vortical structure, similar to the vortex lattice of Abrikosov[14] below the field H_{c2}, but the period of the corresponding structure is in this case an essential function of the angle of inclination θ, increasing with decreasing θ like $1/\sqrt{\theta}$.

2. TEMPERATURE DEPENDENCE OF THE FIELD H_{c3}

In the present section we consider the question of calculating the temperature dependence of the field H_{c3} in parallel orientation ($\theta = 0$). Assuming the superconductor to occupy the region of half-space $z > 0$ and the magnetic field to be oriented along the y axis, we chose the vector potential with a gauge

$$A_x = Hz, \quad A_y = 0, \quad A_z = 0. \tag{2.1}$$

1. The standard method for calculating the critical fields of the superconductors is the Shapoval–de Gennes quasiclassical-trajectory method.[9, 12] In order for this method to be applicable it is necessary that the distance between the Landau levels $\omega_H = eH/mc$ be small compared with the Fermi energy μ, and the radius of the electron orbit in the magnetic field $r_H = cp_0/eH$ be large compared with the correlation length $\xi \sim v_0/\Delta$. These conditions are always satisfied in fields that are of interest for the theory of superconductivity.

Writing the Gor'kov equation (1.1) in the form ($\hbar = 1$)

$$\Delta^*(\mathbf{r}) = |\lambda| T \sum_\omega \int d\mathbf{r}' K_\omega(\mathbf{r}, \mathbf{r}') \Delta^*(\mathbf{r}'), \tag{2.2}$$

we obtain in the quasiclassical approximation the following expression for the kernel $K_\omega(\mathbf{r}, \mathbf{r}')$[12]

$$K_\omega(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^2} \int_0^\infty dt\, e^{-2|\omega|t} \int d\mathbf{p}\, \delta(\varepsilon_\mathbf{p} - \mu) \times \left\langle \delta(\mathbf{r}' - \boldsymbol{\rho}(t)) \exp\left[\frac{2ie}{c} \int_{\mathbf{r}'}^{\mathbf{r}} \mathbf{A}(\boldsymbol{\rho})\, d\boldsymbol{\rho}\right] \right\rangle, \tag{2.3}$$

where $\boldsymbol{\rho}(t)$ is the equation of the classical trajectory of the particle emerging at the instant of time $t = 0$ from the point $\boldsymbol{\rho}(0) = \mathbf{r}$ and having at this point a velocity $\dot{\boldsymbol{\rho}}(0) = \mathbf{v} = \mathbf{p}/m$. In formula (2.3) we take the sum (integral with respect to dp) over all possible electron trajectories on the Fermi surface. The integration in the argument of the exponential in (2.3) is along the classical trajectory of the electron, directed from the point $\mathbf{r}$ to $\mathbf{r}'$, and the angle brackets denote averaging over all possible trajectories.

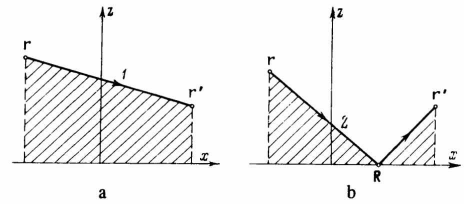

FIG. 1

The type of classical trajectories, in the case of surface superconductivity, can be readily classified (Fig. 1). Trajectories are possible, going directly from the point $\mathbf{r}$ into $\mathbf{r}'$, as well as trajectories corresponding to reflection from the surface at a certain point $\mathbf{R}$. If the vector potential is chosen in the form (2.1), the phase in the argument of the exponential in (2.3) is proportional to the area covered by the classical trajectory of the electron in the (x, z) plane (Fig. 1):

$$\frac{2e}{c} \int_{\mathbf{r}'}^{\mathbf{r}} \mathbf{A}(\boldsymbol{\rho})\, d\boldsymbol{\rho} = \frac{2eH}{c} S(\mathbf{r}, \mathbf{r}'). \tag{2.4}$$

We shall henceforth consider specular and diffuse laws of reflection of electrons from the surface of the metal. In the case of specular reflection, the trajectory is specified uniquely by the value of the initial electron velocity at the point $\mathbf{r}$, and therefore the angle brackets in (2.3) can be left out in this case. In the case of diffuse scattering, the reflection can occur at any point $\mathbf{R}$. It is then necessary to average over the angles of emission of the particle after reflection from the surface.

In accordance with the foregoing, the kernel $K_\omega(\mathbf{r}, \mathbf{r}')$ can be represented in the form of a sum $K_\omega^{(1)} + K_\omega^{(2)}$, where $K_\omega^{(1)}$ corresponds to trajectories of type 1, shown in Fig. 1a, and $K_\omega^{(2)}$ corresponds to trajectories of type 2, shown in Fig. 1b. Proceeding to the calculation of the corresponding contributions, we note that $K_\omega^{(1)}(\mathbf{r}, \mathbf{r}')$ does not depend on the character of the reflection and is determined in accordance with (2.3) by the formula

$$K_\omega^{(1)\mathrm{spec}}(\mathbf{r}, \mathbf{r}') = K_\omega^{(1)\mathrm{diff}}(\mathbf{r}, \mathbf{r}') = \frac{m^3}{(2\pi)^2} \int_0^\infty dt\, e^{-2|\omega|t} \iint_{-\infty}^{\infty} dv_x\, dv_y \times \int_{-z/t}^{\infty} dv_z\, \delta\left(\frac{m\mathbf{v}^2}{2} - \mu\right) \delta(\mathbf{r}' - \mathbf{r} - \mathbf{v}t) \exp\left[\frac{2ie}{c} S(\mathbf{r}, \mathbf{r}')\right], \tag{2.5}$$

where

$$S(\mathbf{r}, \mathbf{r}') = \tfrac{1}{2}(z + z')(x - x').$$

The condition $v_z > -z/t$ separates here the trajectories that do not experience collisions with the plane $z = 0$ at the instant of time t.

The quantity $K_\omega^{(2)}(\mathbf{r}, \mathbf{r}')$, corresponding to the trajectories that collide with the surface, is defined in the case of specular reflection in analogy with (2.5) as

$$K_\omega^{(2)\mathrm{spec}}(\mathbf{r}, \mathbf{r}') = \frac{m^3}{(2\pi)^2} \int_0^\infty dt\, e^{-2|\omega|t} \iint_{-\infty} dv_x\, dv_y \times \int_{-\infty}^{-z/t} dv\, \delta\left(\frac{m\mathbf{v}^2}{2} - \mu\right) \delta(\mathbf{r}' - \boldsymbol{\rho}(t)) \exp\left[\frac{2ie}{c} S(\mathbf{r}, \mathbf{r}')\right], \tag{2.6}$$

and in this case

$$\rho(t)=\mathbf{R}+\bar{\mathbf{v}}(t-\tau),\qquad S(\mathbf{r},\mathbf{r}')=\frac{1}{2}\frac{z^2+z'^2}{z+z'}(x-x'). \tag{2.7}$$

The point $\mathbf{R}$ has coordinates

$$X=x+v_x\tau,\quad Y=y+v_y\tau,\ Z=0,$$

where $\tau = -z/v_z$ is the instant of encounter of the trajectory with the surface, and $\mathbf{v}$ corresponds to replacement of v_z by $-v_z$: $\bar{\mathbf{v}} = (v_x, v_y, -v_z)$. In the case of diffuse reflection, the kernel $K^{(2)}$ is obtained by averaging over all trajectories corresponding to the instant $t > \tau$, with weight $\pi^{-1}\cos\theta$, where θ is the angle characterizing the direction of particle emission after reflection at the point $\mathbf{R}$. We get

$$K_\omega^{(2)\,\mathrm{diff}}(\mathbf{r},\mathbf{r}')=\frac{m^3}{(2\pi)^2}\int_0 dt\, e^{-2|\omega|t}\iint_{-\infty}^{\infty} dv_x\, dv_y\int_{-\infty}^{-z/t} dv_z\,\delta\left(\frac{m\mathbf{v}^2}{2}-\mu\right)$$
$$\cdot\frac{1}{\pi}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\int_0^{2\pi} d\varphi\,\delta(\mathbf{r}'-\boldsymbol{\rho}_1(t))\exp\left[\frac{2ie}{c}S_1(\mathbf{r},\mathbf{r}')\right], \tag{2.8}$$

where $\rho_1(t) = \mathbf{R} + \mathbf{n}v(t-\tau)$, and $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta \times \sin\varphi, \cos\theta)$—unit vector in the direction of particle emission.

The remaining calculations are trivial. For example, in the specular case we obtain on the basis of (2.5) and (2.6), after integration with respect to v_x, v_y, and v_z:

$$K_\omega^{\mathrm{spec}}(\mathbf{r},\mathbf{r}')=\left(\frac{m}{2\pi R}\right)^2 e^{-2|\omega|R/v_0}\exp\left[\frac{ieH}{c}(z+z')(x-x')\right]$$
$$+\left(\frac{m}{2\pi\bar{R}}\right)^2 e^{-2|\omega|\bar{R}/v_0}\exp\left[\frac{ieH}{c}\frac{z^2+z'^2}{z+z'}(x-x')\right], \tag{2.9}$$

where $R = |\mathbf{r}'-\mathbf{r}|$, $\bar{R} = |\mathbf{r}'-\bar{\mathbf{r}}|$, and $\bar{\mathbf{r}}$ is the point specularly reflected with respect to the point $\mathbf{r}$. We note that this expression can be obtained also by the Abrikosov method,[6] if we choose the G-functions in (1.1) in the form (in the quasiclassical approximation)

$$G_\omega(\mathbf{r},\mathbf{r}')=G_\omega^0(\mathbf{r}-\mathbf{r}')\exp\left[\frac{ie}{c}\int_{(1)}\mathbf{A}(s)\,ds\right]$$
$$-G_\omega^0(\mathbf{r}-\bar{\mathbf{r}}')\exp\left[\frac{ie}{c}\int_{(2)}\mathbf{A}(s)\,ds\right], \tag{2.10}$$

where in the first term the integral is taken along the path 1 (Fig. 1a), and in the second along the path 2 (Fig. 1b). Substituting (2.10) in (1.1) and recognizing[6] that crossing terms of the type $G^0_\omega(\mathbf{r}-\mathbf{r}')G^0_\omega(\mathbf{r}-\mathbf{r}')$ give a negligibly small contribution to the integral, we arrive at formulas (2.2) and (2.9).

2. We proceed to solve the equation for the ordering parameter. According to [6,7], $\Delta^*(\mathbf{r})$ should be sought in the form

$$\Delta^*(\mathbf{r})=\Delta(z)\,e^{-ikx}, \tag{2.11}$$

where $\Delta(z)$ is real, and k determines the position of the "center" of the superconducting layer relative to the surface of the metal. The parameter k should then be determined from the condition of maximum field H(k) (H_{C3} = max H(k)).

The integral over space in Eq. (2.2) diverges at small distances. It must be recognized, however, that summation over the frequencies should be terminated at the Debye frequency ω_D. This eliminates the divergence, leading to the appearance of the "large" quantity $\ln\omega_D$, which is cancelled out by the small parameter λ in (2.2). As a result, Eq. (2.2) can be transformed into (see [1,6]):

$$N(0)\Delta(z)\ln\frac{v_0}{\pi eT_c\sigma}=T\sum_\omega\fint_0^\infty dz'\Delta(z')\iint_{-\infty}^{\infty}dx'\,dy'K_\omega(\mathbf{r},\mathbf{r}')\,e^{-ik(x-x')}, \tag{2.12}$$

where $N(0) = mp_0/2\pi^2$ is the density of states on the Fermi surface, and the stroke through the integral sign denotes that the region $|z-z'| < \sigma$ should be excluded, where σ is an infinitesimally small number ($\sigma \to +0$). In the final answer, σ drops out, since the terms $\ln\sigma$ cancel each other in both parts of (2.12).

In the specular case, the equation for $\Delta(z)$ reduces to the form

$$\Delta(z)\ln\frac{v_0}{\pi eT_c\sigma}=\frac{\pi T}{v_0}\fint_0^\infty dz'\Delta(z')\int_0^\infty\frac{\mu\, d\mu}{1+\mu^2}$$
$$\times\left\{\frac{J_0([eHc^{-1}(z+z')-k](z-z')\mu)}{\mathrm{sh}\,[2\pi Tv_0^{-1}|z-z'|\sqrt{1+\mu^2}]}\right.$$
$$\left.+\frac{J_0([eHc^{-1}(z^2+z'^2)/(z+z')-k](z+z')\mu)}{\mathrm{sh}\,[2\pi Tv_0^{-1}|z+z'|\sqrt{1+\mu^2}]}\right\}, \tag{2.13}$$

where J_0 is a Bessel function. When T = 0 this equation can be greatly simplified, since the integrals with respect to μ can be obtained explicitly in this case. Changing over to dimensionless coordinates expressed in units of the characteristic magnetic radius $\rho_0 = (\hbar c/eH)^{1/2}$, we obtain[1)]

$$\Delta(z)\ln\frac{v_0(eH/c)^{1/2}}{\pi eT_c\sigma}=\frac{1}{2}\fint_0^\infty dz'\Delta(z')\left\{\frac{\exp[-|z(z-\alpha)-z'(z'-\alpha)|]}{|z-z'|}\right.$$
$$\left.+\frac{\exp[-|z(z-\alpha)+z'(z'-\alpha)|]}{z+z'}\right\}, \tag{2.14}$$

where $\alpha = k\rho_0$. The value of α should be obtained from the condition that the field $H = H_{C3}$ be maximal. Thus, the problem reduces to finding the largest eigenvalue of Eq. (2.14).

In the diffuse case, the equation for the ordering parameter is obtained from (2.13) by replacing the second term in the curly brackets by the amount

$$\frac{2\pi T}{v_0}\sum_\omega\Phi_\omega(z)\Phi_\omega(z'),$$

where

$$\Phi_\omega(z)=\int_0^\infty\frac{\xi\, d\xi}{(1+\xi^2)^{3/2}}J_0\left(\xi z\left(\frac{eH}{c}z-k\right)\right)\exp\left\{-\frac{2|\omega|}{v_0}z\sqrt{1+\xi^2}\right\}. \tag{2.15}$$

Thus, in the diffuse case part of the kernel $K_\omega(z, z')$, corresponding to reflection from the surface, factors out. In conclusion, we present the form of the integral equation at T = 0 in the diffuse case. Introducing dimensionless variables, just as in Eq. (2.14), we get

$$\Delta(z)\ln\frac{v_0(eH/c)^{1/2}}{\pi eT_c\sigma}=\frac{1}{2}\fint_0^\infty dz'\Delta(z')\frac{\exp[-|z(z-\alpha)-z'(z'-\alpha)|]}{|z-z'|}$$
$$+\int_0^\infty dz'\Delta(z')\int_0^\infty\frac{x\,dx}{(1+x^2)^{3/2}}\int_0^\infty\frac{y\,dy}{(1+y^2)^{3/2}}\frac{J_0(xz(z-\alpha))J_0(yz'(z'-\alpha))}{z\sqrt{1+x^2}+z'\sqrt{1+y^2}} \tag{2.16}$$

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1)In the numberator of the expression under the logarithm sign, e denotes the charge of the electron, and in the denominator it denotes the base of the natural logarithms, but this cannot lead to a misunderstanding, since the electron charge always enters in combination with the magnetic field (eH/c).

3. The obtained expressions are too complicated to admit of an analytic investigation. We therefore present here a numerical calculation, confining ourselves to the case T = 0.

Excluding the infinitesimally small number σ from Eqs. (2.14) and (2.16), and introducing the notation

$$\lambda = \ln \frac{\pi e T_c}{v_0 (eH/c)^{1/2}}, \qquad (2.17)$$

we rewrite these equations in the form

$$\lambda^{\text{spec}} \Delta(z) = L(z)\Delta(z) + \int_0^\infty K_1(z, z') \frac{\Delta(z) - \Delta(z')}{|z - z'|} dz' - \int_0^\infty K_2(z, z') \frac{\Delta(z')}{z + z'} dz', \qquad (2.18)$$

$$\lambda^{\text{diff}} \Delta(z) = L(z)\Delta(z) + \int_0^\infty K_1(z, z') \frac{\Delta(z) - \Delta(z')}{|z - z'|} dz' - \int_0^\infty K_3(z, z') \Delta(z') dz', \qquad (2.19)$$

with

$$K_1(z, z') = \tfrac{1}{2} \exp[-|z(z-\alpha) - z'(z'-\alpha)|], \qquad (2.20)$$

$$K_2(z, z') = \tfrac{1}{2} \exp[-|z(z-\alpha) + z'(z'-\alpha)|], \qquad (2.21)$$

$$K_3(z, z') = \int_0^\infty \Phi_\nu(z) \Phi_\nu(z') d\nu, \qquad (2.22)$$

$$\Phi_\nu(z) = \int_0^\infty \frac{x dx}{(1 + x^2)^{1/2}} J_0(xz(z-\alpha)) e^{-\nu z\sqrt{1+x^2}}, \qquad (2.23)$$

$$L(z) = -K_1(z, 0) \ln z + \int_0^\infty \operatorname{sign}(z' - z) \ln|z' - z| \frac{\partial}{\partial z'} K_1(z, z') dz'. \qquad (2.24)$$

Multiplying both sides of (2.18) and (2.19) by $\Delta(z)$ and integrating with respect to z, we obtain $\lambda^{\text{spec}} = \lambda_1 - \lambda_2$ and $\lambda^{\text{diff}} = \lambda_1 - \lambda_3$, where λ_i are the following integrals:

$$\lambda_1 = \int_0^\infty L(z)\Delta^2(z) dz + \frac{1}{2} \int_0^\infty \int_0^\infty K_1(z, z') \frac{[\Delta(z) - \Delta(z')]^2}{|z - z'|} dz\, dz', \qquad (2.25)$$

$$\lambda_2 = \int_0^\infty \int_0^\infty K_2(z, z') \frac{\Delta(z)\Delta(z')}{z + z'} dz\, dz', \qquad (2.26)$$

$$\lambda_3 = \int_0^\infty F^2(s) ds,$$

$$F(s) = \int_0^\infty dz \Delta(z) \int_0^\infty \frac{x dx}{(1 + x^2)^{1/2}} J_0(xz(z-\alpha)) e^{-sz\sqrt{1+x^2}}. \qquad (2.27)$$

Here $\Delta(z)$ is assumed to be normalized in accordance with the condition

$$\int_0^\infty \Delta^2(z) dz = 1. \qquad (2.28)$$

Since we are interested in the smallest value of λ ($\lambda = \lambda^{\text{spec}}$ and $\lambda = \lambda^{\text{diff}}$), we can calculate it by using a variational method, evaluating the integrals (2.25)–(2.27) for several trial functions $\Delta(z)$. Generally speaking, we obtain here overestimated values of λ, i.e., somewhat underestimated values of the field H_{cp}.

As was noted by Abrikosov,[6] when $T \to T_c$ a similar variational method of solving the Schrödinger equation (the linearized Ginzburg–Landau equation) with trial functions of the type $\exp(-kz^2)$ yields instead of the exact value of the ratio $(H_{c3}/H_{c2})_{T = T_c} = 1.69$, obtained with the aid of rather complicated Weber functions,[7] the value $(1 - 2/\pi)^{-1/2} = 1.66$, which differs by only 2% from the exact value.[2)] In analogy with this, in our case we choose the trial functions at T = 0 in the form

$$\Delta(z) = Ce^{-\beta(z-\gamma)^2}, \qquad (2.29)$$

and obtain C from the normalization condition (2.28).

The integrals (2.25)–(2.27) with this trial function were calculated with the M-20 electronic computer (speed 20 000 operations per second).[3)] By running through a number of values of the parameters α, β, and γ it was found that the minimum value of λ^{spec} is reached near the point $\alpha = 1.0$, $\beta = 0.6$, and $\gamma = 0$, and equals 0.289. The calculation of each point lasts about two minutes. The calculation of λ_3 turned out to be somewhat more complicated. At large values of s, the function F(s) in (2.27) has in asymptotic form $F(s) \approx \Delta(0)/2s$, i.e., the integral $\int_0^\infty F^2(s)ds$ converges quite slowly. Calculation of this integral was carried out to such a value of $s = s_{\max}$, at which the function F(s) differed from its asymptotic form not more than 10%, after which the remaining part of the integral (from $s_{\max}$ to infinity) was calculated with the aid of the asymptotic formula presented above.

The results of the calculations of λ^{spec} and λ^{diff} at certain points are listed in the table. A surprising fact is that the minimum of λ^{diff} is reached at approximately the same point as the minimum of λ^{spec} ($\alpha = 1.0$; $\beta = 0.6$; $\gamma \approx 0$), and amounts to $\lambda^{\text{diff}} = 0.265$. It is seen from the table that the deviation of γ from zero corresponds to somewhat larger values of λ. As follows from [10,11], at z = 0 the derivative $d\Delta/dz$ is in general different from zero if $T \neq T_c$. It is possible that in our case the exact value of γ also differs from zero, but, as seen from the table, this is immaterial for the calculation of H_{c3}, since the corresponding value of λ depends little on γ.

Knowing the value of λ, we can readily find the critical field of the surface superconductivity $H_{c3}(0)$ in accordance with Eq. (2.17). It is more convenient to represent the final result in the form of the ratio H_{c3}/H_{c2} at T = 0. According to Helfand and Werthamer,[2] the critical field H_{c2} at T = 0, obtained by Gor'kov[1] by a variational method, is actually exact. The quantity $H_{c2}(0)$, according to [1,2], for impurity-free superconductors, amounts to

$$\frac{e}{c} H_{c2}(0) = \frac{1}{2\gamma} \left(\frac{\pi e T_c}{v_0} \right)^2, \quad \ln \gamma = C = 0.577. \qquad (2.30)$$

Comparing (2.17) with (2.30) and using the value of λ from the table, we obtain

$$\frac{H_{c3}^{\text{spec}}(0)}{H_{c2}(0)} = 1.99, \qquad \frac{H_{c3}^{\text{diff}}(0)}{H_{c2}(0)} = 2.09. \qquad (2.31)$$

We note that the value of $H_{c3}^{\text{spec}}(0)$ is close to

2) A similar remark is contained also in the book by de Gennes [9].

3) All the numerical calculations were performed by A. A. Motornaya.

α	β	γ	λ spec	λ diff	α	β	γ	λ spec	λ diff
0,9	0,6	0	0,324	0,315	1,1	0,6	0	0,365	0,330
1,0	0,5	0	0,294	0,272	1,0	0,6	+0,1	0,290	0,269
1,0	0,6	0	0,289	0,265	1,0	0,6	−0,1	0,294	0,267
1,0	0,7	0	0,296	0,271					

$$\frac{e}{c} H_{c3}^{\text{spec}}(0) \approx \frac{1}{\gamma}\left(\frac{\pi e T_c}{v_0}\right)^2. \tag{2.32}$$

Thus, the magnitude of the ratio $H_{c3}(T)/H_{c2}(T)$, as well as of $H_{c3}(T)/H_c(t)$ (H_c—thermodynamic critical field) turns out to be temperature dependent, increasing with decreasing T. In the case of specular reflection of the electrons from the surface, this increase from the point $T = T_c$ to the point $T = 0$ is approximately 20%, and in the case of diffuse reflection it is of the order of 25%. In diffuse reflection, the field $H_{c3}(0)$ is approximately 5% higher than the corresponding critical field in specular reflection. This conclusion agrees with the usually encountered situation, when the character of the scattering of the electrons from the surface changes the macroscopic characteristics of the metal only insignificantly.

3. SINGULARITIES OF THE SURFACE SUPERCONDUCTIVITY IN AN OBLIQUE FIELD

The purpose of the present section is to investigate the structure of the solution of the equation for the ordering parameter $\Delta(\mathbf{r})$ in the case of oblique orientation of the magnetic field ($\theta = 0$). Considering for simplicity the case $T \to T_c$, we can go over from the integral equation (1.1) to the Ginzburg–Landau differential equation.[4] Placing the magnetic field in the yz plane at an angle to the surface of the superconductor ($z = 0$) and setting the vector potential equal to

$$A_x = H(z\cos\theta - y\sin\theta), \quad A_y = 0, \quad A_z = 0, \tag{3.1}$$

we obtain in the linear approximation

$$\frac{1}{2m}\left[i\frac{\partial}{\partial x} + \frac{2eH}{c}(z\cos\theta - y\sin\theta)\right]^2\Delta - \frac{1}{2m}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Delta = \alpha\Delta,$$
$$\left(\frac{\partial\Delta}{\partial z}\right)_{z=0} = 0, \tag{3.2}$$

where $\alpha \sim (T_c - T)$. The point at which the nonzero solution of this equation arises first determines the field of the surface superconductivity $H_{c3} = H_{c3}(\theta)$. Unlike the case considered in Sec. 2, we should seek Δ in the form[8, 15]

$$\Delta(\mathbf{r}) = e^{ik_0x}\varphi_0(y, z), \quad (\partial\varphi_0/\partial z)_{z=0} = 0, \tag{3.3}$$

where φ_0 is obtained from the equation

$$\frac{1}{2m}\left[k_0 - \frac{2eH}{c}(z\cos\theta - y\sin\theta)\right]^2\varphi_0 - \frac{1}{2m}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi_0 = \alpha\varphi_0. \tag{3.4}$$

It is clear, at the same time, that Eq. (3.2) is satisfied also by the function

$$\Delta'(\mathbf{r}) = e^{i(k_0+k')x}\varphi_0\left(y + \frac{k'\rho_0^2}{2\sin\theta}, z\right), \quad \rho_0 = \left(\frac{c}{eH}\right)^{1/2}, \tag{3.5}$$

with arbitrary k′. Thus, we encounter here the case of degeneracy characteristic of a superconductor of the second kind below the field H_{c2}.[14] Because of that, in an oblique magnetic field ($\theta \neq 0$), unlike the case of parallel orientation ($\theta = 0$), the superconducting surface layer will have a vortical structure, similar to the Abrikosov vortical lattice. If the angle of inclination of the magnetic field tends to zero, the period of the corresponding vortical lattice greatly exceeds the period of the Abrikosov structure in the field H_{c2}.[4)]

In the general case we should seek the solution for the ordering parameter in the form of a superposition of solutions of the type (3.5)

$$\Delta(\mathbf{r}) = \sum_{n=-\infty}^{\infty} C_n \exp\left\{i\left(k_0 + \frac{2\pi n}{a}\right)x\right\}\varphi_0(y + nb, z), \tag{3.6}$$

where the quantities a and b, which play the role of periods of the vortical structure along the axis x and y, are connected by virtue of (3.5) by the relation ($\Phi_0 = \hbar c/2e$—magnetic-flux quantum)

$$ab\sin\theta = \pi\rho_0^2 = \Phi_0/H, \tag{3.7}$$

which expresses the condition of quantization of the magnetic flux per cell.

According to Abrikosov,[14] the coefficients C_n should satisfy the periodic condition $C_{n+\nu} = C_n$, where ν is a certain integer. The value of ν determines whether the vortex lattice is quadratic or rectangular ($\nu = 1$), triangular ($\nu = 2$), etc. As is well known, in the case of volume superconductivity, the minimum free energy corresponds to a triangular lattice, but the difference in the energy between the quadratic and the triangular lattices is very small (see [9]). For this reason, intending only to estimate the magnitude of the period, we shall consider the case $\nu = 1$, corresponding to a rectangular lattice. The ratio of the periods a/b should then be obtained by minimizing the free energy as a function of this ratio.

When $\nu = 1$ all the coefficients C_n are equal: $C_n = C$. To determine C it is necessary to take into account the nonlinear term $\beta|\mathbf{\Delta}|^2\Delta$ in the Ginzburg–Landau equation (3.2). Writing Δ in the form $\Delta = C\Delta_0 + \Delta_1$, where $\Delta_1 \to 0$ as $H \to H_{c3}$ and Δ_0 is given by

$$\Delta_0(\mathbf{r}) = \sum_{n=-\infty}^{\infty} \exp\left\{i\left(k_0 + \frac{2\pi n}{a}\right)x\right\}\varphi_0(y + nb, z), \tag{3.8}$$

we obtain the following equation for Δ_1:

$$\frac{1}{2m}\left[i\frac{\partial}{\partial x} + \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]^2\Delta_1 - \frac{1}{2m}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Delta_1 - \alpha\Delta_1$$
$$= -\beta C^3|\Delta_0|^2\Delta_0 + \varepsilon C\frac{2eH_0}{mc}(z\cos\theta - y\sin\theta)$$
$$\times\left[i\frac{\partial}{\partial x} + \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]\Delta_0. \tag{3.9}$$

[4)]After this article was written, we learned that similar ideas are contained also in the paper by V. R. Karasik and A. I. Rusinov [16]. Their method, however, does not make it possible to find the period of the corresponding vortical lattice. We take the opportunity to thank V. R. Karasik for reporting his results prior to publication.

Here $H_0 = H_{c3}(\theta)$ is the critical field, H is the external field ($H \to H_0$), which we shall represent in the form $H = (1-\varepsilon)H_0$ with $\varepsilon \to 0$. Equation (3.9) is valid when $\kappa \gg 1$ (κ—Ginzburg–Landau parameter), when the variation of the field due to the superconducting currents can be disregarded compared with the monotonic shift of the field $H_0 - H$ (see [17]) (we note, however, that qualitatively all the obtained results remain valid also when $\kappa \sim 1$; on the other hand, the case $\kappa \ll 1$ is of no interest for the theory of surface superconductivity).

Writing down the condition for the orthogonality of the right side of (3.9) to the solution of the corresponding homogeneous equation, we obtain the following expression for C:

$$C^2 = \frac{\varepsilon \int \left| \left[i\frac{\partial}{\partial x} + \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]\Delta_0 \right|^2 d\mathbf{r}}{m\beta \int |\Delta_0|^4 d\mathbf{r}} \qquad (3.10)$$

In the derivation of (3.10) we have used the relation

$$\int \Delta_0^* \left[i\frac{\partial}{\partial x} + \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]\Delta_0 d\mathbf{r} = 0, \qquad (3.11)$$

which expresses the conditions for the vanishing of the total current in the surface layer at $H = H_0$. This relation can be obtained by a method which is perfectly analogous to that used in [17] in the derivation of (11).

Let us proceed to calculate the free energy. At a given value of the external field H, the quantity

$$G = F - \frac{\mathbf{HB}}{4\pi}V$$

($B = \overline{H}$—induction) should be a minimum, with F given by [4]

$$F = \int d\mathbf{r}\left\{ -\alpha|\Delta|^2 + \frac{1}{2}\beta|\Delta|^4 + \frac{1}{2m}\left|\left(i\nabla + \frac{2e}{c}\mathbf{A}\right)\Delta\right|^2 + \frac{\mathbf{H}^2(\mathbf{r})}{8\pi}\right\}, \qquad (3.12)$$

from which we obtain, with allowance for the Ginzburg–Landau equation

$$G = \int d\mathbf{r}\left\{ -\frac{1}{2}\beta|\Delta|^4 + \frac{1}{8\pi}(\mathbf{H}(\mathbf{r}) - \mathbf{H})^2 - \frac{\mathbf{H}^2}{8\pi}\right\}. \qquad (3.13)$$

When $\kappa \gg 1$, the condition for minimization of this quantity can be represented in the form

$$I_1^2 / I_2 = \max, \qquad (3.14)$$

where

$$I_1 = \int \left| \left[i\frac{\partial}{\partial x} + \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]\Delta_0 \right|^2 d\mathbf{r}, \qquad (3.15)$$

$$I_2 = \int |\Delta_0|^4 d\mathbf{r}. \qquad (3.16)$$

Calculating I_1 and I_2 with the aid of the expansion (3.8) we get

$$I_1 = \frac{1}{b}L_xL_y \int_{-\infty}^{\infty} dy \int_0^{\infty} dz \left[k_0 - \frac{2eH_0}{c}(z\cos\theta - y\sin\theta)\right]^2 \varphi_0^2(y,z), \qquad (3.17)$$

$$I_2 = \frac{1}{b}L_xL_y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dy \int_0^{\infty} dz\, \varphi_0(y,z)\varphi_0(y+mb,z) \times \varphi_0(y+nb,z)\varphi_0(y+(n-m)b,z), \qquad (3.18)$$

where L_x and L_y are the dimensions of the sample in the directions of x and y. The first quantity (I_1) depends on b only like 1/b, and the integral in (3.17) is a certain constant. The dependence of $I_2(b)$ is somewhat more complicated. To calculate the integral I_2, we need the exact functions $\varphi_0(y, z)$, the determination of which is not a simple problem even when $\theta \to 0$ (see [8, 15]). However, in estimating the quantity b, it is sufficient to know only certain general properties of the functions $\varphi_0(y, z)$, which can be established without difficulty on the basis of an analysis of Eq. (3.4). As seen from this equation, $\varphi_0(y, z)$ tends to zero as $y, z \to \infty$, and the characteristic interval in which these functions are essentially different from zero amounts to

$$\Delta z \sim \frac{\rho_0}{\sqrt{\cos\theta}}, \quad \Delta y \sim \frac{\rho_0}{\sqrt{\sin\theta}}, \qquad (3.19)$$

with $\varphi_0(y, z)$ decreasing exponentially outside this interval.

Returning to the calculation of the function $I_2(b)$, we see that when $b \ll \Delta y$ the number of terms which must be taken into account in the sum (3.18) is of the order of $\Delta m \sim \Delta y/b$, $\Delta n \sim y/b$, and the sum itself has an order of magnitude $(\Delta y/b)^2$ (φ_0 is assumed to be normalized to unity). In this case the ratio $I_2(b)/I_1^2(b)$ behaves like 1/b. If, to the contrary, $b \gg \Delta y$, then only one term $m = n = 0$ remains out of the entire sum, as a result of which we get $I_2(b)/I_1^2(b) \sim b$. Thus, the ratio I_2/I_1^2 increase both with decreasing and with increasing b ($b \ll \Delta y$ and $b \gg \Delta y$). It is therefore clear that it should have a minimum (and the quantity (3.14) a maximum) at a certain value $b = b_0 \sim \Delta y \sim \rho_0/\sqrt{\sin\theta}$. From this we conclude on the basis of (3.7) that a is also of the order of $\rho_0/\sqrt{\sin\theta}$. Thus, when $\theta \to 0$ we get an Abrikosov structure with a period that increases asymptotically like

$$a \sim b \sim \frac{\rho_0}{\sqrt{\theta}} \sim \frac{\xi(T)}{\sqrt{\theta}}. \qquad (3.20)$$

To illustrate the foregoing, let us consider a very simple example, when it is possible to calculate (approximately) the function $\varphi_0(y, z)$. To this end we use the already mentioned remark of Abrikosov (see Sec. 2), and seek the solution of (3.4) by a variational method, choosing the trial functions in the form (compare with [8, 15]):

$$\varphi_0(y,z) = Ce^{-y^2/2y_0^2}e^{-z^2/2z_0^2}. \qquad (3.21)$$

Minimizing the "energy" α relative to k_0, y_0, z_0, we arrive at the following values of these parameters:

$$k_0 = \frac{1}{\rho_0}\left(\frac{2\gamma\cos\theta}{\pi}\right)^{1/2}, \quad y_0 = \frac{\rho_0}{\sqrt{2\sin\theta}}, \quad z_0 = \rho_0\left(\frac{\gamma}{2\cos\theta}\right)^{1/2}, \qquad (3.22)$$

which is in agreement with (3.19); γ is the ratio $H_{c3}(\theta = 0)/H_{c2}$, and in the considered approximation $\gamma = (1 - 2/\pi)^{-1/2} = 1.66$. In the same approximation, the value of the derivative $\beta = H_{c3}(dH_{c3}/d\theta)_{\theta=0}$ turns out to be $\beta = -\gamma = -1.66$. For comparison we indicate that an exact value of these quantities amounts to $\gamma_0 = 1.69$[7] and $\beta_0 = -1.35$.[8]

Calculating the integral I_2 with the functions (3.21) and substituting in (3.14), we arrive at the condition

$$xf^2(x) = \min, \quad x = b/y_0, \qquad (3.23)$$

$$f(x) = \sum_{n=-\infty}^{\infty} e^{-n^2x^2/2}, \quad (3.24)$$

which coincide with that obtained by Abrikosov for superconductors of the second kind.[14] The minimum of (3.23) is reached at the point $x = \sqrt{2\pi}$, from which we get, with allowance for (3.7),

$$a = b = \rho_0(\pi / \sin\theta)^{1/2}, \quad (3.25)$$

i.e., in the approximation under consideration the lattice is quadratic, which is not surprising, since the trial functions φ_0 employed by us coincide with the exact solutions of the Ginzburg–Landau equation at $H \approx H_{C2}$.[14]

The character of the vortical state in the surface layer can be clarified by investigating the distribution of the superconducting current near the surface. Substituting the functions (3.21) in formula (3.8) and calculating the current j, we obtain (compare with [14]):

$$j_x = \text{const}\cdot\left[\frac{2e}{m}\left(k_0 - \frac{2z\cos\theta}{\rho_0^2}\right)|\Delta_0|^2 - \frac{e}{m}\frac{\partial}{\partial y}|\Delta_0|^2\right],$$

$$j_y = \text{const}\cdot\frac{e}{m}\frac{\partial}{\partial x}|\Delta_0|^2. \quad (3.26)$$

In view of the relation (3.11) (or formulas (3.22)), the first term in the expression for j_x vanishes in the integration with respect to z. From this we get that "in the mean" the distribution of the current has the same form as in the Abrikosov lattice.[14] Since the periods of the structure (3.8) are much larger than the thickness of the superconducting layer when $\theta \to 0$ ($\delta z \sim \rho_0 \sim \xi$), such an average distribution characterizes to some degree also the distribution of the current as a whole.

Thus, as shown by the foregoing investigation, an Abrikosov structure with a large period: $a \sim b \sim \xi/\sqrt{\theta} \gg \xi(t)$ is produced at small inclination angles ($\theta \to 0$) (see Fig. 2). Unlike the usual Abrikosov lattice, the orientation of which in space (in the plane perpendicular to the applied field) is not distinguished in any way, in this case the directions of the axis are fixed by the projection of the external field H_y.

The existence of a periodic structure of a superconducting surface layer in an inclined field leads to a large number of qualitative effects, one of which can be the possibility of observing the so-called "resistive" effects under conditions of surface superconductivity ($H_{C2} < H < H_{C3}(\theta)$), connected with the dissipative motion of the vortices under the influence of the current flowing parallel to the surface, analogous to the resistive effects in the mixed states of superconductors of the second kind.[18, 19]

In conclusion let us stop to discuss the singularities of the vortical state of a superconducting surface layer in thin films. As shown in the author's earlier paper[17] (see also the analogous paper[20]), in this case, even in the case of parallel orientation of the magnetic field ($\theta = 0$), in not too thin films ($d \gtrsim \xi$), the interference of the superconducting currents near two film surfaces leads to the occurrence of a one-dimensional structure of the field in a film with a period (along the x axis) of the order of $\Delta x_1 \sim \xi$. If we now incline the magnetic field relative to the surface of the film then, as shown earlier, periodicity appears with a period $\Delta x_2 \sim \xi/\sqrt{\theta}$.

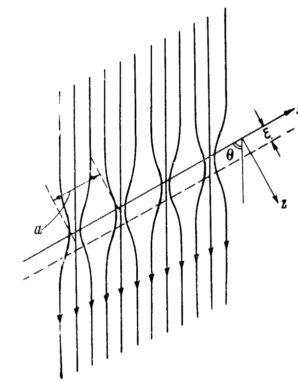

FIG. 2

The multiplicity of the degeneracy of the solution in the case of the film doubles compared with (3.8), so that now Δ has the form (here z is reckoned from the center of the film):

$$\Delta(\mathbf{r}) = C_1\Delta(x, y, z) + C_2\Delta^*(x, -y, -z), \quad (3.27)$$

and considerations analogous to those given in [17] show that $|C_1| = |C_2|$. It is clear that in this case a "competition" will occur between the two periods Δx_1 and Δx_2, and inasmuch as Δx_1 does not depend on θ, while Δx_2 is an essential function of the angle θ, there occur values of the angle at which Δx_2 is a multiple of Δx_1. As a result, different characteristics of the films (for example, their magnetic moments) will oscillate with variation of the angle at small values of the angle. A detailed calculation of this effect, as well as a detailed study of the vortical lattices in the superconducting layer, is beyond the scope of the present article and will be published later.

In conclusion, I take the opportunity to thank A. A. Abrikosov for a discussion of the given work and a number of useful remarks. I am also grateful to A. A. Mortornaya for the numerical calculations connected with the first part of the work.

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