MICROSCOPIC DESCRIPTION OF CRITICAL PHENOMENA

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We have been able to give a basis for and to generalize the earlier proposed phenomenological scaling hypotheses by considering phase transitions using the methods of quantum field theory. We show that the critical behavior of most quantities of physical interest such as the specific heat, correlation radius, magnetic susceptibility, etc., is determined by two unknown parameters which are connected with an infinite number of Feynman diagrams. The parameters are independent of the properties of the binary interaction potential but may change when non-binary interactions are included. These parameters are different for systems of differing symmetry (Bose gas, Ising model, Heisenberg ferromagnet). The behavior of the correlations near the transition point is described not only by the above-mentioned parameters but also by unknown functions. We have calculated the asymptotic behavior of these functions at distances $r \gg r_c$ (where r_c is the correlation radius). Depending on the symmetry of the correlated quantities two kinds of asymptotic behavior may occur: (4.4) or (4.12) in three-dimensional systems and (4.4′) or (4.13) in two-dimensional systems. In the two-dimensional case this result agrees with calculations for the Ising model. We give in the Appendix a simple diagram technique for classical systems such as the Ising model, a lattice of planar dipoles, or a Boltzmann gas.

1. INTRODUCTION

IN the phase transition region fluctuations increase anomalously and interact so strongly that perturbation theory is useless to describe this behavior.

Patashinskiĭ and Pokrovskiĭ [1] were the first to make an attempt to select and sum Feynman diagrams which are important in the phase transition region. However, their methods when applied to the plane Ising model, where an exact solution exists, gave an incorrect answer.[2] Because of this a large number of papers appeared in which phenomenological hypotheses about the structure of the substance in the vicinity of the phase transition were advanced (see the reviews[3, 4]). Accepting these hypotheses one was able to construct a theory with two unknown parameters which were determined by the thermodynamics of the system and by the way the pair-correlations fall off in the transition point. The behavior of most other physical quantities could be expressed in terms of these parameters. Some unknown functions also appeared in the theory which described the behavior of the correlations in the vicinity of the transition.

In the present paper we study the region of the phase transition microscopically using only general properties (such as the Ward identity, Dyson equations, and so on) of many-body systems. We show that all phenomenological laws established earlier follow from these general properties. This result confirms the physical picture which led to the scaling hypotheses.[3,4] A microscopic approach to the problem enables us to proceed further than was possible with a phenomenological analysis. We find the asymptotic behavior of the correlation functions at distances larger than the correlation radius. We elucidate the degree of universality of the unknown parameters and functions. It turns out that the algorithm to obtain the scaling relations is not more complicated than in the phenomenological approach.

2. CORRELATIONS IN THE CRITICAL POINT

We consider the phase transition in the Ising model of arbitrary number of dimensions. We shall use a diagram technique (see [5] and the Appendix of the present paper) for its analysis. The derivation of the general diagram equations is given in the Appendix as it is practically completely the same as the corresponding calculations for a Bose gas given in [1].

In the transition point the equation for the correlation function of the spins $D(\mathbf{k})$ in the momentum representation has the form

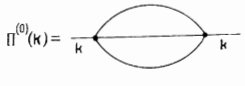
$$D(k) = [\Pi(0) - \Pi(k)]^{-1}. \tag{2.1}$$

$\Pi(\mathbf{k})$ can be expressed as a functional of $D(\mathbf{k})$ as is shown by the following diagram:

(2.2)

Equation (2.1) is valid for $|\mathbf{k}|r_0 \ll 1$ (r_0 is the range of the interaction). To understand the nature of the solutions of Eq. (2.1) we consider its iteration where we take as zeroth approximation the equation

$$D^{-1}(k) = \Pi^{(0)}(0) - \Pi^{(0)}(k), \tag{2.2′}$$



or analytically

$$D^{-1}(k) = -\int \frac{dk_1\,dk_2}{(2\pi)^6}[D(k_1)D(k_2)D(k_1+k_2-k) - D(k_1)D(k_2)D(k_1+k_2)].$$

(Equations such as (2.2′) were solved in Patashinskiĭ and Pokrovskiĭ's paper.[1]) When $|\mathbf{k}|r_0 \sim 1$ [1)] one finds

[1)] When $|\mathbf{k}|r_0 \sim 1$ one must retain in (2.1) the inhomogeneous term—the bare Green function $\sim(\mathbf{k}r_0)^2$. The analysis given below is therefore valid only when $|\mathbf{k}|r_0 \ll 1$.

the solution of (2.2′) from dimensionality considerations. Indeed,

$$[D]^{-1} = [k]^6[D]^3 \tag{2.3}$$

(square brackets indicate the dimensionality). Hence

$$D(\mathbf{k}) \propto |\mathbf{k}|^{-3/2} \quad \text{as} \quad \mathbf{k} \to 0. \tag{2.4}$$

When substituting (2.4) into (2.2′) values of $|\mathbf{k}_1|$ and $|\mathbf{k}_2| \sim |\mathbf{k}| \ll 1/r_0$ are important in the integral and (2.4) is thus indeed the solution of (2.2).

To obtain the next approximation we must substitute (2.4) in the unconsidered diagrams. Logarithmic divergences occur then. Logarithms occur in proportion to the iterations in ever increasing powers and may distort the zeroth approximation (2.4) so that it is no longer recognizable.

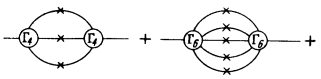
For an analysis of the properties of the exact solution we use a method proposed in a paper by Gribov and Migdal,[6] in which they considered a problem formally analogous to the problem of phase transitions. This method is based upon the unitarity condition for the Green function. In our theory the Green function $D(\mathbf{k}^2)$ is real and there is no unitarity condition for it. However, it can be derived for the analytical continuation of $D(\mathbf{k}^2)$ to negative $\mathbf{k}^2$. Such a possibility is connected with the fact that a field theory with a purely Euclidean metric where $\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2$, is equivalent to a theory with $k^2 = k_x^2 - k_y^2 - k_z^2$ for the Feynman condition for going round poles. Knowing $D^{-1}(\mathbf{k}^2)$ or $\operatorname{Im} D^{-1}(\mathbf{k}^2)$ for negative $\mathbf{k}^2$ we can without any difficulty recover D^{-1} for positive $\mathbf{k}^2$. For instance, if

$$\operatorname{Im} D^{-1}(\mathbf{k}^2) = A(-\mathbf{k}^2)^\alpha \text{ when } \mathbf{k}^2 < 0,$$

we have

$$D^{-1}(\mathbf{k}^2) = A\frac{(\mathbf{k}^2)^\alpha}{\sin \pi\alpha} + \text{const.}$$

The quantity $\operatorname{Im} D^{-1}$ is obtained for $\mathbf{k}^2 < 0$ from the unitarity condition. In a Feynman kind of theory the unitarity condition for it is well known.[7,8] It has the form

$\operatorname{Im} D^{-1}(k) =$  (2.5)

where the imaginary part of the Green function corresponds to the lines with a cross. Analytically

$$\operatorname{Im} D^{-1}(k) = \int |\Gamma_4(k_1, k_2, k - k_1 - k_2)|^2 \times \operatorname{Im} D(k_1) \operatorname{Im} D(k_2) \operatorname{Im} D(k_1 + k_2 - k)\, dk_1\, dk_2 + \ldots \tag{2.5′}$$

In Eqs. (2.5) and (2.1) there is no dimensional parameter which can fix the scale so that D and Γ must be power functions of $\mathbf{k}^2$. The powers or dimensionalities of the Green function D or the many-point diagrams Γ_n must in principle, as was shown in [6], be determined from the unitarity condition and the equation for the vertex part.[11] We have, however, no calculational method to determine them. We shall thus use (2.5) only to express the dimensionalities of the many-point diagrams Γ_n in terms of the dimensionality of the Green function D.

Comparing the dimensionality of the n-th term in Eq. (2.5) with D^{-1} we shall have[2)]

$$D^{-1} \sim \Gamma_n^2 D^{n-1}(k^a)^{n-2}, \tag{2.6}$$

where a is the dimensionality of the space. (We have in (2.6) taken into account the fact that in the term with $n-1$ intermediate particles there are $n-2$ integrations over independent momenta.) If we give the dimensionality of D, assuming that in the coordinate representation $D(r) \propto r^{-\alpha}$ we can use (2.6) and take into account that $D(k) \propto k^{\alpha - a}$ and we find

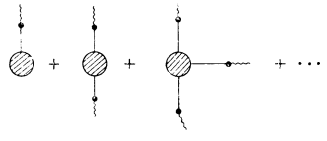
$$\Gamma_n \sim k^{a - n\alpha/2}. \tag{2.7}$$

The contents of Eq. (2.7) consist in the fact that

$$\Gamma_n(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n) = |\mathbf{k}_1|^{a - n\alpha/2} \gamma_n\left(\frac{\mathbf{k}_2}{|\mathbf{k}_1|}, \frac{\mathbf{k}_3}{|\mathbf{k}_1|}, \ldots, \frac{\mathbf{k}_n}{|\mathbf{k}_1|}\right). \tag{2.8}$$

We bear in mind that (2.8) is valid if all $|\mathbf{k}_i| \ll r_0$. The ratios of the $\mathbf{k}_i$ can be arbitrary. Equation (2.8) is of interest because the Γ_n are connected with the correlation functions of n spins which in turn determine the reaction of the system to a magnetic field.

To establish this connection we note that the change in the free energy in a magnetic field is given by the following diagrams:

 (2.9)

The functions $Q_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ defined in Patashinskiĭ and Pokrovskiĭ's paper[9] by the relation

$$Q_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \frac{\delta^n F}{\delta h(\mathbf{r}_1) \ldots \delta h(\mathbf{r}_n)} \tag{2.9′}$$

(F is the free energy of the system in a magnetic field h(r)) are thus connected (in the momentum representation) with the Γ_n by the formula

$$Q_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) = D(\mathbf{k}_1) \ldots D(\mathbf{k}_n) \Gamma_n(\mathbf{k}_1, \ldots, \mathbf{k}_n), \quad \mathbf{k}_1 + \mathbf{k}_2 + \ldots + \mathbf{k}_n = 0. \tag{2.10}$$

Therefore

$$Q_n \sim D^n \Gamma_n \sim k^{a - \alpha n/2 - n(a-\alpha)}. \tag{2.11}$$

The change to the coordinate representation introduces $n-1$ integrations over $d^a k$ and as a result we get

$$Q_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \sim r^{-n\alpha/2} \tag{2.12}$$

(we took into account that $k \sim 1/r$). Written out completely, Eq. (2.12) looks as follows

$$Q_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) = |\mathbf{r}_1|^{-n\alpha/2} q_n\left(\frac{\mathbf{r}_2}{|\mathbf{r}_1|}, \ldots, \frac{\mathbf{r}_n}{|\mathbf{r}_1|}\right). \tag{2.13}$$

The result (2.13) is the same as the phenomenological formula found in [9,10].

The microscopic approach in the transition point based upon the unitarity condition gives thus the formulae of the phenomenological theory.

[2)] In ref. [6] persuasive arguments are adduced to suggest that all terms in the unitarity condition are of the same order. In the problem considered here these arguments can be strengthened if we use the fact that each of the terms in 1.8 [sic!] is positive.

3. VICINITY OF THE PHASE TRANSITION AND THERMODYNAMICS. ALGORITHM FOR THE SCALE RELATIONS

For the critical value of the temperature all terms in the unitarity condition (2.5) had singularities in one point, $k^2 = 0$; these singularities were superimposed upon one another and gave the total singularity of the Green function $D(\mathbf{k}^2)$. When the temperature is slightly raised the position changes. We show in the Appendix that Eq. (2.1) for $T \neq T_c$ has the form

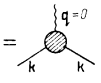
$$D(\mathbf{k},\tau)=\frac{1}{\tau+\Pi(0,0)-\Pi(\mathbf{k},\tau)},\quad \tau\sim\frac{T-T_c}{T_c}. \tag{3.1}$$

Equation (3.1) shows that the Green function does not become infinite at $\mathbf{k} = 0$ when $\tau \neq 0$. Using the field theory language, one can say that when $\tau \neq 0$ the quanta of the emitted field (the field of the fluctuations in the magnetic moment) have a non-vanishing physical mass. The Green function of such a field depends only on the momentum and this physical mass which determines the scaling (see also [7]):

$$D(\mathbf{k})=\frac{1}{|\mathbf{k}|^{a-\alpha}}f\left(\frac{\mathbf{k}^2}{m^2(\tau)}\right). \tag{3.2}$$

(Here, m is the physical mass, the factor $1/|\mathbf{k}|^{a-\alpha}$ was added in order that as $m \to 0$ the function (3.2) changes into the function (2.1).)

The quantity $r_c = 1/m$ is the correlation radius. The problem arises how this radius is connected with τ. So far one thing is clear: $m \to 0$ as $\tau \to 0$. More detailed information about $r_c(\tau)$ can be obtained from the Ward identity. This identity (as shown in the Appendix for the Ising model) has the form

$\mathcal{T}(k,0) =$  $= \frac{\partial D^{-1}}{\partial \tau}$ (3.3)

In the transition point $\mathcal{T}(\mathbf{k}, \mathbf{q})$ must satisfy the unitarity condition and it has thus a well-defined dimensionality γ (we do not evaluate this). If we substitute (3.2) into (3.3) and after integration put $\tau = 0$ we get by equating dimensionalities:

$$[\mathcal{T}]=\frac{[D]^{-1}}{[\tau]}=\frac{[k]^{a-\alpha}}{[\tau]}=[k]^{\gamma},$$
$$[k]=[\tau]^{1/(a-\gamma-\alpha)}\equiv[\tau]^{\beta}=[r_c(\tau)]^{-1},$$
$$r_c(\tau)\sim\tau^{-\beta}. \tag{3.4}$$

The dimensionality of $\mathcal{T}$, as we stated already, is determined by unitarity in the transition point and the same is true of the dimensionality of D. The unitarity condition determines thus, in principle $r_c(\tau)$.

An obvious generalization of the discussions given here leads to the following formula for Q_n in the vicinity of the transition:

$$Q_n(\mathbf{r}_1,\ldots,\mathbf{r}_n)=\frac{1}{|\mathbf{r}_1|^{n\alpha/2}}q_n\left(\frac{\mathbf{r}_2}{|\mathbf{r}_1|},\ldots,\frac{\mathbf{r}_n}{|\mathbf{r}_1|},\frac{\mathbf{r}_1}{r_c(\tau)}\right). \tag{3.5}$$

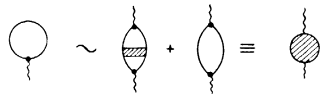
This result again confirms the phenomenological theory.[9,10]

The behavior of the correlations in the neighborhood of the critical point is connected with the thermodynamics of the system. To establish this connection we use the formula for the average energy E in the Ising model:

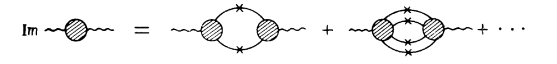
$$E=\int V_{\mathbf{k}}D(\mathbf{k},\tau)\,d\mathbf{k},$$
$$C\sim\frac{\partial E}{\partial\tau}=\int V_{\mathbf{k}}\frac{\partial D(\mathbf{k},\tau)}{\partial\tau}\,d\mathbf{k} \tag{3.6}$$

(C is the specific heat, $V_{\mathbf{k}}$ the interaction potential of the spins). It is impossible to substitute Eq. (3.2) into Eq. (3.6) since in the integrals large momenta $k \sim 1/r_0$, for which (3.2) is incorrect, are important. A renormalization is necessary which makes small momenta $k \ll 1/r_0$ important in the integrals.

To perform this renormalization we note the following. The specific heat C can, thanks to (3.6), be expressed in terms of diagrams of the polarization operator type:

$C = \frac{v}{\partial\tau}$  (3.7)

(the shaded rectangle is the scattering amplitude). The singular part of the polarization operator (3.7) is determined by the unitarity condition:

Im  (3.8)

Values of $k \sim 1/r_c(\tau)$ are important in (3.8) and one can use the self-similar function (3.2). Equating dimensionalities we have

$$C\sim\mathcal{T}^2D^2k^{\alpha}. \tag{3.9}$$

Thanks to (3.3) $\mathcal{T} \sim D^{-1}/\tau$ and hence

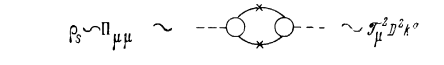
$$C\sim k^{\alpha}/\tau^2. \tag{3.10}$$

Since $k \sim 1/r_c \sim \tau^{\beta}$ we have

$$C\infty\tau^{\beta\alpha-2}. \tag{3.11}$$

Equation (3.11) shows that the specific heat is determined by the function $r_c(\tau)$ and by it alone. Equation (3.11) was assumed earlier[4,5] on the basis of qualitative considerations.

The discussion leading to (3.11) shows how one must obtain general scaling relations. It is necessary to use the unitarity condition and the Ward identity (if it exists). As an example we give the formula for the superfluid component density in helium near the λ-point.[11] The superfluid density is given, as one can easily see, by the vector polarization operator:[12]

$\rho_s \sim \Pi_{\mu\mu}$  $\sim \mathcal{T}_{\mu}^2 D^2 k^{\alpha}$ (3.12)

In the sum over the frequencies ω_n[13] in Eq. (3.12) there remains only the term n = 0 since that is just the one which determines the most singular part of $\Pi_{\mu\mu}$ (see [1]); D in (3.12) may denote any of the two Green functions which characterize the Bose system[13] as one can easily show that their dimensionality is the same; $\mathcal{T}_{\mu}$ is the momentum vertex for which there is a Ward identity:[13]

$$\mathcal{T}_{\mu}=\partial D^{-1}/\partial k_{\mu}\quad(\mu=1,2,3). \tag{3.13}$$

Substituting (3.13) into (3.12) we have

$$\rho_s \sim (D^{-1}/k)^2 D^2 k^a \sim k^{a-2} \sim \tau^{\beta(a-2)}. \qquad (3.14)$$

It is interesting that Eq. (3.14) shows that it is impossible to have a superfluid state in the two-dimensional case[14] since then

$$\rho_s \infty \ln \tau \to \infty \quad \text{as} \quad \tau \to 0,$$

which physically has no meaning.

4. ASYMPTOTIC BEHAVIOR OF THE CORRELATION FUNCTIONS FOR $r \gg r_c(\tau)$

The suggested approach to the problem of phase transitions not only enables us to give a basis of the phenomenological theory but also makes it possible to obtain a number of new results referring to the behavior of the correlation functions in the region $r \gg r_c(\tau) \gg r_0$. In this region the correlation functions are damped exponentially over distances $\sim r_c$. The problem which we pose here consists in finding an exact index of the exponent for correlations of different kinds and in evaluating the factor multiplying the exponential.

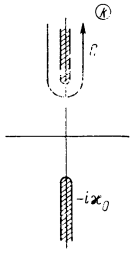
The possibility and the method of solution of such a problem follow from the following considerations. The index of the exponent in the r-representation is determined by the singularity in $|\mathbf{k}|$ in the momentum representation which lies closest to the real axis. The factor is determined by the nature of this singularity. The singular points for negative $\mathbf{k}^2$ after changing to the pseudo-Euclidean metric become singularities of the Green function for positive k^2. The position and nature of the singular points for $\mathbf{k}^2 < 0$ can be studied using the unitarity condition.

We now turn to realizing our planned program. Let $F(\mathbf{r}, \tau)$ be some correlation function and $F(\mathbf{k}^2, \tau)$ its Fourier transform. We have

$$F(r,\tau) = \int e^{i\mathbf{kr}} F(k^2,\tau)\frac{dk}{(2\pi)^3} = \frac{1}{2\pi^2 r}\int_0^\infty k \sin kr\, F(k^2,\tau)\,dk$$

$$= \frac{1}{4\pi^2 i r}\int_{-\infty}^{+\infty} k e^{ikr} F(k^2,\tau)\,dk \qquad (4.1)$$

($r = |\mathbf{r}|$). The singularities in $|\mathbf{k}|$ are situated as



shown in the figure. Changing to integrating along the contour C we have

$$F(r,\tau) = \frac{1}{2\pi^2 r}\int_{\kappa_0}^{\infty} \kappa e^{-\kappa r}\,\mathrm{Im}\,F(-\kappa^2,\tau)\,d\kappa. \qquad (4.2)$$

It is clear that the quantities κ_0 and Im F depend on which correlation function we consider.

We start with the correlations of the magnetic moments, i.e., with the Green function which occurs in the diagrams. The first singularity of such a Green function in $k^2 = -\mathbf{k}^2$ is a simple pole at $k^2 = m^2 = 1/r_c^2(\tau)$. This fact follows from the Lehmann expansion for $D(k^2)$.[8] Hence, the asymptotic behavior of $D(r, \tau)$ has the form

$$D(r,\tau) \infty r^{-1}\exp\{-r/r_c(\tau)\} \quad \text{as} \quad r \to \infty \qquad (4.3)$$

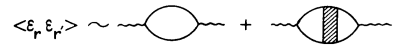
Using (3.5) and (3.4) we can rewrite Eq. (4.3) in the form

$$D(r,\tau) \infty \frac{\tau^{\alpha\beta}}{r\tau^\beta} e^{-r\tau^\beta} \text{ when } r\tau^\beta \gg 1. \qquad (4.4)$$

In a two-dimensional system the multiplying factor in (4.4) would be $r^{-1/2}$. This was confirmed by Kadanoff's calculations[15] in the Ising model near the transition point. The formula from that paper has the form

$$D(r,\tau) \infty \frac{\tau^{1/4}}{(r\tau)^{1/2}} e^{-r\tau} \quad \text{when } r\tau \gg 1. \qquad (4.4')$$

The pole is not always the leading singularity since the residue in it may vanish either by chance or due to symmetry. To explain what was said we consider the energy density correlation function in the Ising model above the transition point. This function is given by the diagrams



(4.5)

One sees easily that the pole term in (4.5)

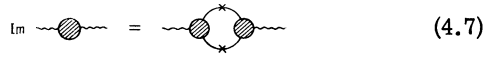
(4.6)

vanishes identically since above the transition ternary vertices are forbidden by the symmetry of the problem.

In a Bose gas the pole term is absent in the particle density correlation function (polarization operator) for the same reason above the transition.

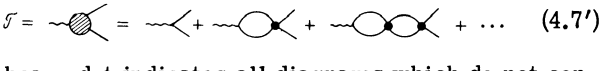
The asymptotic behavior of correlations of this kind is determined by the singularity following after the pole—a two-particle branch point at $k^2 = 4m^2 = 4r_c^2(\tau)$ and is proportional to $\exp(-r/r_c)$.

It is necessary when evaluating the multiplying factor to analyze the nature of the branch point. This analysis should be done strictly using the unitarity condition but a simpler and more straightforward method consists in summing the main diagrams. The correlation function studied can be written in the form



(4.7)

when $4m^2 < q^2 < 16m^2$, where $q^2 = -\mathbf{q}^2$ while



(4.7′)

(The heavy dot indicates all diagrams which do not contain two-particle states.)

We consider the second term in (4.7′), $\mathcal{T}^{(2)}$. Direct calculation gives

$$\mathcal{T}^{(2)} \infty \frac{d^3k}{(k^2 - m^2 + i0)\,((q-k)^2 - m^2 + i0)} \infty \frac{1}{\sqrt{q^2}} \ln\frac{2m - \sqrt{q^2}}{2m + \sqrt{q^2}}$$

$$\sim \ln(4m^2 - q^2) \text{ as } q^2 \to 4m^2. \qquad (4.8)$$

When obtaining Eq. (4.8) we assumed the heavy dot in

(4.7′) to be constant. As $q^2 \to 4m^2$ this is permissible since the diagrams occurring in it have by definition no singularity in that point.

The summation of the geometric series (4.7′), each term of which is logarithmically large as $q^2 \to 4m^2$, gives

$$\mathcal{J} \sim \ln^{-1}(4m^2 - q^2), \quad q^2 \to 4m^2. \tag{4.9}$$

Substituting (4.9) into (4.7) we get

$$\operatorname{Im} F(q^2) \propto \frac{1}{\ln^2(4m^2 - q^2)} \int d^3k\, \delta(k^2 - m^2)\, \delta((q-k)^2 - m^2)$$

$$\sim \frac{\theta(q^2 - 4m^2)}{\ln^2|4m^2 - q^2|} \quad \text{as} \quad q^2 \to 4m^2. \tag{4.10}$$

Equation (2.10) solves the problem of the determination of the nature of the two-particle branch point. Substituting (4.10) into (4.2) we get (as $r \to \infty$)

$$F(r) \propto \frac{1}{r} \int_{2m}^{\infty} e^{-\varkappa r} \frac{d\varkappa}{\ln^2(2m - \varkappa)} \propto \frac{e^{-2mr}}{r^2 \ln^2 r}. \tag{4.11}$$

Using the self-similarity of (3.5) we can rewrite (4.11) in the form

$$F(r, \tau) \sim \frac{\tau^{3\beta-2} \exp(-2r\tau^{\beta})}{(r\tau^{\beta})^2 \ln^2(r\tau^{\beta})} \quad \text{when } r\tau^{\beta} \gg 1. \tag{4.12}$$

When we repeat the discussions given here for the two-dimensional Ising model we get instead of (4.12) the equation

$$F(r, \tau) \sim (r\tau)^{-2} e^{-2r\tau}. \tag{4.13}$$

This is confirmed by a direct evaluation of the energy correlation function (see [15]).

The asymptotic behaviors (4.12) and (4.14) are completely universal and are determined merely by the number of measurements and the symmetry properties of the quantity whose correlations we study. We must bear in mind that the proportionality coefficients in (4.12) and (4.14) may vanish—this does not contradict unitarity. We see no basis for this so long as we are dealing with correlations of scalar quantities such as $\varepsilon_{\mathbf{r}}$. When we consider vectorial transverse correlations threshold zeroes of the vertex parts arise.[8] This changes the asymptotic behavior (4.12). We shall, however, not write down the corresponding formulae as they are only of methodological interest.

5. QUALITATIVE PROPERTIES AND DEGREE OF UNIVERSALITY OF THE CORRELATION FUNCTIONS

We were able in Sec. 3 to connect the thermodynamics of the system and the correlation functions at large distances. Another approach to thermodynamics is possible. In the Ising model the average energy is equal to

$$E = \sum_{\mathbf{r}\mathbf{r}'} V_{\mathbf{r}\mathbf{r}'} \langle \sigma_{\mathbf{r}} \sigma_{\mathbf{r}'} \rangle \tag{5.1}$$

(V is the interaction potential) and is determined by the temperature dependence of the correlation $\langle \sigma_{\mathbf{r}} \sigma_{\mathbf{r}'} \rangle$ for $|\mathbf{r} - \mathbf{r}'| \sim r_0$. In the momentum representation large momenta $\sim 1/r_0$ correspond to distances $\sim r_0$; $D(\mathbf{k}, \tau)$ has for any $\mathbf{k}$ a singularity as $\tau \to 0$. This singularity arises as follows.

Because of unitarity $D(\mathbf{k}^2, \tau)$ has a singularity in $\mathbf{k}^2$ in the points $\mathbf{k}^2 = -n^2m^2 = -n^2/r_c^2(\tau)$ corresponding to the n-particle threshold. If we consider the τ-dependence using the formula $r_c(\tau) \propto \tau^{-\beta}$ we are led to the conclusion that the singularities in τ for fixed $\mathbf{k}^2$ lie in the points

$$\tau_n = (-\mathbf{k}^2/n^2)^{1/2\beta}. \tag{5.2}$$

As $N \to \infty$ the singularities (5.2) bunch at the point $\tau = 0$ and this point is thus singular for all $\mathbf{k}^2$. The singularity arising from the bunching of thresholds creates thermodynamic singularities in (5.1).[3)]

We consider the problem of the dimensionless parameters and functions. Since the bare Green function disappeared in Eqs. (2.1) and (2.5) in the Ising model with binary interactions the critical parameters and functions are independent on the range and strength of this interaction. This fact is confirmed by calculations in the planar Ising model with interactions along two diagonals.[16] The situation is similar to what occurs in quantum field theory. In the latter case, because of the renormalization of the interaction its properties at very small distances determine the magnitude of the (charge and mass) constants and do not affect the form of the Green function at large distances.

The problem of the change in the critical properties of the Ising model when non-binary interactions are included is much more complicated. We give some arguments in favor of the view that there is indeed such a change. We consider the correlation function $\langle \varepsilon_{\mathbf{r}} \varepsilon_{\mathbf{r}'} \rangle$ ($\varepsilon_{\mathbf{r}}$ is the energy density). This function must have singularities at the thresholds in $\mathbf{k}$-space when $\mathbf{k}^2 = -(2n)^2/r_c^2(\tau)$. A direct evaluation of this function in the planar Ising lattice shows that all thresholds with $n > 1$ disappear; the only singularity is for $\mathbf{k}^2 = -4/r_c^2(\tau)$.[4)] We assume that this property is connected with the specific form of the interaction $\ln \cosh \varphi$ (see Appendix) so that the correlation function $\langle \varepsilon_{\mathbf{r}} \varepsilon_{\mathbf{r}'} \rangle$ cannot be universal. Of course, it is impossible to exclude the opposite possibility completely.

We note in conclusion that we see no reasons for expecting that the critical properties of systems with different kinds of symmetry (the Ising model, Bose gas, Heisenberg ferromagnet) should be the same. The same conclusion was earlier reached by Vaks and Larkin[2] on the basis of other considerations.

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APPENDIX

We consider the partition function of the Ising model with an arbitrary interaction potential:

[3)] In the planar Ising model the Green function has for all $\mathbf{k}$ a singularity $\tau \ln \tau$. [16] This is, as we showed above, connected with the logarithmic behavior of the specific heat in this model. In the general case the singularity of the Green function has the form $\tau^{3\beta-1}$.

[4)] This follows from the following considerations. The planar Ising lattice is equivalent to a perfect Fermi gas. [14] The correlation function $\langle \epsilon_{\mathbf{r}} \epsilon_{\mathbf{r}'} \rangle$ is the polarization operator of this perfect gas and has thus only the two-particle threshold.

$$Z=\sum_{(\sigma_r)}\exp\Big\{\frac{1}{2}\sum_{r,r'}\beta V_{rr'}\sigma_r\sigma_{r'}\Big\},\quad \sigma_r=\pm 1. \tag{A.1}$$

To obtain a diagram technique we rewrite (A.1) in the form

$$Z=\sum_{(\sigma_r)}\Big\{e^{\Delta}\prod_r\exp(\sigma_r\varphi_r)\Big\}_{\varphi_r=0}=\Big\{e^{\Delta}\prod_r[\exp(\varphi_r)+\exp(-\varphi_r)]\Big\}_{\varphi_r=0}$$
$$=e^{\Delta}\prod_r\exp(\ln\operatorname{ch}\varphi_r)\,|_{\varphi=0}=e^{\Delta}\exp\Big(\sum_r\ln\operatorname{ch}\varphi_r\Big)\Big|_{\varphi_r=0},$$
$$\Delta\equiv\frac{1}{2}\sum_{r,r'}\beta V_{rr'}\frac{\partial}{\partial\varphi_r}\frac{\partial}{\partial\varphi_{r'}}. \tag{A.2}$$

Comparing (A.2) with the well-known equations of quantum field theory we check that the problem is reduced to an evaluation of the S-matrix of a scalar field with a bare propagator $\beta V_{\mathbf{rr'}}$ and a self force ln cosh φ.

The correlation function $\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r'}}\rangle$ is proportional to the Green function $\langle\varphi_{\mathbf{r}}\varphi_{\mathbf{r'}}\rangle$. To evaluate $\langle\varphi_{\mathbf{r}}\varphi_{\mathbf{r'}}\rangle$ we can use a Feynman diagram technique. The function $\langle\varphi_{\mathbf{r}}\varphi_{\mathbf{r'}}\rangle$ is given by the diagrams in (2.2). The bare propagator $\beta V_{\mathbf{rr'}}$ is associated with the lines. The vertex with n tails is found by n-fold differentiation of ln cosh φ for $\varphi=0$. If in a diagram there is an intersection containing m lines, we must divide it by m! to take identical behavior into account. The vertex coordinates $\mathbf{r_i}$ run through all points of the lattice. If we change to the momentum representation[5)] and introduce the self-energy part Π the connection between D and Π is given by the equation

$$D(\mathbf{k})=\beta V_{\mathbf{k}}+\beta V_{\mathbf{k}}\Pi(\mathbf{k})\beta V_{\mathbf{k}}+\ldots=\frac{\beta V_{\mathbf{k}}}{1-\beta V_{\mathbf{k}}\Pi}=\frac{1}{T/V_{\mathbf{k}}-\Pi(\mathbf{k},T)}, \tag{A.3}$$

where $T=1/\beta$. The transition temperature is determined by the equation

$$T_c/V_0=\Pi(0,T_c). \tag{A.4}$$

Using (A.4) we can write (A.3) in the form

$$D(\mathbf{k},T)=\Big\{\frac{T}{V_{\mathbf{k}}}-\frac{T_c}{V_0}-[\Pi(\mathbf{k},T)-\Pi(0,T_c)]\Big\}^{-1}. \tag{A.5}$$

Introducing the variable $\tau=(T-T_c)/T_c$ we can for small $\mathbf{k}^2$ and τ write in place of (A.5)

$$D(\mathbf{k},\tau)=\{a\tau+b\mathbf{k}^2-[\Pi(\mathbf{k},\tau)-\Pi(0,0)]\}^{-1}. \tag{A.6}$$

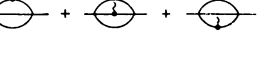
(a and b are constants). When $\tau=0$ we can neglect the term $b\mathbf{k}^2$ in (A.6) compared to Π and we get the result (2.1):

$$D(\mathbf{k},0)=[\Pi(0)-\Pi(\mathbf{k})]^{-1}. \tag{A.7}$$

To derive the Ward identity (3.3) it is sufficient to note that

$$\frac{\partial D_0(\mathbf{k},\tau)}{\partial\tau}\equiv\frac{\partial}{\partial\tau}\frac{1}{a\tau+b\mathbf{k}^2}\sim D_0^2(\mathbf{k},\tau). \tag{A.8}$$

When differentiating the diagrams for Π with respect to τ each line in turn splits into two because of (A.8), for instance:

$\frac{\partial}{\partial\tau}$  (A.9)

5)We take the integrals over the momenta from zero to $2\pi/r_0$ where r_0 is the size of the cell.

The diagrams obtained are diagrams for $\mathcal{T}(\mathbf{k},0)$ as one should prove.

Applying a method analogous to the one used to derive (A.2) we can derive a diagram technique in any classical system. For the dipole lattice of [2] we have

$$Z=\int\prod_r d\mathbf{n}_r\exp\Big(\frac{1}{2}\sum_{r,r'}\beta V_{rr'}\mathbf{n}_r\mathbf{n}_{r'}\Big)=\Big[e^{\hat{\Delta}}\prod_r\int\exp(\mathbf{n}_r\varphi_r)\,d\mathbf{n}_r\Big]_{\varphi_r=0}$$
$$=\Big\{e^{\hat{\Delta}}\exp\Big(\sum_r\ln[I_0(|\varphi_r|)]\Big)\Big\}_{\varphi_r=0},$$
$$\hat{\Delta}=\frac{1}{2}\sum_{r,r'}\beta V_{rr'}\frac{\partial}{\partial\varphi_r}\frac{\partial}{\partial\varphi_{r'}}$$

(I_0 is a Bessel function of imaginary argument). This is the theory of the complex field $\psi=(\varphi_x+i\varphi_y)/\sqrt{2}$ with propagator $\beta V_{\mathbf{rr'}}$ and interaction energy

$$\ln I_0(\sqrt{\psi^+\psi})=\tfrac{1}{2}\psi^+\psi+\ldots$$

For the Boltzmann gas

$$Z=\sum_N\frac{\xi^N}{N!}\int\exp\Big\{-\frac{\beta}{2}\sum_{i,j}V(\mathbf{r}_i-\mathbf{r}_j)\Big\}\prod_{i=1}^N d\mathbf{r}_i$$
$$=e^{\Delta}\sum_N\frac{\xi^N}{N!}\int\exp\Big\{\int\rho_N(\mathbf{r})\varphi(\mathbf{r})\,d\mathbf{r}\Big\}\prod_{i=1}^N d\mathbf{r}_i=e^{\Delta}\sum_{N=0}^{\infty}\frac{\xi^N}{N!}\Big(\int e^{\varphi(\mathbf{r})}d\mathbf{r}\Big)^N,$$

where

$$\Delta=-\frac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'\,\beta V(\mathbf{r}-\mathbf{r}')\frac{\delta}{\delta\varphi(\mathbf{r})}\frac{\delta}{\delta\varphi(\mathbf{r}')}$$
$$\xi=(m/2\pi\beta)^{3/2}e^{\beta\mu},\qquad \rho_N=\sum_{i=1}^N\delta(\mathbf{r}-\mathbf{r}_i)$$

(μ is the chemical potential). In this case the propagator will thus be $-\beta V(\mathbf{r}-\mathbf{r}')$ and the potential energy ξe^{φ}.

In all three cases the diagram technique is very convenient to consider phase transitions.

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