ELECTRON SOUND IN METALS

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It is shown that macroscopic oscillations similar to second sound in dielectrics may propagate in pure metals at low temperatures. These oscillations essentially constitute sound in the electron gas, although the interaction between the electrons occurs as a result of exchange of thermal phonons. Electron sound is possible under certain conditions which, in the case of closed Fermi surfaces, reduce to equality of the number of electrons and holes. The phase velocity of the sound significantly depends on the topology of the Fermi surface. Damping of the sound is also investigated.

INTRODUCTION

LANDAU was the first to point to the existence of undamped temperature waves in liquid He II, called second sound. Microscopically, second sound is "ordinary" sound in a phonon gas. This point of view, naturally, leads to the question of the possible existence of phonon second sound in solids, a question frequently discussed in the literature. This phenomenon was recently observed experimentally by Ackerman, Bertman, Fairbank, and Guyer in crystalline He^4 [1] (their paper contains also references to theoretical work).

For second sound to exist it is necessary that the normal collisions, in which the quasimomentum of the phonons is conserved, be much more probable than the processes in which the quasimomentum is not conserved (scattering by lattice defects, umklapp processes, etc.). Second sound exists in the interval of frequencies ω bounded from above by the frequency of the normal collisions τ_N^{-1} and from below by the frequency of collisions with momentum loss τ_U^{-1}:

$$\tau_U^{-1} \ll \omega \ll \tau_N^{-1}. \tag{1}$$

As noted by Gurevich and Shklovskiĭ [2], phonon second sound in semiconductors is possible only under the additional condition that the number of electrons differ little from the number of holes. The propagation of second-sound wave is not accompanied in this case by appreciable oscillations of the density of the electric charge and by appearance of current, so that the damping connected with the conductivity of the medium is small. The characteristic velocity of second sound in semiconductors, just as in dielectrics, is of the same order as the phonon velocity, i.e., the velocity of ordinary (first) sound.

In the present paper we consider oscillations of the second-sound type in metals. As will be shown below, at sufficiently low temperatures these oscillations constitute sound in an electron gas, although the electron-electron interactions which make the sound possible is realized by exchange of thermal phonons. This occurs because at low temperatures the phonon mean free path relative to collisions with electrons, $l_{pe} \sim \Theta/T$ (Θ - Debye frequency), is much shorter than the electron free path relative to collisions with phonons, $l_{ep} \sim (\Theta/T)^5$. Therefore the phonon emitted by the electron will be absorbed by another electron within a very short time after covering a very short distance. Thus, the electron-phonon interaction leads to effective interelectron collisions (for details see [3]). The characteristic second-sound velocity V is then, naturally, of the same order as the electron Fermi velocity: $V \sim v_F$, although at a certain special topology of the Fermi surface we have $V \sim v_F T/\epsilon_F \ll v_F$ (T - temperature, ϵ_F - Fermi energy). With increasing temperature, the contribution of the phonons becomes appreciable and the sound ceases to be purely electronic.

A necessary condition for the existence of second sound in a metal is the equality of the number of electrons and holes (similar limitations exist in the case of open surfaces). In a metal, the conditions under which the normal collisions prevail over the collisions with umklapp become complicated and depend on the topology of the Fermi surfaces. If the Fermi surfaces are closed and the Brillouin zone can be chosen such that its boundaries do not cross the Fermi surfaces, then the second sound can propagate at all directions in the lattice. On the other hand, if the surfaces are open, then the second sound can propagate only in directions perpendicular to the directions of the openings. Of course, condition (1) on the frequency, as well as analogous conditions on the wavelength and the mean free path, must be satisfied.

SOUND DISPERSION LAW

We write down the system of kinetic equations for the electron ($f(\mathbf{r}, \mathbf{p}, t)$) and phonon ($N(\mathbf{r}, \mathbf{q}, t)$) distribution functions

$$\frac{\partial f}{\partial t} + (\mathbf{v}\nabla) f + e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} = I_{ep}^N(f, N) + I_e^U(f, N), \tag{2}$$

$$\frac{\partial N}{\partial t} + (\mathbf{s}\nabla) N = I_{ep}^N(N, f) + I_p^U(N, f). \tag{3}$$

Here $\mathbf{E}$ - electric field, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ and $\mathbf{s} = \partial\Omega/\partial\mathbf{q}$ - velocities of the electron and phonon, and $\epsilon(\mathbf{p})$ and $\Omega(\mathbf{q})$ - their energies. The operators I_{ep}^N and I_{pe}^N describe the normal electron-phonon and phonon-

electron collisions. The terms I^U correspond to collisions with loss of quasimomentum.

In Eqs. (2) and (3), no account was taken of the interelectron collisions, which are negligible in typical metals, or of interphonon collisions, since at low temperatures phonon-electron collisions are much more probable (see, e.g., [4], p. 159).

We are interested in cases when the terms I^U are small compared with I^N, i.e., the metal is sufficiently pure, and the umklapp processes are much less probable than normal collisions. For closed Fermi surfaces, such a situation is ensured by the exponentially small probability of the umklapp processes at low temperatures. In the case of open Fermi surfaces, we shall be interested in what follows only in the case when the surface is closed in certain directions. In these directions, the probability of the unklapp processes is exponentially small, as before, and the corresponding component of the total quasimomentum is conserved with exponential accuracy.

In the case when the surfaces are closed but cross, the boundaries of the Brillouin zone, the umklapp processes, generally speaking, are just as probable as the normal collisions. However, the very subdivision into N and U processes is arbitrary, since it depends on the choice of the unit cell of the reciprocal lattice. From the physical point of view it is important whether it is possible to choose the unit cell such that its faces do not cross the Fermi surfaces.

Closed Surfaces

We start with the simplest case of closed Fermi surfaces which do not intersect the boundaries of the Brillouin zone. Under the conditions of interest to us, the kinetic equation can be solved by successive approximations. This signifies in fact expansion in powers of the small parameters

$$\omega\tau_N \ll 1, \quad 1/\omega\tau_U \ll 1, \quad \tau_N/\tau_U \ll 1, \quad l_N/\lambda \ll 1, \quad \lambda/l_U \ll 1, \quad l_N/l_U \ll 1. \tag{4}$$

For the distribution function

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \ldots, \quad N = N^{(0)} + N^{(1)} + N^{(2)} + \ldots$$

we obtain a chain of equations that are linearized in the deviation from equilibrium (see [3]):

$$I_{ep}^N(f^{(0)}, N^{(0)}) = 0, \quad I_{pe}^N(N^{(0)}, f^{(0)}) = 0, \tag{5}$$

$$\frac{df^{(0)}}{dt} + e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial\varepsilon} = I_{ep}^N(f^{(1)}, N^{(1)}), \quad \frac{dN^{(0)}}{dt} = I_{pe}^N(N^{(1)}, f^{(1)}), \tag{6}$$

$$\frac{df^{(1)}}{dt} = I_{ep}^N(f^{(2)}, N^{(2)}) + I_{ep}^U(f^{(0)}, N^{(0)}),$$

$$\frac{dN^{(1)}}{dt} = I_{pe}^N(N^{(2)}, f^{(2)}) + I_{pe}^U(N^{(0)}, f^{(0)}). \tag{7}$$

Here

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + (\mathbf{v}\nabla)f, \quad \frac{dN}{dt} \equiv \frac{\partial N}{\partial t} + (\mathbf{s}\nabla)N.$$

Solutions of the zeroth-approximation equations (5), as is well known, are the drift functions

$$f^{(0)} = f_0\left(\frac{\tilde{\varepsilon} - \mathbf{pu} - \delta\mu}{T(1+\vartheta)}\right) \approx f_0(\tilde{\varepsilon}) - \frac{\partial f_0}{\partial\varepsilon}(\mathbf{pu} + \delta\mu + \tilde{\varepsilon}\vartheta), \tag{8}$$

$$N^{(0)} = N_0\left(\frac{\Omega - \mathbf{qu}}{T(1+\vartheta)}\right) \approx N_0(\Omega) - \frac{\partial N_0}{\partial\Omega}(\mathbf{qu} + \Omega\vartheta), \tag{9}$$

corresponding to local equilibrium of the system of quasiparticles; this equilibrium is established under the influence of the normal collisions. The quantity $\mathbf{u}(\mathbf{r}, t)$ has the meaning of the velocity of ordered motion, while $T\vartheta(\mathbf{r}, t)$ denotes the deviation of the local temperature from the temperature T of the thermostat, and $\delta\mu(\mathbf{r}, t)$ denotes the change of the chemical potential of the electrons[1)]

$$f_0(\tilde{\varepsilon}) = [e^{\tilde{\varepsilon}/T} + 1]^{-1}, \quad N_0(\Omega) = [e^{\Omega/T} - 1]^{-1}, \quad \tilde{\varepsilon} = \varepsilon - \mu.$$

The drift solution (8), in view of the equality of the number of electrons and holes, does not lead to the occurrence of an electric current (or charge). Indeed, using the fact that the dispersion law $\epsilon(\mathbf{p})$ is even, we obtain for the electric current density

$$\mathbf{j}^{(0)} = e(n_- - n_+)\mathbf{u},$$

where n_- and n_+ are the numbers of electrons and holes, inasmuch as for closed surfaces (which do not intersect the boundaries of the Brillouin zone) we have

$$\frac{2}{h^3}\int d\mathbf{p}\, p_i v_k\left(-\frac{\partial f_0}{\partial\varepsilon}\right) = (n_- - n_+)\delta_{ik}. \tag{10}$$

Current is produced when $n_+ = n_-$ only as the result of collisions with loss of quasimomentum and of spatial and temporal inhomogeneity of the parameters of the local equilibrium u, ϑ, and $\delta\mu$, i.e., in the next higher approximation in the small parameters (4). Accordingly the term with the electric field is included in (6).

The first-approximation equation (6) assumes the following form:

$$-\frac{\partial f_0}{\partial\varepsilon}[\dot{\mathbf{pu}} + (\mathbf{v}\nabla)(\mathbf{pu}) + \delta\dot{\mu} + (\mathbf{v}\nabla)\delta\mu + \tilde{\varepsilon}\dot{\vartheta} + \tilde{\varepsilon}(\mathbf{v}\nabla)\vartheta] = I_{ep}^N(f^{(1)}, N^{(1)}),$$

$$-\frac{\partial N_0}{\partial\Omega}[\dot{\mathbf{qu}} + (\mathbf{s}\nabla)(\mathbf{qu}) + \Omega\dot{\vartheta} + \Omega(\mathbf{s}\nabla)\vartheta] = I_{pe}^N(N^{(1)}, f^{(1)}). \tag{11}$$

Let us write the conditions for the solvability of these equations, which are the consequence of the conservation of the number of electrons, of the total energy, and of the quasimomentum in normal collisions:

$$\int d\mathbf{p}\, I_{ep}^N = \int d\mathbf{p}\,\varepsilon I_{ep}^N + \int d\mathbf{q}\,\Omega I_{pe}^N = \int d\mathbf{p}\,\mathbf{p} I_{ep}^N + \int d\mathbf{q}\,\mathbf{q} I_{pe}^N = 0. \tag{12}$$

Here and henceforth we imply summation over the electron bands and over the phonon polarizations. We obtain the following system of equations describing the quantities $\delta\mu$, ϑ, and u in the zeroth approximation in the small parameters (4):

$$\delta\dot{\mu}\langle 1\rangle + \dot{\vartheta}\langle\tilde{\varepsilon}\rangle = 0,$$

$$\delta\dot{\mu}\langle\tilde{\varepsilon}\rangle + \dot{\vartheta}\{\langle\tilde{\varepsilon}^2\rangle + \langle\Omega^2\rangle\} + {}^1/_3\,\mathrm{div}\,\mathbf{u}\{\langle\mathbf{pv}\tilde{\varepsilon}\rangle + \langle\mathbf{qs}\Omega\rangle\} = 0, \tag{13}$$

$$\frac{1}{3}\frac{\partial\vartheta}{\partial x_i}\{\langle\mathbf{pv}\varepsilon\rangle + \langle\mathbf{qs}\Omega\rangle\} + \dot{u}_k\{\langle p_i p_k\rangle + \langle q_i q_k\rangle\} = 0.$$

We have introduced here the notation

[1)] We assume for the time being that the collisions with the phonons can cause the electron to go over from any electron or hole surface to any other surface. Only the total number of electrons is conserved here.

$$\langle\varphi(\mathbf{p})\rangle \equiv \frac{2}{h^3}\int d\mathbf{p}\,\varphi(\mathbf{p})\left(-\frac{\partial f_0}{\partial\varepsilon}\right), \quad \langle\psi(\mathbf{q})\rangle \equiv \frac{1}{h^3}\int d\mathbf{q}\,\psi(\mathbf{q})\left(-\frac{\partial N_0}{\partial\Omega}\right),$$

and the integration is carried out over the Brillouin zone. The fact that the electron and phonon mean values are denoted by the same bracket cannot lead to any misunderstanding; $\langle 1\rangle \equiv 2h^{-3}\int d\mathbf{p}\,(-\partial f_0/\partial\epsilon)$, and a corresponding expression for phonons is not encountered below. In deriving (13) we used the central symmetry of the dispersion laws, and also the following properties of the mean values:

$$\langle p_i v_k\rangle = \delta_{ik}(n_+ - n_-) = 0,$$

$$\langle p_i v_k\tilde{\varepsilon}\rangle = \frac{\delta_{ik}}{3}\langle \mathbf{pv}\tilde{\varepsilon}\rangle, \quad \langle q_i s_k\Omega\rangle = \frac{\delta_{ik}}{3}\langle \mathbf{qs}\Omega\rangle.$$

In order to find the dispersion of the second sound, we seek the solution of the system (13) in the form $\delta\mu$, ϑ, and $\mathbf{u} \sim \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. For the velocity $V = \omega/k$ we obtain

$$V^2 = \frac{1}{9}\frac{(\hat{\alpha}^{-1})_{\varkappa\varkappa}(\langle \mathbf{pv}\tilde{\varepsilon}\rangle + \langle \mathbf{qs}\Omega\rangle)^2}{\langle\varepsilon^2\rangle + \langle\Omega^2\rangle - \langle\tilde{\varepsilon}\rangle^2/\langle 1\rangle}, \tag{14}$$

where

$$\alpha_{ik} = \langle p_i p_k\rangle + \langle q_i q_k\rangle, \qquad \varkappa = \mathbf{k}/k.$$

The relations between the amplitudes ("polarization" of the wave) are as follows:

$$\delta\mu = -\vartheta\frac{\langle\tilde{\varepsilon}\rangle}{\langle 1\rangle}, \quad u_i = \vartheta\frac{(\hat{\alpha}^{-1})_{i\varkappa}}{3V}(\langle \mathbf{pv}\tilde{\varepsilon}\rangle + \langle \mathbf{qs}\Omega\rangle). \tag{15}$$

Open Surfaces

For concreteness, we consider the case when the umklapp processes are impossible in some direction z. This means that the faces of the Brillouin zone, which intersect the Fermi surface, should be parallel to the z axis. It is clear the the drift solution (8) is now valid only if the drift velocity u is directed along the z axis, for only in this direction is the quasimomentum conserved. (Formally this is expressed by the fact that the terms describing the unklapp processes in directions perpendicular to the z axis are not small and should be taken into account in Eqs. (5)–(7) together with the terms I^N.)

Let us ascertain under which conditions the drift solution does not lead to the occurrence of the current $\mathbf{j} = e\langle p_z\mathbf{v}\rangle u_z$. (For open surfaces, the condition $n_+ = n_-$ becomes meaningless.) Let us assume that the direction z has a sufficiently high symmetry, so that the drift current is directed also along this axis; $\langle p_z v_k\rangle \sim \delta_{zk}$. (It is sufficient, for example, to have two vertical symmetry planes. This case is realized in a simple hexagonal lattice.) It is easy to show that

$$\langle p_z v_z\rangle = \frac{2}{h^3}\int f_0\,d\mathbf{p} - \frac{2}{h^3}\int dS_z f_0 p_z,$$

where $d\mathbf{S}$ is the element of the surface of the Brillouin zone. The first term, obviously, is simply the total number of electrons n. The second term differs from zero only for the "hole" bands, in which the states adjacent to the faces and not intersecting the Fermi surface are occupied, and therefore this term equals the product of the number of states in the Brillouin zone N by the number of hole bands σ. As a result we get

$$j_z = e(n - \sigma N)u_z.$$

Thus, the condition for the vanishing of the current in the case of drift has a simple physical meaning: the total number of electrons equals the number of all the states σN in the hole bands. In the case of closed surfaces we get from this, obviously, that $n_- = n_+$.

Under the conditions considered above, second sound can propagate along the direction z of the opening. The dispersion equation is obtained in analogy with the foregoing, with account taken of the fact that now only the z-component of the quasimomentum is conserved. We present the result for the velocity and polarization of the wave:

$$V^2 = \frac{(\langle p_z^2\rangle + \langle q_z^2\rangle)^{-1}\{\langle p_z v_z\tilde{\varepsilon}\rangle + \langle q_z s_z\Omega\rangle\}^2}{\langle\tilde{\varepsilon}^2\rangle + \langle\Omega^2\rangle - \langle\tilde{\varepsilon}\rangle^2/\langle 1\rangle}$$

$$\delta\mu = -\vartheta\frac{\langle\tilde{\varepsilon}\rangle}{\langle 1\rangle}, \quad u_z = \frac{\vartheta}{V}\frac{\langle p_z v_z\tilde{\varepsilon}\rangle + \langle q_z s_z\Omega\rangle}{\langle p_z^2\rangle + \langle q_z^2\rangle}. \tag{16}$$

We have used the fact that by virtue of the assumed symmetry properties

$$\langle p_z v_k\tilde{\varepsilon}\rangle \sim \delta_{kz}. \tag{17}$$

In perfect analogy, we can consider the case when the z axis is the only direction of the opening. The wave vector of the sound $\mathbf{k} = k\boldsymbol{\varkappa}$ is located in this case in the xy plane, and the result is obtained from (16) by replacing the index z by κ.

Closed Surfaces Intersecting the Brillouin-zone Boundaries

As already mentioned, in this case it is important that it is possible to choose the unit cell in the p-space in such a way that its boundaries no longer intersect the Fermi surfaces. It is clear that the dispersion law can no longer be assumed centrally symmetrical and, for example, the mean values $\langle\mathbf{p}\rangle$, $\langle p_i p_k v_l\rangle$, etc., taken over the new cell, differ from zero. We note that the mean values of the periodic functions remain unchanged, for example, $\langle\mathbf{v}\rangle = 0$, etc.

In this section we shall take into account the fact that a multiply-connected Fermi surface can consist of individual surfaces (or groups of surfaces) separated from one another by distances much larger than the temperature momentum of the phonon. Then the electrons are practically unable to go from one group to the other, and the particle-number conservation law is satisfied for each group separately (with exponential accuracy). In the preceding section it was tacitly assumed that there is only one group of Fermi surfaces that are close to one another.

We break down the unit cell into region, each of which contains only one of the isolated groups. Then $\langle I_{ep}^N\rangle^a = 0$, where $\langle\,\rangle^a$ denotes averaging over the a-th region. Accordingly, the solution in the zeroth approximation has the same form (8) as before, but now we must assume that the change of the chemical potential $\delta\mu^a$ is different for different regions.

The equations for the quantities $\delta\mu^a$, ϑ, and $\mathbf{u}$, as can be readily seen, have the following form:

$$\delta\mu^a\langle 1\rangle^a + \vartheta\langle\tilde{\varepsilon}\rangle^a + \mathbf{u}\left(\langle\mathbf{p}\rangle^a - \frac{1}{V}\langle p_\varkappa\rangle^a\right) = 0,$$

$$\sum_a \delta\mu^a \langle\tilde{\varepsilon}\rangle^a + \vartheta(\langle\tilde{\varepsilon}^2\rangle + \langle\Omega^2\rangle) + u\left[\langle p\tilde{\varepsilon}\rangle - \frac{1}{V}(\langle pv_x\tilde{\varepsilon}\rangle + \langle qs_x\Omega\rangle)\right] = 0,$$

$$\sum_a \delta\mu^a \left(\langle p_i\rangle^a - \frac{1}{V}\langle p_i v_x\rangle^a\right) + \vartheta\left[\langle p_i\tilde{\varepsilon}\rangle - \frac{1}{V}(\langle p_i v_x\tilde{\varepsilon}\rangle + \langle q_i s_x\Omega\rangle)\right]$$
$$+ u_k\left[\langle p_i p_k\rangle + \langle q_i q_k\rangle - \frac{1}{V}\langle p_i p_k v_x\rangle\right] = 0. \qquad (18)$$

Here, as before, $V = \omega/k$ and $\mathbf{k} = k\kappa$. The phonon mean values are taken over the undisplaced Brillouin zone, in which the dispersion law of the phonons has a symmetry center. After eliminating $\delta\mu^2$ and ϑ we get

$$\beta_{ik}u_k = 0, \quad \beta_{ik} = V^2 a_{ik} + V b_{ik} - c_{ik}. \qquad (19)$$

Here

$$a_{ik} = \langle p_i p_k\rangle + \langle q_i q_k\rangle - \sum_a \frac{\langle p_i\rangle^a\langle p_k\rangle^a}{\langle 1\rangle^a} - \xi_i\xi_k,$$

$$b_{ik} = -\langle p_i p_k v_x\rangle + \sum_a \frac{\langle p_i\rangle^a\langle p_k v_x\rangle^a + \langle p_k\rangle^a\langle p_i v_x\rangle^a}{\langle 1\rangle^a} + \xi_i\eta_k + \xi_k\eta_i,$$

$$c_{ik} = \sum_a \frac{\langle p_i v_x\rangle^a\langle p_k v_x\rangle^a}{\langle 1\rangle^a} + \eta_i\eta_k,$$

where

$$\xi = \langle p\varepsilon'\rangle - \sum_a \frac{\langle p\rangle^a\langle\varepsilon'\rangle^a}{\langle 1\rangle^a},$$

$$\eta = \langle pv_x\varepsilon'\rangle + \langle qs_x\Omega'\rangle - \sum_a \frac{\langle pv_x\rangle^a\langle\varepsilon'\rangle^a}{\langle 1\rangle^a},$$

$$\varepsilon' = \frac{\tilde{\varepsilon}}{\Delta}, \quad \Omega' = \frac{\Omega}{\Delta}, \quad \Delta = \left[\langle\tilde{\varepsilon}^2\rangle + \langle\Omega^2\rangle - \sum_a \frac{(\langle\tilde{\varepsilon}\rangle^a)^2}{\langle 1\rangle^a}\right]^{1/2}$$

We now use the fact that the dispersion law $\epsilon(p)$ is centrally symmetrical in the expanded p-space: $\epsilon(p) = \epsilon(-p)$. Therefore, in the cell chosen by us, each group of surfaces a corresponds to a centrally-symmetrical group $\bar{a}$ (see Fig. 1). We denote the group equivalent to $\bar{a}$ but located in our cell by $\tilde{a}$. It is easy to understand that the surfaces a and $\tilde{a}$ are symmetrical with respect to a certain point $\mathbf{p}^a$, since a parallel transfer of one of the centrally-symmetrical surfaces does not violate their relative central symmetry. The vector $\mathbf{p}^a$ is different for each pair of surfaces a and $\tilde{a}$. This symmetry property allows us to to greatly simplify the coefficients of (19). We consider the first two terms in the expression for b_{ik}:

$$-\langle p_i p_k v_x\rangle + \sum_a \frac{\langle p_i\rangle^a\langle p_k v_x\rangle^a + \langle p_k\rangle^a\langle p_i v_x\rangle^a}{\langle 1\rangle^a}$$
$$= -\frac{1}{2}\sum_a (\langle p_i p_k v_x\rangle^a + \langle p_i p_k v_x\rangle^{\tilde{a}})$$
$$+ \frac{1}{2}\sum_a \left(\frac{\langle p_i\rangle^a\langle p_k v_x\rangle^a + \langle p_k\rangle^a\langle p_i v_x\rangle^a}{\langle 1\rangle^a}\right.$$
$$\left. + \frac{\langle p_i\rangle^{\tilde{a}}\langle p_k v_x\rangle^{\tilde{a}} + \langle p_k\rangle^{\tilde{a}}\langle p_i v_x\rangle^{\tilde{a}}}{\langle 1\rangle^{\tilde{a}}}\right).$$

For the pair a and $\tilde{a}$ we choose the origin at the symmetry center of this pair:

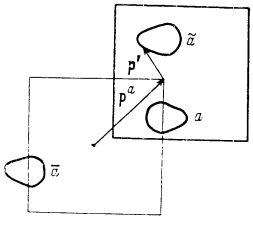

FIG. 1. Undisplaced and displaced Brillouin zones in the case when the Fermi surfaces intersect the faces of the undisplaced cell. The surfaces a and $\bar{a}$ in the expanded p-space are symmetrical with respect to the origin. The surfaces a and $\tilde{a}$ in the displaced cell are symmetrical with respect to the point $\mathbf{p}^a$.

$$\mathbf{p} = \mathbf{p}^a + \mathbf{p}'.$$

Using the obvious relations

$$\langle p_i'\rangle^a = -\langle p_i'\rangle^{\tilde{a}}, \quad \langle p_k' v_x\rangle^a = \langle p_k' v_x\rangle^{\tilde{a}}, \quad \langle 1\rangle^a = \langle 1\rangle^{\tilde{a}},$$

we can readily verify that the expression given above vanishes. Analogously we can show that $\xi = 0$. Thus,

$$a_{ik} = \langle p_i p_k\rangle + \langle q_i q_k\rangle - \sum_a \frac{\langle p_i\rangle^a\langle p_k\rangle^a}{\langle 1\rangle^a}, \quad b_{ik} = 0. \qquad (20)$$

Further, inasmuch as the tensors $\langle \mathbf{p_i v_k}\rangle^a$ and $\langle \mathbf{p_i v_k}\epsilon\rangle$ are diagonal, the matrix c_{ik} has only one nonzero element c_{KK}, equal to

$$c_{xx} = \sum_a \frac{|\langle p_x v_x\rangle^a|^2}{\langle 1\rangle^a} + \left(\langle p_x v_x\varepsilon'\rangle + \langle q_x s_x\Omega\rangle - \sum_a \frac{\langle\varepsilon'\rangle^a\langle p_x v_x\rangle^a}{\langle 1\rangle^a}\right)^2 \qquad (21)$$

Using the obtained relations, we obtain from the dispersion equation $\det\hat{\beta} = 0$ the velocity of sound

$$V^2 = (\hat{a}^{-1})_{xx}c_{xx}. \qquad (22)$$

As seen from (21), $c_{KK} > 0$. In addition, $\langle a^{-1}\rangle_{KK} > 0$, since the diagonal matrix elements a (20) are positive by virtue of the Bunyakovskiĭ-Schwartz inequality. It is easy to see that if there is only one group of surfaces, formula (22) goes over into (14), inasmuch as $\langle \mathbf{p_i v_k}\rangle = 0$.

The polarization of the wave is characterized by the following relations:

$$\delta\mu^a = -\frac{u}{\langle 1\rangle^a}\left\{\langle p\rangle^a - \frac{1}{V}(\langle\varepsilon'\rangle^a\eta - \langle pv_x\rangle^a)\right\}, \quad \vartheta = \frac{u\eta}{V\Delta}, \qquad (23)$$

$$u_i = \frac{\vartheta}{V}(\hat{\gamma}^{-1})_{ik}\eta_k\Delta, \quad \gamma_{ik} = a_{ik} - \delta_{ik}\frac{1}{V^2}\sum_a \frac{|\langle p_x v_x\rangle^a|^2}{\langle 1\rangle^a}.$$

VELOCITY OF SECOND SOUND

We investigate the expressions derived above for the speed of sound. The electronic mean values which enter in formulas (14)–(22) are of the following order of magnitude:

$$\langle p_i\rangle^a \sim p_F\langle 1\rangle, \quad \langle p_i p_k\rangle \sim p_F^2\langle 1\rangle, \quad \langle p_i v_k\rangle^a \sim \varepsilon_F\langle 1\rangle,$$

$$\langle\tilde{\varepsilon}\rangle \sim \frac{T^2}{\varepsilon_F}\langle 1\rangle, \quad \langle\tilde{\varepsilon}^2\rangle \sim T^2\langle 1\rangle, \quad \langle p_i v_k\tilde{\varepsilon}\rangle \sim T^2\langle 1\rangle.$$

For simplicity we assume that all the Fermi surfaces, and also the distances between them, have the same characteristic dimension p_F, $\epsilon_F \sim p_F v_F$, and $\langle 1\rangle^a \sim \langle 1\rangle$. Let us calculate, for example, $\langle \mathbf{p_i v_k}\tilde{\epsilon}\rangle$:

$$\langle p_i v_k\tilde{\varepsilon}\rangle = \frac{2}{h^3}\int dp\, p_i v_k(\varepsilon - \mu)\left(-\frac{\partial f_0}{\partial\varepsilon}\right) = -T\frac{2}{h^3}\int p_i\frac{\partial\psi}{\partial p_k}dp$$
$$= -T\frac{2}{h^3}\int p_i\psi dS_k + \delta_{ik}T\frac{2}{h^3}\int\psi dp.$$

Here $\psi(x) = x(f_0(x) - 1) - \ln f_0(x)$, where $x = \tilde{\epsilon}/T$; it is easy to see that $\psi(-x) R \psi(x)$, $\psi|_{x\gg 1} \sim e^{-x}$, and $\psi(x) > 0$. For closed surfaces, the first term in the right side of the equality can be neglected, since ψ is exponentially small on the surface of the Brillouin zone. Thus,

$$\langle p_i v_k\tilde{\varepsilon}\rangle = \delta_{ik}T^2\langle 1\rangle\int_{-\infty}^{\infty}\psi(x)\,dx.$$

This result is valid also for open surfaces, if i and k coincide with direction of the opening z. Indeed, when k = z, the integral $\int p_i\psi dS_k$ is exponentially small, as

before, by virtue of the smallness of ψ, and the integral vanishes when i = z but k ≠ z (see (17)).

The phonon mean values are estimated without difficulty, since the main contribution to the integrals is made by the "temperature" phonons, for which $\Omega \sim T$:

$$\langle q_x s_x \Omega\rangle \approx \langle \Omega^2\rangle \approx s^2\langle q_i q_k\rangle \approx$$
$$\approx T^2\left(\frac{v_F}{s}\right)^3\left(\frac{T}{\varepsilon_F}\right)^2\langle 1\rangle,$$

where as before $\langle 1\rangle \equiv -2h^{-3}\int dp\,\partial f_0/\partial\epsilon$. The factor following T^2 in this formula is equal to $-h^{-3}\int dq\,\partial N_0/\partial\Omega$. We assume also that the temperature momentum of the phonon is much smaller than the Fermi momentum: $T \ll T_0 = \epsilon_F s/v_F$. In the case of several isolated electron groups, those terms of expression (21) for c_{KK} which are small with decreasing temperature at $T \ll T_0$ can be neglected and

$$c_{xx} = \sum_a \frac{[\langle p_x v_x\rangle^a]^2}{\langle 1\rangle^a} \sim \varepsilon_F^2\langle 1\rangle \sum_a \left(\frac{n_+^a - n_-^a}{n}\right)^2. \qquad (24)$$

According to (20) and (22), we get

$$V \sim \frac{v_F}{1+(T/T_1)^4}, \quad T_1 = \varepsilon_F\left(\frac{s}{v_F}\right)^{3/4}, \qquad (25)$$

and when $T \gtrsim T_1$ the phonon mean values from the expression (20) for a_{ik} become significant. Naturally, when $T \ll T_1$ and pure electron sound is produced, we have $V \approx v_F$, and when $T \gtrsim T_1$, owing to the influence of the phonons, the phase velocity decreases (see Fig. 2).

For small electron groups, when the distance p_0 between the Fermi surfaces is much larger than their characteristic dimension p_F, it is easy to show that

$$V \sim \left(\frac{p_F}{p_0}\right)^2 \frac{v_F}{1+(T/T_1)^4(p_F/p_0)^2}. \qquad (26)$$

In the case of one group of surfaces, the temperature-independent term (24) in c_{KK} vanishes. Then

$$V \sim v_F\frac{T}{\varepsilon_F}\left[\frac{1+(T/T_2)^2}{1+(T/T_1)^4}\right]^{1/2}, \quad T_2 = \varepsilon_F\left(\frac{s}{v_F}\right)^{3/2} \qquad (27)$$

We see therefore that at the very lowest temperature ($T \ll T_2$) we get pure electronic sound propagating with velocity $V \sim v_F T/\epsilon_F$. When $T \gg T_1$, the sound becomes purely-phonon and, naturally, $V \approx s$. In the intermediate region, which is relatively narrow ($T_1/T_2 \sim (v_F/s)^{1/4}$), $V \sim T^2$ and is determined both by the electron and by the phonon characteristics (see Fig. 2).

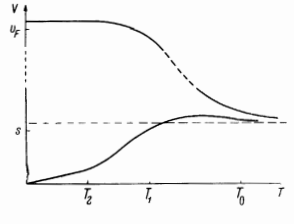

FIG. 2. Dependence of the velocity of second sound on the temperature. The lower curve pertains to the case when there is one group of surfaces. The upper curve pertains to several groups of surfaces (separated by a distance greatly larger than the temperature momentum of the phonon).

We present also the order of magnitude of the ratio of the amplitudes for pure electron sound. For several groups, we get from (23)

$$\vartheta \sim \frac{u}{V}, \quad \delta\mu \sim u\left(p_0 + \frac{v_F}{V}p_F\right),$$

and in the case of one group of surfaces

$$\vartheta \sim \frac{u}{v_F}\frac{\varepsilon_F}{T}, \quad \delta\mu \sim up_F\frac{T}{\varepsilon_F}.$$

DAMPING OF SOUND

As is well known, the damping of the sound is connected with the finite time and finite mean free path of the normal collisions and with the presence of processes which do not conserve the quasimomentum. Therefore, to obtain the damping, it is necessary to derive for the quanities u, ϑ, and $\delta\mu$ "hydrodynamic" equations with allowance for the "viscous" terms that are linear in the small parameters (4) (similar to the Navier-Stokes equation), and also to take into account the electric field that leads to Joule losses. It is first necessary to solve (11) with respect to the functions $f^{(1)}$ and $N^{(1)}$, and substitute the result in (7). The conditions for the solvability of the latter equations yield together with relations (12) a system of homogeneous equations for the drift parameters and the electric field:

$$\int d\mathbf{p}\frac{d}{dt}(f^{(0)}+f^{(1)}) = \int d\mathbf{p}\,\tilde{\varepsilon}\frac{d}{dt}(f^{(0)}+f^{(1)}) + \int d\mathbf{q}\,\Omega\frac{d}{dt}(N^{(0)}+N^{(1)}) = 0,$$
$$\int d\mathbf{p}\,\mathbf{p}\frac{d}{dt}(f^{(0)}+f^{(1)}) + \int d\mathbf{q}\,\mathbf{q}\frac{d}{dt}(N^{(0)}+N^{(1)})$$
$$= \int d\mathbf{p}\,\mathbf{p}I_{ep}^{U}(f^{(0)},N^{(0)}) + \int d\mathbf{q}\,\mathbf{q}I_{pe}^{U}(N^{(0)},f^{(0)}).$$

These equations and Maxwell's equations describe the sound with allowance for damping. We omit the rather cumbersome and complicated calculations[2)].

The result has the following order of magnitude

[2)]The main difficulty arises in solving the equation for $f^{(1)}$, which is obtained from (11) after eliminating $N^{(1)}$. (The operator I_{pe} does not contain integration over the phonon momenta q, and therefore the second equation of (11) is algebraic with respect to $N^{(1)}$.) This equation can be written in the form

$$I_{epe}^{N}(f^{(1)}) = F(\mathbf{p}),$$

where I_{epe}^{N} has the meaning of the collision integral between the electrons via the phonons, and F(p) is expressed in terms of the drift functions $f^{(0)}$ and $N^{(0)}$. From the structure of the equation it is easy to see that $f^{(1)}$ is a "fast" function of the variable $\tilde{\varepsilon}$ (which changes significantly over distances ~T) and a smooth function of the momentum $\mathbf{p}^*$ on the constant-energy surface. Using the fact that the thermal momentum of the phonon is small compared with the dimensions of the Fermi surface ($T/s \ll p_F$), we can show that the operator I_{epe}^{N} is integral with respect to the variable $\tilde{\varepsilon}/T$ and differential with respect to $\mathbf{p}^*$. Then the form of the integral operator and the order of magnitude of the coefficients in the equation are not connected with the dispersion laws of the electrons and phonons, which affect only the dimensionless differential operators that do not contain the temperature. Therefore the order of magnitude of the function $f^{(1)}$ and its dependence on $\tilde{\varepsilon}$ can be obtained for arbitrary laws of electron and phonon dispersion.

$$\frac{\operatorname{Im}\omega}{\omega} \sim \omega\tau^N\left(\frac{v_F}{V}\right)^2 + \frac{1}{\omega\tau^U}. \qquad (28)$$

The first term describes the viscous damping ($\operatorname{Im}\omega \sim \nu/k^2$, where $\nu \sim l^N v_F$ is the electronic viscosity) and is connected, for example, with the appearance of terms of the type

$$\tau^N\langle \mathbf{p} p_i v_k v_l\rangle \frac{\partial^2 u_i}{\partial x_k \partial x_l}$$

in the third equation of (13). As seen from (28), electron sound in metals is possible if the following inequality is satisfied

$$(\tau^U)^{-1} \ll \omega \ll \tau_N^{-1}(V/v_F)^2, \qquad (29)$$

which is particularly difficult to satisfy for slow waves. The latter condition, when rewritten in the form of the inequality for the wavelength

$$\lambda^2 \gg v_F l^N/\omega,$$

acquires a simple physical meaning, since $(v_F l^N/\omega)^{1/2}$ is the diffusion displacement of the electron during one period of the field. If the velocity of the wave is of the order of the electron velocity we arrive naturally, just as in the case of dielectrics, to the condition (1), i.e., the condition on the lengths coincides with the condition on the frequencies.

Let us estimate the contribution made to the damping by the Joule losses in the metal. To this end it is simplest to use the energy formula for the absorption coefficient, according to which the latter is the ratio of the dissipated power to the energy density in the wave. Confining ourselves to the case of purely electronic sound, we have

$$\operatorname{Im}\omega \sim \overline{\mathbf{jE}}/nm\bar{u}^2.$$

In order to find the electric field $\mathbf{E}$, which occurs when sound propagates in the metal, we should use Maxwell's equations

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = -\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t},$$

and calculate the current that enters in it with the aid of the kinetic equation. As seen from (11), the current part of the distribution function $f^{(1)}$ is of the order of

$$f^{(1)} \sim \tau^N \frac{\partial f_0}{\partial \varepsilon}\left[e\mathbf{E}\mathbf{v} + \sum_a \mathbf{v}\nabla\delta\mu^a + \tilde{\varepsilon}(\mathbf{v}\nabla)\vartheta + \dot{\mathbf{p}}\mathbf{u}\right].$$

The corresponding current density is

$$\mathbf{j} = \hat{\sigma}\mathbf{E} + \mathbf{j}',$$

where[3)]

$$\sigma_{ik} = e^2\langle v_i v_k \tau^N\rangle \sim ne^2\tau^N/m,$$

$$f_i' = e\sum_a \langle v_i v_k \tau^N\rangle^a \nabla_k \delta\mu^a + e\langle v_i v_k \tilde{\varepsilon}\tau^N\rangle \nabla_k\vartheta + e\langle v_i p_k\rangle \dot{u}_k.$$

The current $\mathbf{j}'$ plays the role of an inhomogeneity in Maxwell's equations. Solving these equations under the condition that the longitudinal current vanishes, $j_\kappa = 0$, we obtain for the transverse components of the electric field

$$E_\alpha = -\rho_{\alpha\beta}^* j_\beta'^*, \quad \hat{\rho}^* = \left(\hat{\sigma}^* + i\frac{k^2c^2}{4\pi\omega}\right)^{-1},$$

where

$$\sigma_{\alpha\beta}^* = \sigma_{\alpha\beta} - \frac{\sigma_{\alpha\kappa}\sigma_{\kappa\beta}}{\sigma_{\kappa\kappa}}, \quad j_\alpha'^* = j_\alpha' - j_\kappa'\frac{\sigma_{\alpha\kappa}}{\sigma_{\kappa\kappa}}.$$

The longitudinal field is

$$E_\kappa = -E_\alpha\sigma_{\kappa\alpha}/\sigma_{\kappa\kappa}.$$

Using the relations between $\delta\mu$, ϑ, and $\mathbf{u}$, we can easily show that both in the case (15) and in the case of (23) we have, in order of magnitude,

$$E_\alpha \sim \frac{m\omega}{e}\left[1 + \left(\frac{k\delta}{\omega\tau^N}\right)^2\right]^{-1} u, \quad E_\kappa \sim \frac{m\omega}{e}u, \quad \delta^{-2} = \frac{4\pi ne^2}{mc^2}.$$

Using the energy formula, we get

$$\operatorname{Im}\omega/\omega \sim \omega\tau^N.$$

It is easy to show that with increasing temperature, when the contribution of the phonons becomes appreciable, this result is conserved in order of magnitude.

Thus, the dissipation due to the Joule losses does not exceed the dissipation connected with the non-electromagnetic mechanisms (of course, provided that $n_+ = n_-$), and the absorption of sound in metals will be small if the conditions (29) are satisfied.

[3)] Inasmuch as the relaxation time τ^N cannot be introduced rigorously, these results should not be taken literally. In particular, the "mean" value which enters here is $\langle v_i p_k\rangle \neq 0$.

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