DISLOCATION MODEL OF FERROMAGNETISM IN NONMAGNETIC CRYSTALS

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An analysis of the magnetic properties of a plane array, each cell of which is formed by sufficiently long linear spin chains with positive exchange interaction between the neighbors, is carried out on basis of the Sharp and Avery[1] hypothesis. The dependence of the Curie point of such a system on the number of spins in each segment of the array is determined within the framework of the Ising model, and the temperature dependence of spontaneous magnetization is elucidated. Some features of such a temperature dependence are discussed for models imitating, in a certain temperature range, the behavior of three-dimensional arrays consisting of linear chains.

INTRODUCTION

SHARP and Avery[1], in an interpretation of their experimental results, advanced the hypothesis that alkali-halide crystals of the NaCl type can have ferromagnetism due entirely to dislocations distributed in the crystals. According to this hypothesis, each unit cell containing the dislocation has an elementary spin whose exchange interaction with its "neighbors with respect to dislocation" is described by a positive exchange integral. An estimate of the corresponding Curie temperature leads to a value on the order of several hundred degrees.

It is known that dislocations in a crystal subject to weak plastic deformation forms a certain spatial structure, called a dislocation array. From the point of view of the Sharp and Avery hypothesis, each link of such an array (each side of the cell) is a sufficiently long linear chain of spins. Indeed, at low dislocation density, there are 10^4–10^5 unit cells for each dislocation segment. However, an infinitely long linear chain of spins cannot have ferromagnetism. Therefore, in spite of the large lengths of the links, the ferromagnetism is brought about by the presence of nodes of the dislocation arrays joining the linear chains into a two- or three-dimensional structure.

In this paper we analyze the ferromagnetic properties of the simplest model based on the Sharp and Avery hypothesis, namely, a periodic two-dimensional array consisting of linear spin chains. We call attention to the basic problem raised, in our opinion, by the formulated hypothesis; this problem reduces to an elucidation of the dependence of the Curie temperature of such a structure on the number of spins in one linear link. Only a very weak dependence on the number of spins, which is tremendous in one segment, can explain the observed rather high Curie temperature. We believe that this question can be discussed qualitatively within the framework of the Ising model.

In Sec. 1 we consider the magnetic properties of a linear chain consisting of a large number of spins, under the condition that the outermost spins are fixed. In Sec. 2 it is shown that the problem of a two-dimensional array consisting of a linear chain of spins can be reduced to the Onsager problem (cf., e.g.,[2]) with a certain effective exchange integral and an effective magnetic field. In this connection, both the very phase transition in such a system and the temperature dependence of the spontaneous magnetic moment lend themselves to a simple analysis (the latter on the basis of Yang's solution[3]). In Sec. 3 we discuss the structure imitating a three-dimensional array of linear chains at temperatures much below the Curie point.

1. LINEAR ISING CHAIN WITH FIXED TERMINAL SPINS

Let us consider an Ising chain of $n+2$ spins, in which the values of the two terminal spins σ_1 and σ_2 are fixed, and all the internal spins s_k ($k = 1, 2, \ldots, n$) can assume values ± 1. The energy of a certain spin configuration $\{\sigma_2, s_1, s_2, \ldots, s_n, \sigma_1\}$ in an external magnetic field H is conveniently written in the form

$$E(\sigma_1, \sigma_2, n) = -\varepsilon \sum_{k=1}^{n-1} s_k s_{k+1} - \frac{H}{2} \sum_{k=1}^{n-1} (s_k + s_{k+1})$$

$$- \varepsilon[\sigma_1 s_n + \sigma_2 s_1] - \frac{H}{2}[s_1 + s_n + 2(\sigma_1 + \sigma_2)],$$

where ϵ is the interaction energy of the neighboring spins, and the magnetic moment of the spin is assumed equal to unity.

The partition function of such a chain is

$$Z(\sigma_1, \sigma_2, n) = \sum_{\{s_i\}} \exp\{-\beta E(\sigma_1, \sigma_2, n)\}, \quad \beta = 1/kT,$$

where the summation is over all the s_i, which take on the values ± 1. This partition function can be written in the following matrix form [2]:

$$Z(\sigma_1, \sigma_2, n) = e^{\beta H(\sigma_1+\sigma_2)} \sum_{s_1 s_n} \langle s_1 | \hat{P}^{n-1} | s_n \rangle \langle s_n | \hat{T} | s_1 \rangle$$

$$= e^{\beta H(\sigma_1+\sigma_2)} \mathrm{Sp}(\hat{P}^{n-1}\hat{T}),$$

where the matrices $\hat{P}$ and $\hat{T}$ should be defined as follows:

$$\hat{P} = \begin{Vmatrix} e^{\gamma+\alpha} & e^{-\gamma} \\ e^{-\gamma} & e^{\gamma-\alpha} \end{Vmatrix}, \quad \hat{T} = \begin{Vmatrix} e^{\gamma_1+\gamma_2+\alpha} & e^{\gamma_1-\gamma_2} \\ e^{-\gamma_1+\gamma_2} & e^{-(\gamma_1+\gamma_2+\alpha)} \end{Vmatrix}.$$

In writing out the matrices we used the following dimensionless parameters: $\alpha = \beta H$, $\gamma = \beta\epsilon$, $\gamma_1 = \gamma\sigma_1$, and $\gamma_2 = \gamma\sigma_2$. The trace $\mathrm{Sp}(\hat{P}^{n-1}\hat{T})$ can be readily cal-

culated in the representation in which the matrix $\hat{P}$ is diagonal, when the following equality holds:

$$\mathrm{Sp}\,(\hat{P}^{n-1}\hat{T}) = \mathrm{Sp}\,(\hat{P}_0^{n-1}\hat{S}^{-1}\hat{T}\hat{S}) = \sum_{i=1}^{2}\lambda_i^{n-1}(\hat{S}^{-1}\hat{T}\hat{S})_{ii},$$

where λ_i are the known expressions (see[2]) for the eigenvalues of the matrix $\hat{P}$:

$$\lambda_{1,2} = \mathrm{ch}\,\alpha \pm \sqrt{\mathrm{ch}^2\alpha + e^{-4\gamma} - 1}, \tag{1}$$

and the matrix $\hat{S}$, as can be readily verified, is given by

$$\hat{S} = \Delta \begin{Vmatrix} 1 & \varkappa e^{2\gamma} \\ -\varkappa e^{2\gamma} & 1 \end{Vmatrix}, \tag{2}$$

where

$$\Delta = [1 + (\varkappa e^{2\gamma})^2]^{-1/2}, \qquad \varkappa = \mathrm{sh}\,\alpha - \sqrt{\mathrm{ch}^2\alpha + e^{-4\gamma} - 1}.$$

The diagonal elements of the matrix T in the corresponding representation are then equal to $(\hat{S}^{-1}\hat{T}\hat{S})_{ii} = \Delta^2 T_{ii}$, with

$$\begin{aligned} T_{11}(\sigma_1,\sigma_2) &= e^{\alpha}e^{\gamma_1+\gamma_2} - 2\varkappa e^{2\gamma}\,\mathrm{ch}\,(\gamma_1-\gamma_2) + (\varkappa e^{2\gamma})^2 e^{-(\gamma_1+\gamma_2)}e^{-\alpha}, \\ T_{22}(\sigma_1,\sigma_2) &= e^{-\alpha}e^{-(\gamma_1+\gamma_2)} + 2\varkappa e^{2\gamma}\,\mathrm{ch}\,(\gamma_1-\gamma_2) + (\varkappa e^{2\gamma})^2 e^{\gamma_1+\gamma_2}e^{\alpha}. \end{aligned} \tag{3}$$

Using the chosen representation, we obtain for the partition function

$$Z(\sigma_1,\sigma_2,n) = e^{\alpha(\sigma_1+\sigma_2)}(\lambda_1 e^{\gamma})^{n-1}\Delta^2 T(\sigma_1,\sigma_2), \tag{4}$$

$$T(\sigma_1,\sigma_2) = T_{11}(\sigma_1,\sigma_2) + \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1} T_{22}(\sigma_1,\sigma_2). \tag{5}$$

It can be shown that

$$\ln T(\sigma_1,\sigma_2) = I\sigma_1\sigma_2 + b(\sigma_1+\sigma_2) + c,$$

where the coefficients, I, b, and c can be readily determined by sorting out the possible values of the pair combinations $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$. As a result it turns out that

$$I = \frac{1}{4}\ln\frac{T_1 T_{-1}}{T_0^2}, \qquad b = \frac{1}{4}\ln\frac{T_1}{T_{-1}}, \qquad c = \frac{1}{4}\ln(T_1 T_{-1} T_0^2),$$

where

$$\begin{aligned} T_1 &\equiv T(+1,+1), \quad T_{-1} \equiv T(-1,-1), \\ T_0 &\equiv T(-1,+1) = T(+1,-1). \end{aligned} \tag{6}$$

In the main approximation in the weak magnetic field ($\alpha \ll 1$), the quantities I, b, and c are given by

$$\begin{aligned} I(n) &= \mathrm{Arth}\,(\mathrm{th}\,\gamma)^{n+1} + O(\alpha^2), \qquad b(n) = \frac{\alpha}{2}\frac{(e^{2\gamma}-1)[1-(\mathrm{th}\,\gamma)^n]}{1+(\mathrm{th}\,\gamma)^{n+1}} \\ c(n) &= \ln\left[4\sqrt{1-(\mathrm{th}^2\gamma)^{n+1}}\,\mathrm{ch}^2\gamma\right] - \alpha e^{2\gamma}. \end{aligned} \tag{7}$$

Using the expansion (7), we write out the partition function of a finite chain of n free spins with two fixed terminal spins σ_1 and σ_2 in a weak external field ($\alpha \ll 1$):

$$Z(\sigma_1,\sigma_2,n) = Q(n)\exp\{I(n)\sigma_1\sigma_2 + [\alpha + b(n)]\,(\sigma_1+\sigma_2)\}, \tag{8}$$

where

$$Q(n) = {}^1/_2(2\,\mathrm{ch}\,\gamma)^{n+1}[1-(\mathrm{th}^2\gamma)^{n+1}]^{1/2}. \tag{9}$$

Having the partition function (8) as a function of the temperature and the field, we can easily obtain the dependence of the magnetization of the given chain on the temperature. The spontaneous magnetic moment is given by

$$M(n) = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\ln Z(\sigma_1,\sigma_2,n),$$

from which we get the following expression for the magnetic moment of the chain:

$$M(n) = (\sigma_1+\sigma_2)\left[1 + \frac{1}{2}\frac{(e^{2\gamma}-1)[1-(\mathrm{th}\,\gamma)^n]}{1+(\mathrm{th}\,\gamma)^{n+1}}\right]. \tag{10}$$

We recall that $\gamma = \beta\epsilon = \epsilon/kT$.

It is seen from (10) that the moment M(n) is a rather complicated function of n, and naturally differs from zero only in the case when the spins σ_1 and σ_2 have the same direction and $|\sigma_1+\sigma_2| = 2$.

Let us find the limiting values of the magnetic moment (10). When $\gamma \to \infty$ ($T = 0^\circ K$) we obtain $M(n) = n + 2$. In other words, there is one electronic magneton for each node, and consequently the chain is completely magnetically polarized. When $\gamma \ll 1$, which is equivalent to $kT \gg \epsilon$, the proportionality of the magnetic moment to the number of spins in the chain vanishes: $M(n) = 2(1+\gamma)$, and therefore at large n ($n \gg 1$) the magnetic moment per node becomes very small: $M(n)/n \cong 2/n$. Thus, the spin system becomes practically completely disordered. If we are interested in only very large n, to which we confine ourselves henceforth, then it is easy to estimate the temperature at which a rather abrupt transition from "order" to "disorder" in the spin system takes place. This temperature corresponds to the condition

$$(\mathrm{th}\,\gamma)^n \sim 1. \tag{11}$$

At very large n, relation (11) can hold for only sufficiently large γ, for which $\tanh\gamma \cong 1 - 2e^{-2\gamma}$. Therefore the value of the parameter γ, at which condition (11) is satisfied has, with logarithmic accuracy, the following order of magnitude:

$$\gamma \sim {}^1/_2\ln n \gg 1. \tag{12}$$

Consequently, the estimate for the characteristic temperature below which it is meaningful to speak of a ferromagnetic state of the chain takes the form

$$kT \sim \varepsilon / \ln\sqrt{n} \ll \varepsilon.$$

It is very important for what follows that the temperature below which "order" exists in the spin system decreases very slowly with the length of the linear chain, i.e., with the quantity n. Of course, as $n \to \infty$ we obtain the well known result, according to which "order" in an infinitely long linear chain is possible only when $T = 0^\circ K$. However, even for very large n the characteristic temperature remains quite high.

2. PHASE TRANSITION IN A TWO-DIMENSIONAL ARRAY CONSISTING OF LINEAR SPIN CHAINS

Let us consider a two-dimensional periodic array (plane array) made up of linear Ising chains. We assume that each chain (link) contains, besides the node spins, also n spins interacting in standard fashion only with the nearest neighbors along the chain. For $n = 0$, this model goes over in natural fashion into the usual two-dimensional array of chains, for which the partition function was first calculated by Onsager[4]. Onsager has shown that in a usual plane array of spins there takes place a phase transition whose tempera-

ture T_0 is determined by the equation (cf., e.g.,[2])

$$2\mathrm{th}^2\frac{2\varepsilon}{kT}=1, \tag{13}$$

where ϵ is the energy of the nearest-neighbor exchange interaction. We shall show that the problem considered by us, concerning an array of linear chains, can be reduced to the problem solved by Onsager.

Let a quadratic two-dimensional array have $N = m^2$ modes (m "rows" and m "columns"). We denote by $s^{\nu}_{\mu k}$ ($k = 0, 1, \ldots, n+1$) the spin in row μ between columns ν and $\nu+1$. Then $s^{\nu}_{\mu,n+1} = s^{\nu+1}_{\mu,0}$. We introduce analogously the symbol $s^{\nu i}_{\mu}$ ($i = 0, 1, \ldots, n+1$) for the spin in column ν between rows μ and $\mu+1$, with $s^{\nu,n+1}_{\mu} = s^{\nu,0}_{\mu+1}$. For convenience in notation, we separate the symbols for the node spins $s^{\nu}_{\mu} \equiv s^{\nu 0}_{\mu}$. The partition function of the two-dimensional grid in an external magnetic field H can then be written in the form

$$Z=\mathrm{Sp}\exp\Big\{-\beta\sum_{\mu=1}^{m}E_\mu-\beta\sum_{\nu=1}^{m}E^\nu+\alpha\sum_{\mu,\nu=1}^{m}s_\mu{}^\nu\Big\}, \tag{14}$$

$$E_\mu=-\varepsilon\sum_{\nu=1}^{m}\sum_{k=0}^{n}s_{\mu k}{}^\nu s^\nu_{\mu,k+1}-H\sum_{\nu=1}^{m}\sum_{k=1}^{n}s_{\mu k}{}^\nu,$$

$$E^\nu=-\varepsilon\sum_{\mu=1}^{m}\sum_{i=0}^{n}s_\mu{}^{\nu i}s_\mu^{\nu,i+1}-H\sum_{\mu=1}^{m}\sum_{i=1}^{n}s_\mu{}^{\nu i},$$

where the symbol Sp stands for summation over the set of spin variables (each spin can take on values ± 1).

We use the toroidal boundary condition $s^{\nu}_{m+1,k} = s^{\nu}_{1,k}$ and $s^{m+1,i}_{\mu} = s^{1,i}_{\mu}$ and sum in (14) over all possible configuration of non-nodal spins (i, k = 1, 2, 3, ..., n). Then (14) can be transformed into

$$Z=Q^{2N}\sum_{\{s_\mu^\nu\}}\exp\Big\{I(n)\sum_{\mu,\nu}(s_\mu{}^\nu s^\nu_{\mu+1}+s_\mu{}^\nu s_\mu^{\nu+1})+B\sum_{\mu,\nu}s_\mu{}^\nu\Big\}, \tag{15}$$

$$B=\alpha+b(n) \tag{16}$$

where Q, I(n), and b(n) are given by (9) and (7).

Comparing the form of the partition function (15) with the corresponding expression in Onsager's problem, we see that the quantity I(n) plays the role of the effective energy of exchange interaction (in units of kT) of the nodes of a plane array, connected by linear Ising chains of n spins. The quantity B (in the same units) plays the role of the effective magnetic field acting on these spins.

If there is no external magnetic field (H = 0), then, using the Onsager condition (13), we find that as $m \to \infty$ the temperature of the phase transition is given by the formula $2\tanh^2 2I = 1$, or

$$(\mathrm{th}\,\gamma_c)^{n+1}\equiv a_0=\sqrt{2}-1. \tag{17}$$

Then, when $n \gg 1$, we have for the phase-transition temperature of the two-dimensional array, in the main approximation in $1/n$, $\gamma_c = (1/2)\ln n$ or $kT_c = \epsilon/\ln\sqrt{n}$. We recall that for an ordinary quadratic lattice, where each spin interacts with four nearest neighbors, the phase transition temperature is given by the relation $\gamma_0 = \tanh^{-1}a_0 \cong 0.44$ or $kT_0 \cong \epsilon/0.44$. The dependence of T_c on n obtained by us agrees in natural fashion with the result of the analysis of the linear chain considered above, namely, the characteristic temperature of the transition from "order" to "disorder" for the latter is also proportional to $\ln\sqrt{n}$.

Thus, for a plane array, in which each side of the cell contains n spins ($n \gg 1$), the phase-transition temperature decreases logarithmically with increasing n

$$T_c\sim T_0/\ln\sqrt{n}. \tag{18}$$

If we recognize that the average number of spins per dislocation is of the order of 10^4–10^5, then it turns out that a two-dimensional array of such dislocations has a phase-transition temperature lower by only one order of magnitude than the ordinary quadratic Onsager lattice.

As is well known, certain thermodynamic quantities have singularities at the phase-transition point. Let us elucidate, in particular, the character of the singularities of the specific heat c in the model under consideration. To this end, using (15) and the Onsager solution [cf. [2]], we write down the free energy of the array per node (to which 2n + 1 spins belong):

$$\beta F(T)=-\ln Q^2(n)-\ln(2\,\mathrm{ch}\,2I)-\frac{1}{2\pi}\int_0^\pi d\varphi\ln\frac{1}{2}\left(1+\sqrt{1-\varkappa^2\sin^2\varphi}\right),$$

where $\kappa = 2\sinh 2I/\cosh^2 2I$, and the expressions for I(n) and Q(n) are given in (7) and (9). Then the total energy per node is given by

$$U(T)=\frac{d}{d\beta}[\beta F(T)]=-\frac{d}{d\beta}(\ln Q^2)+\mathscr{E}\frac{dI}{d\beta},$$

where

$$\mathscr{E}=-\left[1+\frac{2}{\pi}\varkappa' K(\varkappa)\right]\mathrm{cth}\,2I,\qquad \varkappa'=2\,\mathrm{th}^2 2I-1.$$

Finally, the specific heat per node equals

$$c(T)=\frac{d\beta}{dT}\left\{-\frac{d^2}{d\beta^2}(\ln Q^2)+\frac{d^2I}{d\beta^2}\mathscr{E}+\left(\frac{dI}{d\beta}\right)^2\mathscr{E}'\right\}, \tag{19}$$

where

$$\mathscr{E}'=-\frac{2}{\pi}\left\{2K(\varkappa)-2E(\varkappa)-(1-\varkappa')\left[\frac{\pi}{2}-\varkappa' K(\varkappa)\right]\right\}\mathrm{cth}^2 2I,$$

and $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind, respectively:

$$K(\varkappa)=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\varkappa^2\sin^2\varphi}},\qquad E(\varkappa)=\int_0^{\pi/2}d\varphi\sqrt{1-\varkappa^2\sin^2\varphi}.$$

If n = 0 (the Onsager case), then $d^2I/d\beta^2 = 0$, $Q = 1$, and only the last term of the curly brackets in (19) remains; this term yields the well known expression for the specific heat per spin[2]. The elliptic integral $K(\kappa)$ has a singularity at $\kappa = 1$ (or $\kappa' = 0$). In this vicinity of this point $K(\kappa) \cong \ln(4/\kappa')$. It is easy to show that as $T \to T_c$ the first and second terms in (19) are finite, and the third term contains a term that increases without limit as $T \to T_c$. Therefore the main dependence of the specific heat on the temperature is described near T_c by the expression

$$c(T)=k\ln\frac{4}{\varkappa'}\left[\frac{4\beta^2}{\pi}\left(\frac{dI}{d\beta}\right)^2\mathrm{cth}\,2I\right]_{T=T_c},$$

which takes the following form when $|1 = T/T_c| \ll 1/\ln n$

$$c(T) \cong \frac{2}{\pi}(\ln n)^2\left\{-\ln\left|1-\frac{T}{T_c}\right|\right\},$$

Thus, the model considered by us retains the logarithmic character obtained by Onsager[4] for the singularity of the specific heat at the phase transition point. However, with increasing n the width of the temperature region in which the logarithmic dependence of the specific heat on $|T - T_c|$ is decisive decreases the more the inequality $|1 = T/T_c| \ll 1/\ln n$ is satisfied.

Being interested in the magnetism of the investigated model, let us find the spontaneous magnetization of the two-dimensional array. To this end, we can use Yang's solution[3], in which the magnetic field should be replaced by the corresponding effective field. Then, at a positive exchange integral, the phase transition corresponds to a transition from the ferromagnetic to the paramagnetic state, and therefore when $T > T_c$ the spontaneous magnetic moment vanishes in the absence of the field ($H = 0$), and equals in the case of $T < T_c$

$$M = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\ln Z = M_0(x)\left\{1+\frac{2(e^{2\gamma}-1)[1-(\mathrm{th}\,\gamma)^n]}{1+(\mathrm{th}\,\gamma)^{n+1}}\right\}, \quad (20)$$

$$M_0(x) = \left[\frac{1+x^2}{(1-x^2)^2}(1-6x^2+x^4)^{1/2}\right]^{1/4}, \quad x \equiv e^{-2I}$$
$$= \frac{1-(\mathrm{th}\,\gamma)^{n+1}}{1+(\mathrm{th}\,\gamma)^{n+1}}. \quad (21)$$

Just as in the case of the specific heat, the magnetic moment is referred to a single node. The phase-transition temperature T_c corresponds to the value $x = x_c \equiv a_0$.

The first factor in (20), as a function of x, coincides with the expression obtained by Yang[3] for the magnetic moment, if we put $x = e^{-2\beta\epsilon}$ in lieu of (21). A unique feature of our problem is the appearance of the second factor in (20); we shall discuss its behavior separately. When $\gamma \to \infty$ ($T = 0°K$), this factor equals $2n + 1$, and the magnetic moment corresponds to the nominal magnetization (complete polarization of all the spins). When $T = T_c$, for large n ($n \gg 1$), the second factor equals, with a high degree of accuracy,

$$\frac{2n}{|\ln a_0|}\frac{1-a_0}{1+a_0} \cong 0.95\cdot 2n,$$

i.e., it actually differs little from its value at $T = 0°K$. Thus, the change of M with temperature is described almost completely by the first factor of (20). We can therefore calculate the spontaneous magnetic moment per spin ($n \gg 1$), with a high degree of accuracy, by using the standard formula[2] and putting

$$M/2n \cong M_0(x),$$

where x is given by (21).

3. MAGNETIZATION OF A PLANE ARRAY WITH ADDITIONAL TRANSVERSE LINKS

We believe it useful to consider the magnetic properties of one more model, made up of linear chains of spins; the analysis of this model also reduces to the Onsager problem[4]. Let us consider the plane array described in Sec. 2, lying in the XOY plane, and assume that two additional linear chains parallel to the Z axis are connected to each of its nodes (one chain lies above the plane of the array, and the other below). We shall assume that each such chain contains in addition m spins, and their number differs from the number of spins in one link in the XOY plane ($m \neq n$).

A simple analysis of the partition function of such a system, similar to the analysis that led us to formula (15), shows that formula (15) holds also for the "two-sided brush" made up of linear chains; the effective exchange energy $I(n)$ remains the same as before, and the effective magnetic field B takes into account the action of the neighbors both in the plane of the array and in the vertical direction:

$$B = a + 4b(n) + 2b^*(m), \quad (22)$$

$$b^*(m) = \frac{a}{2}(e^{2\gamma}-1)[1-(\mathrm{th}\,\gamma)^m]. \quad (23)$$

The value of Q is now given by

$$Q = {}^1/_2(2\mathrm{ch}\gamma)^{n+m+1}[1-(\mathrm{th}^2\gamma)^{n+1}]^{1/2}, \quad (24)$$

i.e., it acquires an additional factor $(2\cosh\gamma)^m$.

Having reduced the partition function to the form (15), we have again arrived at the problem analyzed by Onsager[4] and Yang[3]. On this basis, we can write down immediately the expression for the spontaneous magnetization (per node) below the phase-transition temperature ($T < T_c$):

$$M = M_0(x)\left\{1+(e^{2\gamma}-1)\left[\frac{2(1-[\mathrm{th}\,\gamma]^n)}{1+(\mathrm{th}\,\gamma)^{n+1}}+1-(\mathrm{th}\,\gamma)^m\right]\right\}, \quad (25)$$

where the notation is the same as before (in particular, the phase-transition temperature T_c is unchanged). If $m \sim n$, then the temperature dependence of the magnetic moment of the "two-sided brush" differs little from that of the plane array. However, the situation changes radically when $m \gg n$. Let us consider this limiting case in greater detail.

When $\gamma \to \infty$ ($T = 0°K$), the factor in the curly brackets of (25) becomes $2n + 2m + 1$, corresponding to complete polarization of all the spins (nominal magnetization). At the phase-transition point, the factor in (25) equals

$$\frac{2n}{|\ln a_0|}\frac{3-a_0}{1+a_0} \cong 2\cdot 2n.$$

The spontaneous magnetic moment per spin ($m \gg n \gg 1$), at temperatures close to T_c, is therefore

$$\frac{M}{2(n+m)} \cong \frac{M}{2m} \cong 2\frac{n}{m}M_0(x) \ll M_0(x). \quad (26)$$

Thus, at temperatures close to critical, the magnetic moment per spin decreases sharply. This is connected with the fact that at such temperatures the correlation in the spin orientation extends only over a distance on the order of n steps along the linear chains. Therefore, the bulk of the length of the vertical linear chains, the relative fraction of which is determined by the ratio $(m - n)/m = 1 - n/m \sim 1$, is in the disordered state.

As the temperature decreases, the aforementioned correlation encompasses an ever increasing fraction of the vertical sections, and "order" along the vertical chains is established at a certain temperature, which can be estimated in the same manner as in Sec. 1. The corresponding temperature (which we shall designate

T*) is estimated by a formula of the type (12) and its order of magnitude is

$$T^* \sim \left(\frac{\ln n}{\ln m}\right) T_c, \qquad m \gg n \gg 1. \tag{27}$$

The temperature dependence of the spontaneous magnetic moment is illustrated in Fig. 1.

We deem it very important that when $n \ll m$ the fraction of the spin entering in the plane grid and initiating the ferromagnetic state when $T < T^* < T_c$ is very small compared with the total number of spins that determine the magnetization of the system. In a plane array the spins play the role of the bare structure on which the bulk of the spins of the model in question become ordered. The presence of such a "base" makes possible the existence, at an arbitrary large but finite number of spins m in the transverse linear chains, of a temperature T*, defined by (27), below which the chain is in the ferromagnetic state.

Let us imagine now a three-dimensional structure consisting of parallel plane arrays described in Sec. 2, whose nodes are connected by long transverse chains of spins (in other words, we assume that the transverse filaments, which in our earlier analysis were connected with one plane, now terminate on other plane arrays). We assume as before that each cell of the plane array is made up of linear chains of n spins ($n \gg 1$), and we assign to each transverse filament m additional spins, with $m \gg n$. If $\ln m > \ln n$, then, at temperatures much lower than the Curie temperature of this system but higher than the temperature T* defined by (27), each plane array can be regarded as "polarized" only as a result of the correlation of the spins in its plane. Therefore, in calculating the magnetic moment of such a system at $T < T_c$, we can choose the following approximation: the plane arrays have a nominal magnetization (all spins of one plane have the same direction), and the interaction between them is via the transverse linear chains.

Inasmuch as all spins in one plane array have the same direction, the thermodynamics of such a structure coincides with the thermodynamics of the next system. Let us consider a bundle of linear spin chains tied into nodes at a distance of m spins from one another (Fig. 2). Let N be the number of filaments in the beam, which equals the number of cells in each plane array, and let K be the number of "knots" in the beam (number of nodal spins); we are interested in the transition to the limit as $N, K \to \infty$. In each node there is only one spin σ_i ($i = 1, 2, \ldots, K$) interacting with the nearest spins of all the linear chains in the

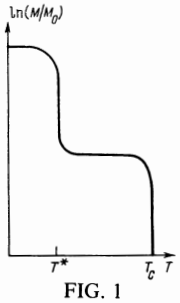

FIG. 1

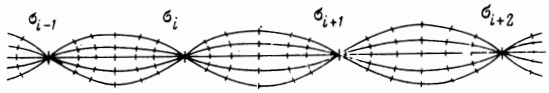

FIG. 2

node. However, in an external magnetic field H it is necessary to assign to each node an additional energy $-HN\sigma_i$. It is then easy to verify that the partition function of such a system (with allowance for the cyclic boundary conditions $\sigma_{K+1} = \sigma_1$) is of the following form:

$$Z = \sum_{\{\sigma_1 \sigma_2 \ldots \sigma_K\}} \exp\left\{N\alpha \sum_{i=1}^{K} \sigma_i\right\} [(e^{\gamma}\lambda_1)^{m-1}\Delta^2]^{NK} \prod_{i=1}^{K} T^N(\sigma_i, \sigma_{i+1}), \tag{28}$$

where the notation is the same as before, and $T(\sigma_1, \sigma_2)$ is determined by (5).

Using the matrix method employed in Sec. 1, we rewrite (28) in the form

$$Z = [(e^{\gamma}\lambda_1)^{m-1}\Delta^2]^{KN} \operatorname{Sp} \hat{R}^K, \tag{29}$$

where the matrix $\hat{R}$ is given by

$$\hat{R} = \begin{Vmatrix} (e^{\alpha}T_1)^N & T_0^N \\ T_0^N & (e^{-\alpha}T_{-1})^N \end{Vmatrix}$$

and the quantities T_1, T_{-1}, and T_0 are given in (6). We denote by μ_1 and μ_2 the eigenvalues of the matrix $\hat{R}$

$$\mu_{1,2} = {}^1/_2\{(e^{\alpha}T_1)^N + (e^{-\alpha}T_{-1})^N \pm \pm ([(e^{\alpha}T_1)^N - (e^{-\alpha}T_{-1})^N]^2 + 4(T_0^2)^N)^{1/2}\}. \tag{30}$$

We can verify that in the entire range of possible values of the temperature and of the external magnetic field, the following inequalities hold:

$$T_{-1} < T_1, \quad T_0 < T_1 \quad \text{if} \quad \alpha > 0,$$
$$T_1 < T_{-1}, \quad T_0 < T_{-1} \quad \text{if} \quad \alpha < 0.$$

For concreteness, we confine ourselves to one sign of α and assume that $\alpha > 0$. Then we get, in the limit as $N, K \to \infty$,

$$\operatorname{Sp}(\hat{R}^K) = \mu_1^K = (e^{\alpha}T_1)^{NK}.$$

The partition function (28) therefore reduces to the very simple expression

$$Z = [(e^{\gamma}\lambda_1)^{m-1}\Delta^2 e^{\alpha}T_1]^{NK}, \tag{31}$$

from which it follows that the magnetic moment of the system is proportional to the product NK, i.e., to the number of "unit cells" of the three-dimensional structure. The magnetic moment of one such unit cell equals

$$M(m) = 1 + \frac{(e^{2\gamma} - 1)[1 - (\operatorname{th}\gamma)^m]}{1 + (\operatorname{th}\gamma)^{m+1}}.$$

The layered anisotropic structure considered by us ($m \gg n$) should be characterized by a sharp variation of the temperature dependence of the spontaneous magnetization in the vicinity of the temperature (see the estimate (12))

$$kT \sim \varepsilon / \ln \gamma m,$$

which, by assumption, is much lower than the Curie temperature. This temperature dependence is represented schematically by the central part of the plot of Fig. 1.

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[1] E. J. Sharp and D. A. Avery, Phys. Rev. 158, 511 (1967).
[2] K. Huang, Statistical Mechanics, Wiley, 1963.
[3] C. N. Yang, Phys. Rev. 85, 809 (1952).
[4] L. Onsager, Phys. Rev. 65, 117 (1944).

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