TRANSVERSE INSTABILITY OF LASER EMISSION

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The transverse field distribution E = E(x, t) sin kz is determined near the generation threshold in the geometric optics approximation for a two-dimensional laser model.

THE interaction of a strong electromagnetic field with a nonlinear medium is known to cause an instability of the original field distribution with respect to transverse perturbations[1-3]. For example, this instability is associated with waveguide propagation and self-focusing of strong light beams in a medium whose refraction coefficient depends on the field

$$n = n_0 + n_2E^2 + \dots \quad (n_2E^2 / n_0 \ll 1). \tag{1}$$

Because of the small coefficients in expansion (1) (weakly nonlinear media) the self-focusing and self-trapping effects become significant only when the fields are sufficiently strong[4]. The resonant interaction between the medium and the field can significantly change the magnitude of the coefficients in (1) (strongly nonlinear media[1]) and even bring them to the same order of magnitude ($n_2E^2/n_0 \lesssim 1$).

The simplest example of a medium that is strongly nonlinear and in a comparatively weak field is the active medium (laser amplifier or oscillator). The problem of the propagation of a plane wave in an amplifier was discussed by Javan and Kelly[5] who primarily considered the stationary distribution of the field. The nonstationary distribution of the field at large energies and short time lengths (giant pulses) was considered by Letokhov and Suchkov[6].

Of interest is the problem of the transverse distribution of the field in a laser operating in a continuous mode.

In the present paper the problem is considered under the following assumptions: (1) the Fabry-Perot resonator is represented as an infinite band filled with an active two-level medium (the two-dimensional problem); and (2) the generation is close to the threshold ($\xi \ll 1$, formula (5)).

The self-consistent system of equations for the density matrix ρ_{mn} (m, n = 1, 2) of the medium and of Maxwell's equations has the form

$$\left(\frac{\partial}{\partial t} + i\omega_0 + \gamma_0\right)\rho = iVN,$$

$$\left(\frac{\partial}{\partial t} + \gamma\right)N = I + 2i(V^*\rho - \rho^*V),$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} - c^2\frac{\partial^2}{\partial x^2} + \frac{\omega}{Q}\frac{\partial}{\partial t}\right)E = 8\pi N_0\frac{\partial^2}{\partial t^2}\,\mathrm{Re}\,\rho D. \tag{2}$$

Here $\rho_{12} = \rho_{21}^* = \rho$, $N = \rho_{22} - \rho_{11}$, $V = ED/\hbar$, D is the dipole moment of transition, I is the flux of particles due to the external pumping, Q is the quality factor of the resonator, γ^{-1} and γ_0^{-1} are the longitudinal and transverse relaxation times, N_0 is the active atom density, and ω_0 is the transition frequency.

The field inside the resonator is sought in the form of a standing wave with slowly time-varying amplitude and arbitrary transverse distribution:

$$E = \sin kz(E(x, t)e^{-i\omega t} + \text{c.c.}). \tag{3}$$

In solid-state lasers usually γ_0, $\omega/Q \gg \gamma$. We consider only this limiting case and assume that all slow time processes are determined by frequencies of the order of γ.

It is convenient to introduce dimensionless variables

$$A = \frac{3}{2}\frac{V}{\sqrt{\xi\gamma\gamma_0}}\frac{1}{1-i\delta}, \quad P = \frac{1-\xi}{\xi} - \frac{\gamma N}{I\xi},$$

$$\tau = \gamma t, \quad x_0 = kx\left(\frac{\xi}{Q}\right)^{1/2}, \tag{4}$$

where $\delta = (\omega_0 - \omega)/\gamma_0$ is the dimensionless detuning and

$$\xi = 4\pi N_0 D^2 QI / \hbar\gamma\gamma_0(1+\delta^2) - 1 \tag{5}$$

is a quantity determining the generation threshold $\xi = 0$.

Substituting (3) into (2) and averaging over the high-frequency space-time oscillations ($\omega \approx ck$) we obtain an abbreviated system of equations

$$i\frac{2Q}{\xi}\frac{\gamma}{\omega}\frac{\partial A}{\partial\tau} + \frac{\partial^2 A}{\partial x_0^2} + (i+\delta)PA = 0, \quad \frac{\partial P}{\partial\tau} + P = |A|^2 - 1. \tag{6}$$

It is apparent from (6) that the real part of the index of refraction $\mathrm{Re}(n - n_0)$ is determined by the quantity $P\delta$. Consequently the electromagnetic field is compressed (focused) in the transverse direction when $P\delta > 0$ and is diverging when $P\delta < 0$. The gain (the imaginary part of dielectric permittivity) is determined by P. When $P > 0$ the field is absorbed and when $P < 0$, it is amplified. The stationary solution with P = 0 and A = 1 naturally corresponds to a homogeneous standing wave.

In problems dealing with the propagation of plane waves in weakly nonlinear media it is usually assumed (see[7] for example) that $P \sim |A|^2$ and therefore the sign of $\mathrm{Re}(n - n_0)$ is determined only by the sign of δ and does not depend on the magnitude of the field. For this reason a weakly nonlinear medium can form a waveguide channel that conducts the light beam and allows for the self-focusing phenomenon.

According to (6), taking self-action effects into account, self-focusing in the usual sense is not possible in an active medium because the sign of $\mathrm{Re}(n - n_0)$ depends on the field and varies in response to whether $A > 1$ or $A < 1$. Therefore the transverse field structure in our model can have a periodic (or quasi-periodic)

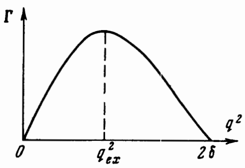

FIG. 1.

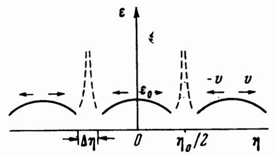

FIG. 2.

character. This becomes quite clear if we remember that pumping in our model is uniformly distributed over the entire medium. We note a further characteristic of (6). Absorption can usually be neglected in a weakly nonlinear medium because it has a non-resonance character[4,7]. In the active medium $\mathrm{Im}(n - n_0) \sim \mathrm{Re}(n - n_0)$ due to resonance interaction. The absorption of emission can significantly change the results particularly in regions where the transverse component of the field is small. The analysis given below shows that a strong competition between diffraction and resonance losses takes place in these regions.

We consider the problem of stability of a homogeneous solution of (6): $A = 1$ and $P = 0$. We set $A = 1 + A'$, A', $P \sim \exp(\Gamma\tau + iqx_0)$ and linearize (6) with respect of small perturbations of A' and P. As a result we obtain a dispersion equation determining the frequency interval of transverse plane waves that are amplified by the active medium

$$\left(\frac{\gamma}{\sigma\xi}\right)^2(\Gamma^3+\Gamma^2)+\Gamma\left(2\frac{\gamma}{\xi\sigma}+q^4\right)+q^4-2\delta q^2=0, \quad \sigma=\omega/2Q. \tag{7}$$

According to this equation when $\delta > 0$ and $q^2 < 2\delta$ the homogeneous standing wave is unstable with respect to perturbations with the given q.

Figure 1 shows the growth increment as a function of q^2 in which the maximum value

$$\Gamma_{max}=-\frac{1}{2}+\left[\frac{1}{4}+\frac{\xi\sigma}{\gamma}\left(\sqrt{1+\delta^2}-1\right)\right]^{1/2} \tag{8}$$

is reached at

$$q_{ex}^2=\delta\,/\,(\Gamma_{\max}+1). \tag{9}$$

The form of the function $\Gamma(q^2)$ has a simple physical meaning: in a resonance medium the potential energy decreases by the amount 2δ[1]; the resulting potential well can support undamped waves (particles) whose energy q^2 is less than the depth of the well.

We note that a similar criterion of space-time instability ($q^2 < 2\delta$) was obtained by Bespalov and Talanov[3] in the course of analyzing the problem of a plane wave in a cubic medium. This coincidence of results stemming from apparently different problems is understandable if we consider that the role of the active medium in linear approximation is simply reduced to the postulate of effective dielectric permittivity that depends only on the stationary value ξ of the field energy.

In a general case it is not possible to find a solution of (6) that decreases or is limited for $|x| \to \infty$. For this reason our analysis of (6) is limited to the neighborhood of the instability threshold $2\delta - q^2 \ll 2\delta$ about the stationary state $|A - 1| \ll 1$. The growth increment in this case is $0 < \Gamma \ll 1$ and in dimensional units has the form

$$\Gamma=\gamma\left(1-\frac{\omega\gamma_0}{4\xi\sigma(\omega_0-\omega)}\frac{k_\perp^2}{k^2}\right). \tag{10}$$

The detuning values considered here $\delta^2 \gg \gamma/\xi\sigma$ are only moderately small.

In stationary state the solution of (6) linearized near $|A|^2 = 1$ has the form

$$A=[1+C_1\sin(\sqrt{2\delta}\,x_0+\vartheta)]\exp\left\{i\left[\frac{C_1}{\delta}\sin(\sqrt{2\delta}\,x_0+\vartheta)+\int C_2\,dx_0\right]\right\}, \tag{11}$$

where C_1, C_2, and ϑ are integration constants. For the next approximation we consider the constants C_i and ϑ as slow functions of x_0 and τ. Substituting (11) into (6) and averaging over the explicit dependence on x_0 we obtain equations for the slow functions

$$\frac{\partial\varepsilon}{\partial x_0}=\Phi\varepsilon+\sqrt{\frac{\delta}{2}}\,\varepsilon\frac{\partial\vartheta}{\partial\tau},\quad \sqrt{\frac{\delta}{2}}\frac{\partial\varepsilon}{\partial\tau}=-\varepsilon\frac{\partial\vartheta}{\partial x_0},$$
$$\frac{\partial\Phi}{\partial x_0}=-(\Phi^2+\varepsilon^2), \tag{12}$$

where for convenience we get

$$C_1=\frac{\delta^{3/2}}{\sqrt{1+\delta^2}}\,\varepsilon(x_0,\tau),\quad C_2=\Phi(x_0,\tau)\,\delta.$$

The equations are written with an accuracy to ϵ^2 (assuming that ϵ and $\Phi \ll 1$). In linear approximation the dispersion equation (10) follows from (12) when

$$\vartheta=-\sqrt{\frac{\delta}{2}}\frac{\Gamma}{\gamma}x_0.$$

The term with Φ^2 in the last equation of (12), due to the strong resonant absorption $\mathrm{Im}(n - n_0)$, prevents the usual waveguide propagation of light[2,4,7]. This can be proved by dropping Φ^2 from (12) and solving the stationary system.

The solution of (12) is sought in the form of waves spreading from point $x_0 = 0$. Let a perturbation with an amplitude $\epsilon_0 \ll 1$ arise at time $\tau = 0$ in the origin of coordinates ($x_0 = 0$). The system (12) describes the propagation of this perturbation along the x_0 axis at $\tau > 0$. We introduce a new variable $\eta = x_0 \pm v\tau$; the (–) sign corresponds to propagation in the direction of increasing x_0 and the (+) sign designates a wave traveling in the direction of decreasing x_0.

The solution of (12) with the initial conditions

$$\varepsilon(\eta)\,|_{\eta=0}=\varepsilon_0,\quad \Phi(\eta)\,|_{\eta=0}=0 \tag{13}$$

has the form

$$\Phi^2|\varepsilon|^{2\beta}+\frac{\beta}{1+\beta}\left(|\varepsilon|^{2(1+\beta)}-|\varepsilon|^{2(1+\beta)}\right)=0,$$
$$\frac{d\varepsilon}{d\eta}=\pm\frac{\varepsilon}{\beta^{1/2}|\varepsilon|^{\beta}}\frac{\left(|\varepsilon_0|^{2(1+\beta)}-|\varepsilon|^{2(1+\beta)}\right)^{1/2}}{(1+\beta)^{1/2}},$$
$$\vartheta=\vartheta_0\pm v(\delta/2)^{1/2}\ln|\varepsilon|,\quad \beta=1+\delta v^2/2. \tag{14}$$

The obtained solution can be interpreted as follows: In a nonlinear medium energy considerations favor[1] the waveguide propagation of light with $\mathrm{Re}(n - n_0) > 0$. In an active medium $\mathrm{Re}(n - n_0) \approx P\delta$, $\mathrm{Im}(n - n_0) \approx P$. In our case ($\delta > 0$) if a random fluctuation at some point x leads to an increasing field ($P \sim |A|^2 - 1 > 0$) the latter is focused and absorbed at this point ($\mathrm{Re}(n - n_0) > 0$ and $\mathrm{Im}(n - n_0) > 0$). As the result of absorption, P can become negative and the gain positive. The field in the course of amplification is "expelled" from this region. ($\mathrm{Re}(n - n_0) < 0$). In the neighboring regions the field increases which again causes absorption and subsequent "expulsion." In other words the region with enhanced overpopulation ($P < 0$) has the most favorable conditions for generation; the channel formed along the z axis undergoes emission causing the formation of channels in neighboring (along the x axis) regions, etc. (see Fig. 2).

The solution (14) shows that the geometric optics approximation ($\epsilon_x \ll \sqrt{2\delta}\epsilon$) used in the derivation of (12) is disturbed in the region of small ϵ where Φ becomes large. The period of variation of ϵ and the width of the region of disruption of the geometric optics approximation can be evaluated in the following manner. The approximate solution of (12) for Φ has the following form

$$\Phi \approx -\varepsilon_0 \tan \varepsilon_0\eta. \tag{15}$$

This solution is applicable if $\Phi \ll \sqrt{2\delta}$. The period of variation of Φ and ϵ is $\eta_0 \approx \pi/\epsilon_0$. The width $\Delta\eta$ of the region in which (12) are invalid is in the order of magnitude found from the condition $\Phi_x \sim \sqrt{2\delta}\Phi$

$$\Delta\eta / \eta_0 \approx 2\varepsilon_0 / \pi\sqrt{2\delta} \ll 1. \tag{16}$$

A more precise determination of η_0 is obtained from (14) by integrating the equation for ϵ with respect to the turning points

$$\eta_0 = \frac{2\sqrt{\pi}}{\varepsilon_0}\left(\frac{\beta}{1+\beta}\right)^{1/2} \frac{\Gamma(1/2 - 1/2(1+\beta))}{\Gamma(1 - 1/2(1+\beta))}. \tag{17}$$

It follows hence that η_0 is weakly dependent on β ($\beta \gg 1$).

Within the region $\Delta\eta$ wide the field seems to grow and become of the order of Φ, i.e. the excess energy stored in the resonator in the standing wave nodes (along the z axis) is emitted in regions of the order of $\Delta\eta$. In the intermediate regions $|A|^2$ little differs from unity. Such a mechanism can be one of the causes of the "spiking" regime. At the output of the laser these effects are manifested as emission pulsation of frequency $T_0^{-1} = v/\eta_0$ and length ΔT of the emitted "spikes" which are of the order of $\Delta T/T_0 \sim \Delta\eta/\eta_0$. The velocity of propagation v of the transverse waves is in the order of magnitude equal to $v \sim ck_\perp/k_\parallel$. Determining $k_\perp$ from (10) for $\Gamma = 0$ and expressing η_0 in dimensional units we find that $T_0^{-1} \sim \sqrt{2\delta}\epsilon_0\xi\sigma$.

To bring this into agreement with the experimentally observed relationship $T_0^{-1} \sim \sqrt{\xi}$ we set the value of the ϵ_0 parameter as $\epsilon_0 \approx \sqrt{\gamma/\sigma\xi 2\delta}$. Then $T_0^{-1} \approx \sqrt{\xi\sigma\gamma}$ and the "spike" length is $\Delta T - 1/2\ \delta\xi\sigma$. The linear dimensions of the "spike" along the x axis are $\Delta x \sim \lambda\sqrt{\omega/2\delta\xi\sigma}$ and the distance between the "spikes" $x_0 \sim \lambda\sqrt{2\delta\omega/\gamma}$ does not depend on the emission energy ξ. We note that the values for T_0 and x_0 given above can be obtained directly from (8) and (9).

Assuming that for a solid-state laser $\sigma \sim 10^6 - 10^8\ \text{sec}^{-1}$, and $\gamma \sim 10^3 - 10^4\ \text{sec}^{-1}$, we obtain the following quantitative results:

$T_0^{-1} \sim \sqrt{\xi}(10^5 - 10^6)\ \text{sec}^{-1}$, $\Delta T^{-1} \sim 2\delta\xi(10^6 - 10^7)\ \text{sec}^{-1}$, $x_0 \sim \sqrt{\delta}(1-10)$ cm, $\Delta x \sim (2\delta\xi)^{-1/2}(10^{-1} - 10^{-2})$ cm, and $v \sim \sqrt{2\delta\xi}(10^6 - 10^7)$ cm/sec.

In the usual crystals the transverse dimension $d \sim 1$ cm and the resulting inhomogeneity is weakly expressed as $x_0 \gtrsim d$. However, the suggested region of excess overpopulation emission is small, $\Delta x \ll d$, and this weak inhomogeneity is sufficient to cause a nonstationary generation regime.

In conclusion we note that in the general case the initial perturbation (the starting point of transverse instability) can be caused by a mechanical inhomogeneity of the crystal or by the inhomogeneity of external pumping. For this reason a more precise determination of the ϵ_0 parameter requires the solution of the problem of emission behavior in the active medium with random distribution of inhomogeneities.

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