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# RELATIVISTICALLY INVARIANT EXPANSION OF THE HELICITY SCATTERING AMPLITUDE

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An expansion of a single-particle helicity state is obtained in terms of states which transform according to an irreducible unitary representation of the homogeneous Lorentz group. An analogous expansion is carried out for a two-particle helicity state in the center-of-inertia system. A relativistically invariant expansion of the helicity scattering amplitude is given. The threshold behavior of the partial helicity amplitude is determined.

1. As is known, the expansion of a single-particle helicity state $|\mathbf{p}s\lambda\rangle$ in terms of states $|EJMs\lambda\rangle$ which transform according to an irreducible representation of the rotation group is given by the Jacob-Wick formula[1]

$$|\mathbf{p}s\lambda\rangle=\sum_{J,M}N_J^{-1/2}|EJMs\lambda\rangle D^{(J)}_{M\lambda}(\varphi,\theta,-\varphi),$$

$$|EJMs\lambda\rangle=N_J^{-1/2}\int|\mathbf{p}s\lambda\rangle D^{*(J)}_{M\lambda}(\varphi,\theta,-\varphi)d\Omega, \quad (1)$$

where

$$N_J=4\pi/(2J+1),\qquad d\Omega=\sin\theta d\theta d\varphi.$$

The problem consists in obtaining an expansion of these states in terms of the states $|\rho mJMs\rangle$ which transform according to an irreducible representation of the Lorentz group.[2,3] In the Lorentz group there are two group invariants, or Casimir operators:

$$F=\mathbf{M}^2-\mathbf{N}^2=1+(\rho^2-m^2)/4,\qquad G=\mathbf{MN}=m\rho/4;$$

$\mathbf{M}$ and $\mathbf{N}$ are infinitesimal operators of the group. Therefore, the representations of this group are characterized by the two numbers $(\rho, m)$. In the unitary representation $\rho$ takes arbitrary real values and m takes integer values. The representations $(\rho, m)$ and $(-\rho, -m)$ are equivalent, and we shall therefore consider representations in which $\rho$ takes positive values.

In order to solve our problem, we use the definition of the helicity state $|\varkappa\mathbf{p}s\lambda\rangle$ with nonvanishing mass $\varkappa$:[1]

$$|\varkappa\mathbf{p}s\lambda\rangle=R_{\varphi,\theta,-\varphi}Z_p|\varkappa s\lambda\rangle, \quad (2)$$

i.e., the state $|\varkappa\mathbf{p}s\lambda\rangle$ is obtained from the state at rest $|\varkappa s\lambda\rangle$ by a Lorentz transformation $Z_p$ along the z axis and a rotation $R_{\varphi,\theta,-\varphi}$. Under the Lorentz transformation L(g), this state transforms in the following way:

$$L(g)|\varkappa\mathbf{p}s\lambda\rangle=\sum_{\lambda'}D^{(s)}_{\lambda'\lambda}[r(g,\mathbf{p})]|\varkappa g\mathbf{p}s\lambda'\rangle. \quad (3)$$

The relation (2) defines the state $|\varkappa\mathbf{p}s\lambda\rangle$ as a function f(g) on the group g. As is known,[2,3] the expansion of the unitary regular representation

$$T_{g_0}f(g)=f(g_0g) \quad (4)$$

into irreducible components is realized by expanding f(g) in terms of the matrix elements of the representation of the group g. Hence, the relativistic generalization of the expansion (1) can be obtained by expanding the single-particle helicity state in terms of the complete orthogonal system of functions

$$\sum_\lambda D^{(J)}_{M\lambda}(\varphi,\theta,-\varphi)\Phi^{(\rho,m)}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right), \quad (5)$$

where $\Phi^{(\rho m)}_{J,\lambda,s}(|\mathbf{p}|/E)$ are the unitary matrix elements of the representation of the Lorentz group.[4-8] The orthogonality condition for these functions is

$$\sum_{\lambda,\lambda'}\int D^{*(J')}_{M'\lambda'}(\varphi,\theta,-\varphi)\Phi^{*(\rho',m')}_{J',\lambda',s}\left(\frac{|\mathbf{p}|}{E}\right)D^{(J)}_{M\lambda}(\varphi,\theta,-\varphi)\Phi^{(\rho,m)}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right)\frac{d\mathbf{p}}{E}$$
$$=N_JN^{(\rho,m)}_{J,s}\delta_{MM'}\delta_{JJ'}\delta_{mm'}\delta(\rho-\rho'), \quad (6)$$

and, hence,

$$\sum_\lambda\int_0^\infty\Phi^{*(\rho',m')}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right)\Phi^{(\rho,m)}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right)\frac{\mathbf{p}^2dp}{E}=N^{\rho m}_{Js}\delta_{mm'}\delta(\rho-\rho')$$

follows from the expansion of the representation (3) into irreducible components and from the existence of the analog of the Plancherel formula. This expansion has been considered first by Shapiro[9] and Dolginov[10] for the case of particles with spin zero. The generalization to the case of particles with arbitrary spin was obtained by Chou Kuang-chao and Zastavenko[11] and by Popov.[12]

Let us now turn to the calculation of the normalization factor $N^{\rho m}_{Js}$. It is seen from the orthogonality condition (6) that we must separate a $\delta(\rho-\rho')$-like singularity from the integral

$$\int_0^\infty\Phi^{*(\rho',m')}_{J,\lambda,s}(\alpha)\Phi^{(\rho,m)}_{J,\lambda,s}(\alpha)\,\mathrm{sh}^2\alpha\,d\alpha, \quad (7)$$

where tanh = $|\mathbf{p}|/E$. To this end we use the explicit expression for the function $\Phi^{\rho,m}_{J,\lambda,s}(\alpha)$, obtained by Dao Wong-Duc and Nguyen Van Hieu[4]

$$\Phi^{(\rho,m)}_{J,\lambda,s}(\alpha)=$$
$$=C^{\rho m}_{J\lambda s}\sum_{d,d'}(-1)^{d+d'}\left(J+s-d-d'-\frac{m}{2}-\lambda\right)!\left(d+d'+\lambda-\frac{m}{2}\right)!$$
$$\times\left[d!d'!(J-\lambda-d')!(s-\lambda-d)!\left(s+\frac{m}{2}-d\right)!\right.$$

$$\times\left(J+\frac{m}{2}-d'\right)!\left(\lambda-\frac{m}{2}+d\right)!\left(\lambda-\frac{m}{2}+d'\right)!\Big]^{-1}$$

$$\times\exp\left\{-\left(2d'+\lambda-\frac{m}{2}+1-\frac{i}{2}\rho\right)\alpha\right\}$$

$$\times F\left(J+1-\frac{i}{2}\rho,\ d+d'+\lambda-\frac{m}{2}+1,\ J+s+2;\ 1-e^{-2\alpha}\right); \qquad (8)$$

here d, d′ run through the integers which leave all the factors under the factorial sign nonnegative. The factor $C^{\rho,m}_{J,\lambda,s}$ contains factorials which are unimportant for the further discussion. Using the following functional relation for the hypergeometric function,[13,14]

$$F(\alpha,\beta,\gamma;1-e^{-2\alpha})=\frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}F(\alpha,\beta,1+\alpha+\beta,e^{-2\alpha})$$
$$+e^{-2(\gamma-\alpha-\beta)\alpha}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta,1-\alpha-\beta+\gamma;e^{-2\alpha}), \qquad (9)$$

we can show that for $\alpha\to\infty$

$$\exp\left\{-\left(2d'+\lambda-\frac{m}{2}-\frac{i}{2}\rho\right)\alpha\right\}\cdot F\left(J+1-\frac{i}{2}\rho,d+d'+\lambda-\frac{m}{2}+1,J+s+2;1-e^{-2\alpha}\right)\to\exp\left\{-\left(2d'+\lambda-\frac{m}{2}-\frac{i}{2}\rho\right)\alpha\right\}$$
$$\times\frac{\Gamma(J+s+2)\Gamma(s-d-d'-\lambda+m/2+i\rho/2)}{\Gamma(s+1+i\rho/2)\Gamma(J+s+1-d-d'-\lambda+m/2)}$$
$$+\exp\left\{-\left(s-d-\lambda+s-d+\frac{m}{2}+\frac{i}{2}\rho\right)\alpha\right\}$$
$$\times\frac{\Gamma(J+s+2)\Gamma(d+d'+\lambda-m/2-s-i\rho/2)}{\Gamma(J+1-i\rho/2)\Gamma(d+d'+\lambda-m/2+1)}, \qquad (10)$$

where $s-d-\lambda\geq 0$, $s-d-m/2\geq 0$, $d'+\lambda-m/2\geq 0$, as follows from (8); we have also used $F(\alpha,\beta,\gamma;0)=1$.

Thus, the limit of the function $e^{(i\rho/2+1)\alpha}\Phi^{(\rho,m)}_{J,\lambda,s}(\alpha)$ for $\alpha\to\infty$ is different from zero and tends to a constant value only if $\lambda=m/2$. Therefore the integral (7) has a $\delta(\rho-\rho')$-like singularity only if $\lambda=m/2$. Thus (7) can be written for $\rho\to\rho'$

$$\lim_{\rho\to\rho'}\int_0^\infty\Phi^{*(\rho',m')}_{J,\lambda,s}(\alpha)\Phi^{(\rho,m)}_{J,\lambda,s}(\alpha)\operatorname{sh}^2\alpha\,d\alpha$$
$$=2\pi\delta_{\lambda,m/2}|\tilde{\Phi}^{(\rho,m)}_{J,m/2,s}(\infty)|^2\lim_{\rho\to\rho'}\frac{1}{\pi}\int_0^\infty\cos(\rho-\rho')\alpha\,d\alpha. \qquad (11)$$

The sum over λ of the coefficients of this divergent integral is the normalization factor $N^{(\rho,m)}_{J,s}=2\pi|\Phi^{(\rho,m)}_{J,m/2,s}(\infty)|^2$. Its explicit form is most conveniently obtained, not from (8), but with the help of the recurrency formula obtained in[8]. In the Appendix we calculate the normalization factor (A.8) and find

$$N^{(\rho,m)}_{J,s}=2\pi\frac{(J-s)!\,[2(J+1)!]^2(s+|m|/2)!\,(s-|m|/2)!}{(J+s)!\,(J+1+|m|/2)!\,(J-|m|/2)!\,(J+1+|m|/2)!\,2s!}$$
$$\times\prod_{k=|m|/2+1}^{s}\left(\frac{\rho^2}{4}+k^2\right)\left|\frac{\Gamma(i\rho/2+|m|/2)}{\Gamma(i\rho/2+J+1)}\right|^2. \qquad (12)$$

Now, using the orthogonality condition (6), we write the relativistic generalization of the expansion (1):

$$|\varkappa p s\lambda\rangle=\sum_{J,M,m}\int_0^\infty N_J^{-1/2}(N^{\rho m}_{J,s})^{-1/2}|\varkappa\rho mJMs\rangle$$
$$\times D^{(J)}_{M,\lambda}(\varphi,\theta,-\varphi)\Phi^{(\rho,m)}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right)d\rho,$$

$$|\varkappa\rho mJMs\rangle$$
$$=N_J^{-1/2}(N^{\rho m}_{J,s})^{-1/2}\sum_\lambda\int|\varkappa ps\lambda\rangle\cdot D^{*(J)}_{M,\lambda}(\varphi,\theta,-\varphi)\Phi^{*(\rho,m)}_{J,\lambda,s}\left(\frac{|\mathbf{p}|}{E}\right)\frac{d\mathbf{p}}{E} \qquad (13)$$

In this expansion, m/2 takes the values s, s − 1, ..., −s, as follows from (11). The same selection rule for the quantum number m was obtained in[12]. In the special case when the spin of the particle s = 0, this formula goes over into the formula for the expansion of the state of a single spinless particle in terms of a system of functions on a hyperboloid, which was obtained by Vilenkin and Smorodinskiĭ.[15] Indeed, as was shown in[5,8], the function $D^{(J)}_{M,\lambda}(\varphi,\theta,-\varphi)\Phi^{(\rho,m)}_{J,\lambda,s}(\alpha)$ goes over, for s = 0, into the eigenfunction of the angular part of the d'Alembert operator $\Box_{\alpha,\theta,\varphi}$.

With the help of the usual orthogonality condition for the single-particle states $\langle\lambda' s p'|p s\lambda\rangle=E\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}$ and formula (6), we can obtain from the expansion (13) the orthogonality condition for the states $|\rho mJMs\rangle$:

$$\langle sM'J'm'\rho'|\rho mJMs\rangle=\delta_{MM'}\delta_{JJ'}\delta_{mm'}\,\delta(\rho-\rho'). \qquad (14)$$

2. Let us now consider the two-particle helicity states. They are defined as the product of two one-particle states:

$$|\varkappa_1\mathbf{p}_1s_1\lambda_1\varkappa_2\mathbf{p}_2s_2\lambda_2\rangle=|\varkappa_1\mathbf{p}_1s_1\lambda_1\rangle|\varkappa_2\mathbf{p}_2s_2\lambda_2\rangle. \qquad (15)$$

As noted above, the one-particle states $|\kappa_i\mathbf{p}_i s_i\lambda_i\rangle$ (i = 1, 2) with nonvanishing mass $\kappa_i$ can be obtained[1,16] from the state at rest by the Lorentz transformation $Z_{|\mathbf{p}_i|}$ along the z axis and a subsequent rotation $R_{\varphi_i,\theta_i,-\varphi_i}$ about the angles $\varphi_i$, $\theta_i$ defining the direction of the momentum $\mathbf{p}_i$ of the particle, i.e.,

$$|\varkappa_i\mathbf{p}_is_i\lambda_i\rangle=R^{(i)}_{\varphi_i,\theta_i,-\varphi_i}Z^{(i)}_{|\mathbf{p}_i|}|\varkappa_is_i\lambda_i\rangle. \qquad (16a)$$

We rewrite this equality in terms of the infinitesimal operators **M** and **N**:

$$|\varkappa_i\mathbf{p}_is_i\lambda_i\rangle=\exp\{-i\varphi_iM_z^{(i)}-i\theta_iM_y^{(i)}+i\varphi_iM_z^{(i)}-i\alpha_iN_z^{(i)}\}|\varkappa_is_i\lambda_i\rangle \qquad (16b)$$

where $\tanh\alpha_i=|\mathbf{p}_i|/E_i$, $\cosh\alpha_i=E_i/\kappa_i$.

Using this relation, we can write the two-particle states in the system of their center of inertia ($\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p}$) in the following form:

$$|\varkappa_1,\varkappa_2,\mathbf{p}s_1\lambda_1s_2\lambda_2\rangle=\exp\{-i\varphi M_z-i\theta M_y$$
$$+i\varphi M_z-i\alpha_1N_z\}|A_{s_1\lambda_1,s_2\lambda_2}\rangle=R_{\varphi,\theta,-\varphi}Z_{|\mathbf{p}|/E_1}|A_{s_1\lambda_1,s_2\lambda_2}\rangle, \qquad (17)$$

where the angles φ, θ determine the direction of the relative momentum **p**: $\mathbf{M}=\mathbf{M}^{(1)}+\mathbf{M}^{(2)}$, $\mathbf{N}=\mathbf{N}^{(1)}+\mathbf{N}^{(2)}$. The states $|A_{s_1\lambda_1,s_2\lambda_2}\rangle$ are states of the particles in a coordinate system where the first particle is at rest; they can be obtained by the following transformation from the state at rest of the two particles:

$$|A_{s_1\lambda_1,s_2\lambda_2}\rangle=\exp\{i(\alpha_1+\alpha_2)N_z^{(2)}-i\pi M_y^{(2)}\}|\varkappa_1s_1\lambda_1,\varkappa_2s_2\lambda_2\rangle.$$

It follows from this formula that the two-particle states in the system of their center of inertia can be described by the four-velocity $U_1=(E_1/\kappa_1,\mathbf{p}_1/\kappa_1)$ of particle 1. In the same way one can show that these states can be described by the four-velocity $U_2=(E_2/\kappa_2,-\mathbf{p}/\kappa_2)$ of particle 2, and also by the four-velocity

$$U=\left(\operatorname{ch}\frac{\alpha_1+\alpha_2}{2},\quad \frac{\mathbf{p}}{|\mathbf{p}|}\operatorname{sh}\frac{\alpha_1+\alpha_2}{2}\right),$$

which is symmetric under exchange of particles 1 and 2.

The two-particle states in the laboratory system are obtained from these states by the Lorentz transformation $L_{-\mathbf{v}}$ with the velocity $\mathbf{v} = (\mathbf{p}_1 + \mathbf{p}_2)/(E_1 + E_2)$ of the center of inertia:[16]

$$|\varkappa_1\mathbf{p}_1 s_1\lambda_1,\ \varkappa_2\mathbf{p}_2 s_2\lambda_2\rangle =$$
$$=\sum_{(\lambda')} D^{(s_1)}_{\lambda_1'\lambda_1}(l,\ \mathbf{p}_1)\, D^{(s_2)}_{\lambda_2'\lambda_2}(l,\ \mathbf{p}_2)\, L_{-\mathbf{v}} R_{\varphi,\,\theta,\,-\varphi} Z_{|\mathbf{p}|/E_1} | A_{s_1\lambda_1' s_2\lambda_2'}\rangle. \tag{18}$$

Defining the final states $|\kappa_3 \mathbf{p}_3 s_3 \lambda_3,\ \kappa_4 \mathbf{p}_4 s_4 \lambda_4\rangle$ in an analogous way, we write the helicity scattering amplitude (HA) in the following form:

$$\langle\lambda_4 s_4 \mathbf{p}_4\varkappa_4,\ \lambda_3 s_3 \mathbf{p}_3\varkappa_3 | T | \varkappa_1\mathbf{p}_1 s_1\lambda_1,\ \varkappa_2\mathbf{p}_2 s_2\lambda_2\rangle = \sum_{(\lambda')} D^{*(s_4)}_{\lambda_4'\lambda_4}(l,\ \mathbf{p}_4)\, D^{*(s_3)}_{\lambda_3'\lambda_3}(l,\ \mathbf{p}_3)$$
$$\times\langle B_{s_4\lambda_4' s_3\lambda_3'} | Z^{-1}_{|\mathbf{p}'|/E_3} R^{-1}_{\varphi',\,\theta',\,-\varphi'} L^{-1}_{-v} T L_{-v} R_{\varphi,\,\theta,\,-\varphi} Z_{|\mathbf{p}|/E_1} | A_{s_1\lambda_1' s_2\lambda_2'}\rangle$$
$$\times D^{(s_1)}_{\lambda_1'\lambda_1}(l,\ \mathbf{p}_1)\, D^{(s_2)}_{\lambda_2'\lambda_2}(l,\ \mathbf{p}_2), \tag{19}$$

where the angles $\varphi'$, $\theta'$ determine the direction of the relative momentum $\mathbf{p}'$ of the final particles, and

$$|B_{s_4\lambda_4 s_3\lambda_3}\rangle = \exp\{i(\alpha_3+\alpha_4)N_z^{(4)}\}\exp\{-i\pi M_y^{(4)}\}\,|\varkappa_3 s_3\lambda_3,\ \varkappa_4 s_4\lambda_4\rangle.$$

The right-hand side of (19) can be written, using the fact that T is a scalar operator,

$$\langle B_{s_3\lambda_3 s_4\lambda_4} | Z^{-1}_{|\mathbf{p}'|/E_3} R^{-1}_{\varphi',\,\theta',\,-\varphi'} L^{-1}_{-v} T L_{-v} R_{\varphi,\,\theta,\,-\varphi} Z_{|\mathbf{p}|/E_1} | A_{s_1\lambda_1 s_2\lambda_2}\rangle$$
$$=\langle B_{s_3\lambda_3 s_4\lambda_4} | T Z^{-1}_{|\mathbf{p}'|/E_3} R^{-1}_{\varphi',\,\theta',\,-\varphi'} R_{\varphi,\,\theta,\,-\varphi} Z_{|\mathbf{p}|/E_1} | A_{s_1\lambda_1 s_2\lambda_2}\rangle$$
$$\equiv\langle B_{s_3\lambda_3 s_4\lambda_4} | T L(G) | A_{s_1\lambda_1 s_2\lambda_2}\rangle, \tag{20}$$

$$G = l_z^{-1}\left(\frac{|\mathbf{p}|}{E_3}\right) r^{-1}_{\varphi',\,\theta',\,-\varphi'} r_{\varphi,\,\theta,\,-\varphi} l_z\left(\frac{|\mathbf{p}|}{E_1}\right). \tag{20$'$}$$

In order to expand the HA in terms of the matrix elements of the representation $D^{(\rho,\,m)}_{J'M';\,JM}(G)$ of the Lorentz group G, we must first go over from the states $|A_{s_1\lambda_1 s_2\lambda_2}\rangle$ ($|B_{s_3\lambda_3 s_4\lambda_4}\rangle$) to the states $|A_{s_1 s_2\sigma\lambda}\rangle$ ($|B_{s_3 s_4\sigma'\mu}\rangle$) which are defined by the total spin of the two particles and its projection on the direction of the relative momentum of these particles. This is effected with the help of the Clebsch-Gordan coefficients:

$$|A_{s_1\lambda_1 s_2\lambda_2}\rangle = \sum_{\sigma} |A_{s_1 s_2\sigma\lambda}\rangle\langle s_1 s_2\sigma\lambda | s_1\lambda_1 s_2 - \lambda_2\rangle,$$

$$|B_{s_3\lambda_3 s_4\lambda_4}\rangle = \sum_{\sigma'} |B_{s_3 s_4\sigma'\mu}\rangle\langle s_3 s_4\sigma'\mu | s_3\lambda_3 s_4 - \lambda_4\rangle, \tag{21}$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$. Thus we obtain the following expansion for the HA:

$$\langle B_{s_3 s_4\sigma'\mu} | TL(G) | A_{s_1 s_2\sigma\lambda}\rangle = {}_{\sigma'\mu}T_{\sigma\lambda}(s,t)$$
$$=\sum_{m/2=-\min(\sigma,\,\sigma')}^{\min(\sigma,\,\sigma')}\int_0^\infty {}_{\sigma'}T_{\sigma}^{(\rho,\,m)}(s) D^{(\rho,\,m)}_{\sigma'\mu;\,\sigma\lambda}[G(s,t)]\,d\rho, \tag{22}$$

where the matrix elements of the representation of the Lorentz group have, according to (20), the form

$$D^{(\rho,\,m)}_{\sigma'\mu;\,\sigma\lambda}[G(s,t)] = \sum_{J=0}^{\infty}\Phi^{(\rho,\,m)}_{\sigma'\mu J}(-\alpha_3)\, d^{(J)}_{\mu\lambda}(+\theta)\,\Phi^{\ \ m)}_{J\lambda\sigma}(\alpha_1). \tag{23}$$

The angle $\theta$ in (23) is the angle between the relative momenta of the initial and final particles, i.e., the scattering angle

$$\operatorname{th}\alpha_1 = |\mathbf{p}|/E_1,\quad \operatorname{th}\alpha_3 = |\mathbf{p}'|/E_3.$$

Using the representation for the Lorentz group element $G = r_2 l_z r_1$, the expression for the coordinate transformation matrices, and expression (20′), one can write the matrix elements

$$D^{(\rho,\,m)}_{\sigma'\mu;\,\sigma\lambda}[G(s,t)]$$

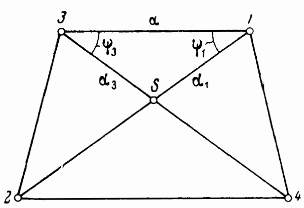
in the form

$$D^{(\rho,\,m)}_{\sigma'\mu;\,\sigma\lambda}[G(s,t)]$$
$$=\sum_M d^{(\sigma')}_{\mu M}(\pi-\psi_3)\,\Phi^{(\rho,\,m)}_{\sigma' M\sigma}(\alpha)$$
$$\times d^{(\sigma)}_{M\lambda}(\psi_1), \tag{24}$$

where the angles $\psi_3$, $\alpha$, $\psi_1$ are expressed through $\alpha_3$, $\theta$, $\alpha_1$ in the following way:

$$\operatorname{ch}\alpha = \operatorname{ch}\alpha_1\operatorname{ch}\alpha_3 - \operatorname{sh}\alpha_1\operatorname{sh}\alpha_3\cos\theta,\qquad \operatorname{sh}\alpha\cos\psi_1 = \operatorname{ch}\alpha_3\operatorname{sh}\alpha_1 - \operatorname{sh}\alpha_3\operatorname{ch}\alpha_1\cos\theta,$$
$$\operatorname{sh}\alpha\cos\psi_3 = \operatorname{ch}\alpha_1\operatorname{sh}\alpha_3 - \operatorname{sh}\alpha_1\operatorname{ch}\alpha_3\cos\theta. \tag{25}$$

These relations give a simple geometrical meaning to the angles $\psi_3$, $\alpha$, $\psi_1$, which is indicated in the kinematic graph of the reaction 1 + 2 → 3 + 4 (cf. the figure).[17]




Substituting (24) in (22), we find that the required expansion of the HA has the form

$$ {}_{\sigma'\mu}T_{\sigma\lambda}(s,t) = \sum_{m/2=-\min(\sigma,\,\sigma')}^{\min(\sigma,\,\sigma')}\int_0^\infty {}_{\sigma'}T_{\sigma}^{(\rho,\,m)}(s)\sum_{M=-\min(\sigma,\,\sigma')}^{\min(\sigma,\,\sigma')} d^{(\sigma')}_{\mu M}(\pi-\psi_3)$$
$$\cdot\Phi^{(\rho,\,m)}_{\sigma' M\sigma}(\alpha)\, d^{(\sigma)}_{M\lambda}(\psi_1)\,d\rho. \tag{26}$$

Expressing the energies and momenta of the particles through the invariant variables s and t, we obtain a manifestly invariant form for the angles $\psi_3$, $\alpha$, $\psi_1$:

$$\operatorname{ch}\alpha = \frac{p_1 p_3}{\varkappa_1\varkappa_3} = \frac{\varkappa_1^2+\varkappa_3^2-t}{2\varkappa_1\varkappa_3},$$

$$\cos\psi_1 = \frac{(s+\varkappa_1^2-\varkappa_2^2)(\varkappa_1^2+\varkappa_3^2-t)-2\varkappa_1^2(s+\varkappa_3^2-\varkappa_4^2)}{\Delta^{1/2}(s,\varkappa_1^2,\varkappa_2^2)\,\Delta^{1/2}(t,\varkappa_1^2,\varkappa_3^2)},$$

$$\cos\psi_3 = \frac{(s+\varkappa_3^2-\varkappa_4^2)(\varkappa_1^2+\varkappa_3^2-t)-2\varkappa_3^2(s+\varkappa_1^2-\varkappa_2^2)}{\Delta^{1/2}(s,\varkappa_3^2,\varkappa_4^2)\,\Delta^{1/2}(t,\varkappa_1^2,\varkappa_3^2)} \tag{27}$$

where

$$\Delta(a,b,c) = a^2+b^2+c^2-2ab-2bc-2ac.$$

Let us consider the special case where the spins of the particles are zero. Then $\sigma' = \sigma = 0$, $\mu = \lambda = M = 0$, $m = 0$, the matrix elements of the Lorentz transformation are

$$\Phi^{\rho,0}_{0,0,0}(\alpha) = \sin(1/2\rho\alpha)/1/2\rho\operatorname{sh}\alpha$$

and expression (26) for the HA goes over into the integral expansion of a single scalar amplitude with respect to the invariant variable t:

$$T(s,t) = \int_0^\infty T(s,\rho)\frac{\sin(\rho\alpha)}{1/4\rho\operatorname{sh}\alpha}\,d\rho, \tag{28}$$

which was first obtained by Dolginov and Toptygin.[10]

Substituting now (23) in (22) and performing first the integration over $\rho$ and the summation over m, we obtain an expansion of the partial helicity scattering amplitude:

$$\langle\lambda_4 s_4\lambda_3 s_3|T_J(s)|s_1\lambda_1 s_2\lambda_2\rangle=\sum_{m/2=-\min(\sigma,\sigma')}^{\min(\sigma,\sigma')}\int_0^\infty\sum_{\sigma,\sigma'}\langle-\lambda_4 s_4\lambda_3 s_3|s_3 s_4\sigma'\mu\rangle$$

$$\times{}_{\sigma'}T_\sigma^{\rho m}(s)\langle\lambda\sigma s_2 s_1|s_1\lambda_1 s_2-\lambda_2\rangle\Phi_{\sigma'\mu J}^{\rho m}\Big(-\frac{|p'|}{E_3}\Big)\Phi_{J\lambda\sigma}^{\rho m}\Big(\frac{|p|}{E_1}\Big)d\rho. \quad (29)$$

Using (A.11), we find that the partial HA behaves like $|p'|^{J-s_3-s_4}|p|^{J-s_1-s_2}$ when $|p'|$ and $|p|$ tend to zero.

3. Let us consider the symmetry properties of the scattering amplitude. To this end we write the expansion for the two-particle state, in the system of their center of inertia, in terms of states which transform according to an irreducible unitary representation of the Lorentz group, using (19):

$$|\mathbf{p}s_1\lambda_1 s_2\lambda_2\rangle=\sum_m\int_0^\infty|\rho mJMs_1s_2\sigma\rangle\langle s_1s_2\sigma\lambda|s_1\lambda_1 s_2-\lambda_2\rangle$$
$$\times D_{M\lambda}^{(J)}(\varphi,\theta,-\varphi)\Phi_{J,\sigma,\lambda}^{(\rho m)}\Big(-\frac{|\mathbf{p}|}{E_1}\Big)d\rho,$$

$$|\rho mJMs_1s_2\sigma\rangle=\int\sum_{(\lambda)}(N_{J,\sigma}^{\rho m}N_J)^{-1}|\mathbf{p}s_1\lambda_1s_2\lambda_2\rangle\langle-\lambda_2s_2\lambda_1s_1|s_1s_2\sigma\lambda\rangle$$
$$\times D_{M\lambda}^{*(J)}(\varphi,\theta,-\varphi)\Phi_{J,\sigma,\lambda}^{*(\rho m)}\Big(\frac{|\mathbf{p}|}{E_1}\Big)\frac{dp}{E_1}. \quad (30)$$

Let us determine the action of the operators of reflection P, of particle exchange $P_{12}$, and time reversal T on the states $|\rho mJMs_1s_2\sigma\rangle$. Using the definition of the helicity state with definite total angular momentum J,[1] we write formula (30) in the form

$$|\rho mJMs_1s_2\sigma\rangle=(N_{J,\sigma}^{\rho m})^{-1}\sum_{\lambda_1,\lambda_2}\int_0^\infty|EJMs_1\lambda_1s_2\lambda_2\rangle\langle-\lambda_2s_2\lambda_1s_1|s_1s_2\sigma\lambda\rangle$$
$$\times\Phi_{J,\sigma,\lambda}^{*(\rho m)}\Big(\frac{|\mathbf{p}|}{E_1}\Big)\frac{p^2dp}{E_1}. \quad (31)$$

The effect of the action of the operators P, $P_{12}$, and T on the states $|EJMs_1\lambda_1s_2\lambda_2\rangle$ are defined by[1]

$$P|EJMs_1\lambda_1s_2\lambda_2\rangle=\eta_1\eta_2(-1)^{J-s_1-s_2}|EJMs_1-\lambda_1s_2-\lambda_2\rangle, \quad (32)$$

where $\eta_1$ and $\eta_2$ are the intrinsic parities of particles 1 and 2, respectively,

$$P_{12}|EJMs_1\lambda_1s_2\lambda_2\rangle=(-1)^{J-s_1-s_2}|EJMs_2\lambda_2s_1\lambda_1\rangle, \quad (33)$$

$$T|EJMs_1\lambda_1s_2\lambda_2\rangle=(-1)^{J-M}|EJ-Ms_1\lambda_1s_2\lambda_2\rangle. \quad (34)$$

With the help of (32) and (33) it is now easy to obtain from (31)

$$P|\rho mJMs_1s_2\sigma\rangle=\eta_1\eta_2(-1)^{J-\sigma}|\rho-mJMs_1s_2\sigma\rangle. \quad (35)$$

[here we have used (A.4)];

$$P_{12}|\rho mJMs_1s_2\sigma\rangle=(-1)^{J-\sigma}|\rho-mJMs_1s_2\sigma\rangle. \quad (36)$$

Hence, for identical particles, $s_1 = s_2 = s$, $(s_1 + s_2)^2 = s_{12}^2$, the states with a definite symmetry with respect to particle exchange are defined in the following way:

$$\{1+(-1)^{2s}P_{12}\}|\rho mJMs_{12}\rangle=|\rho mJMs_{12}\rangle+(-1)^{J+2s-s_{12}}|\rho-mJMs_{12}\rangle.$$

The difference in the signs for Bose-Einstein and Fermi-Dirac statistics is taken into account by the factor $(-1)^{2S}$.

Analogously, we have from (34) and (31)

$$T|\rho mJMs_1s_2\sigma\rangle=(-1)^{J-M}|pmJ-Ms_1s_2\sigma\rangle. \quad (37)$$

Thus we obtain the following symmetry property for the $(\rho, m)$ amplitude from the parity conservation law $P^{-1}SP = S$:

$$\langle\sigma'|T^{(\rho,m)}(s)|\sigma\rangle=\frac{\eta_1\eta_2}{\eta_3\eta_4}(-1)^{\sigma-\sigma'}\langle\sigma'|T^{(\rho,-m)}(s)|\sigma\rangle, \quad (38)$$

the properties of the S matrix with regard to time reversal $T^{-1}ST = S^{-1}$ imply that the matrix $T^{\rho,m}$ is symmetric:

$$\langle\sigma'|T^{(\rho,m)}(s)|\sigma\rangle=\langle\sigma|T^{(\rho,m)}(s)|\sigma'\rangle. \quad (39)$$

It seems to us that the considerations of this paper are of some interest for the reason that, using the analytic properties of the $(\rho, m)$ amplitudes, one may attempt to determine the asymptotic value of the scattering amplitude for large values of the energy.

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## APPENDIX

Using the results of[8], we calculate the asymptotic form of the function $e^a e^{-i\rho a/2}\Phi_{J,s;\lambda=m/2}^{(\rho,m)}(a)$ for $a\to\infty$ $[\tilde\Phi_{J,s;\lambda=m/2}^{(\rho,m)}(\infty)]$. Introducing the new notation $J_1\equiv J$, $J_2 = w$, $J\equiv s$, we rewrite the explicit expression for $\Phi_{J,s;\lambda=s}^{(\rho,m)}(a)$ [cf. formula (12) of[8]] in the form

$$\Phi_{J,s;\lambda=s}^{(\rho,m)}(a)=i^{J-s}(1-e^{-2a})^{J-s}\exp\Big\{-\Big(s+1-\frac{m}{2}-\frac{i}{2}\rho\Big)a\Big\}$$
$$\times F\Big(J+1-\frac{i}{2}\rho,J+1-\frac{m}{2},2J+2;1-e^{-2a}\Big). \quad (A.1)$$

It follows from the definition of the generalized Legendre function of the second kind,[18]

$$Q_{\mu\nu}^{(J)}(z)=e^{i\pi(\mu-\nu)}\frac{\Gamma(J+1+\mu)\Gamma(J+1-\nu)}{2\Gamma(2J+2)}\Big(\frac{z-1}{2}\Big)^{-(J+1)}\Big(\frac{z+1}{z-1}\Big)^{(\nu+\mu)/2}$$
$$\times F\Big(J+1+\nu;J+1+\mu;2J+2;\frac{2}{1-z}\Big) \quad (A.2)$$

that the function $\Phi_{J,s;\lambda=s}^{(\rho,m)}(a)$ is simply expressed through the functions $Q_{-m/2,-i\rho/2}^{(J)}(1-2/(1-e^{-2a}))$. As is easily seen from formula (7) of[8], the expression for the function $\Phi_{J,s;\lambda=-s}^{(\rho,m)}(a)$ is obtained from (A.1) by replacing m by −m. Let us rewrite also the recurrency formula (10):[8]

$$\gamma_\lambda\Phi_{J,s;\lambda-1}^{(\rho,m)}(a)=2\operatorname{sh}a\Big(\lambda\frac{\partial}{\partial a}+\lambda\operatorname{cth}a-\frac{im\rho}{4}\Big)\Phi_{J,s;\lambda}^{(\rho,m)}(a)$$
$$+\gamma_{\lambda+1}\Phi_{J,s;\lambda+1}^{(\rho,m)}(a), \quad (A.3a)$$

where

$$\gamma_\lambda=\alpha_\lambda^{(J)}\alpha_\lambda^{(s)}=[(J+\lambda)(J-\lambda+1)(s+\lambda)(s-\lambda+1)]^{1/2}.$$

Replacing in this formula $\lambda$ by $-\lambda$, and using $\gamma_\lambda=\gamma_{-\lambda+1}$, we obtain

$$\gamma_\lambda \Phi^{(\rho, m)}_{J, s; 1-\lambda}(a) = 2\,\mathrm{sh}\, a\left(\lambda \frac{\partial}{\partial a} + \lambda \,\mathrm{cth}\, a + \frac{im\rho}{4}\right)\Phi^{(\rho, m)}_{J, s; -\lambda}(a) + \gamma_{\lambda+1}\Phi^{(\rho, m)}_{J, s; -\lambda-1}(a). \quad \text{(A.3b)}$$

This leads to the relation

$$\Phi^{(\rho, m)}_{J, s; -\lambda}(a) = \Phi^{(\rho, -m)}_{J, s; \lambda}(a). \quad \text{(A.4)}$$

The asymptotic form of the function $\Phi^{(\rho, m)}_{J, s; \lambda = s}(a)$ for $a \to \infty$ is simply found from (A.1):

$$\Phi^{(\rho, m)}_{J, s; \lambda=s}(\infty) = i^{J-s}\exp\left\{-\left(s+1-\frac{m}{2}-\frac{i}{2}\rho\right)a\right\} \times F\left(J+1-\frac{i}{2}\rho, J+1-\frac{m}{2}; 2J+2; z=1\right) \quad \text{(A.5)}$$

Substituting (A.1) in (A.3a) and going to the limit $a \to \infty$, we now find

$$\Phi^{(\rho, m)}_{J, s, \lambda=s-1}(\infty) = \frac{1}{\gamma_s}\left(s-\frac{m}{2}\right)\left(\frac{i}{2}\rho - s\right)\exp\left\{-\left(s-\frac{m}{2}-\frac{i}{2}\rho\right)a\right\} \times F\left(J+1-\frac{i}{2}\rho, J+1-\frac{m}{2}, 2J+2; z=1\right). \quad \text{(A.6)}$$

Repeating this procedure, we obtain the following asymptotic form of the function $\Phi^{(\rho, m)}_{J, s; \lambda}(a)$ for $a \to \infty$ (for positive $\lambda$):

$$\Phi^{(\rho, m)}_{J, s; \lambda}(\infty) = \prod_{k=\lambda+1}^{s}\frac{1}{\gamma_k}\left(k-\frac{m}{2}\right)\left(\frac{i}{2}\rho - k\right)\exp\left\{-\left(\lambda+1-\frac{m}{2}-\frac{i}{2}\rho\right)a\right\} \times F\left(J+1-\frac{i}{2}\rho, J+1-\frac{m}{2}, 2J+2; z=1\right). \quad \text{(A.7)}$$

Indeed, we can see by substituting (A.7) in (A.3a) that the asymptotic expression for the function $\Phi^{(\rho, m)}_{J, s; \lambda}(a)$ satisfies the recurrence relation for $a \to \infty$. Formula (A.7) leads to the very important assertion that this asymptotic expression differs from zero only for $m/2 \le \lambda$. Finally, we find an expression for the normalization factor from (A.4) and (A.7):

$$N^{(\rho, m)}_{J, s} = 2\pi|\tilde{\Phi}^{(\rho, m)}_{J, s; \lambda=m/2}(\infty)|^2 = 2\pi\left[\left(s-\frac{|m|}{2}\right)!\right]^2 \times \prod_{k=|m|/2+1}^{s}\frac{(k^2+\rho^2/4)}{\gamma_k^2}\left|\frac{\Gamma(2J+1)\Gamma(i\rho/2+|m|/2)}{\Gamma(J+1+i\rho/2)\Gamma(J+1+|m|/2)}\right|^2. \quad \text{(A.8)}$$

Here we have used formula (9.34) of[14]:

$$F(\alpha, \beta, \gamma; z=1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

where $\mathrm{Re}(\gamma-\alpha-\beta) > 0$.

Let us now calculate the asymptotic form of the function $\Phi^{(\rho, m)}_{J, s; \lambda}(a)$ for $a \to 0$. The asymptotic form of this function for $\lambda = s$ is easily obtained from (A.1):

$$\Phi^{(\rho, m)}_{J, s, \lambda=s}(a) = (2ia)^{J-s} \quad \text{(A.9)}$$

For $\lambda = s - 1$, we find the asymptotic form by substituting (A.1) in (A.3) and going to the limit $a \to 0$:

$$\Phi^{(\rho, m)}_{J, s, \lambda=s-1}(a) = \frac{\gamma_s}{J+s}(2ia)^{J-s}, \quad a \to 0. \quad \text{(A.10)}$$

Repeating this procedure, we obtain the following asymptotic form of the function $\Phi^{(\rho, m)}_{J, s; \lambda}(a)$ for $a \to 0$:

$$\Phi^{(\rho, m)}_{J, s; \lambda}(a) = \frac{\gamma_s\gamma_{s-1}\ldots\gamma_{\lambda+1}}{(J+s)\ldots(J+|\lambda|+1)(s-|\lambda|)!}(2ia)^{J-s} = \frac{1}{(s-|\lambda|)!}\prod_{k=|\lambda|+1}^{s}\frac{\gamma_k}{J+k}(2ia)^{J-s}. \quad \text{(A.11a)}$$

The validity of this formula can again be proved by substituting (A.10) in (A.3a). We note that the coefficient in this asymptotic expression is different from zero only for $J \ge s$, and for $J = s$ it is equal to unity. Small values of a correspond to small values of the momentum $|\mathbf{p}|$: $\sinh a = |\mathbf{p}|/\kappa$. Therefore formula (A.10) can be rewritten in the form

$$\Phi^{(\rho, m)}_{J, s; \lambda}\left(\frac{|\mathbf{p}|}{E}\right) = \frac{(2i)^{J-s}}{(s-|\lambda|)!}\prod_{k=|\lambda|+1}^{s}\frac{\gamma_k}{(J+k)}\left(\frac{|\mathbf{p}|}{\varkappa}\right)^{J-s}, \quad |\mathbf{p}| \to 0. \quad \text{(A.11b)}$$

It is of interest to consider the asymptotic form of the function $\Phi^{(\rho, m)}_{J, s; \lambda}(a)$ for large $\rho$. For simplicity we consider the case where $s = 0$. Then m and $\lambda$ are zero and $J = l$. The value of this function is easily obtained from (A.1). As is known, the asymptotic form of the hypergeometric function with respect to the parameter $l$[13] is

$$F(a, b, c; z) = e^{-i\pi a}\frac{\Gamma(c)}{\Gamma(c-a)}(bz)^{-a}[1+O(|bz|^{-1})] + \frac{\Gamma(c)}{\Gamma(a)}e^{bz}(bz)^{a-c}[1+O(|bz|^{-1})]; \quad \text{(A.12)}$$

z, c, and a are fixed and $-3\pi/2 \le \arg bz \le \pi/2$. In the case $a = l + 1$, $b = l + 1 - i\rho/2$, $c = 2l + 2$ we have

$$F\left(l+1, l+1-\frac{i}{2}\rho, 2l+2; z\right) = \frac{\Gamma(2l+2)}{\Gamma(l+1)}\left(-\frac{i}{2}\rho z\right)^{-(l+1)}\{e^{-i(l+1)\pi} + e^{(l+1-i\rho/2)z}\}, \quad \rho \to \infty. \quad \text{(A.13)}$$

Hence, for

$$\left|F\left(l+1, l+1-\frac{i}{2}\rho, 2l+2; z\right)\right|^2 = \left(\frac{\Gamma(2l+2)}{\Gamma(l+1)}\right)^2\left(\frac{\rho z}{4}\right)^{-2(l+1)} \times\left\{1+e^{2(l+1)z}+(-1)^{l+1}e^{(l+1)z}\cos\frac{\rho z}{2}\right\}. \quad \text{(A.14)}$$


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