DETERMINATION OF THE DISTRIBUTION FUNCTION OF PLASMA ELECTRONS FROM THE BREMSSTRAHLUNG SPECTRUM

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A stable method of constructing approximate solutions of integral equations of the first kind (regularization method) is used for the analysis of the bremsstrahlung spectrum of the electrons of a plasma heated by electron-cyclotron resonance in an adiabatic trap. This yields the electron energy distribution function $f(\epsilon)$, (assuming that the function is isotropic), and a clearly pronounced second local maximum is observed in the energy region $\epsilon \cong (80-85)$ keV ("two-temperature" behavior of the plasma).

1. The plasma-electron energy distribution function $f(\epsilon)$ is one of the principal characteristics of a high-temperature plasma. It can be determined by experimentally measuring the bremsstrahlung spectrum $N(E_0)$. To this end it is necessary to solve a Fredholm integral equation of the first kind in the form

$$\int_a^b H(E_0, \varepsilon) f(\varepsilon) d\varepsilon = N(E_0).$$

it is known that the solution of such a problem entails considerable difficulties due to the absence of a continuous dependence of the solution on the right-hand side $N(E_0)$, namely, small changes of $N(E_0)$ can correspond to arbitrarily large changes of the solution. Thus, if the integral is replaced by a finite sum and the resultant system of linear algebraic equations is solved, then as the number of terms in this sum increases the result becomes worse and can lead to absurd conclusions. Since the experimentally obtained bremsstrahlung spectrum $N(E_0)$ contains uncontrolable measurement errors, the foreoing difficulties are inherent in the problem of determining the electron distribution function $f(\epsilon)$ from the bremsstrahlung spectrum $N(E_0)$. In the present article, to solve this problem, we use a stable method of solving integral equations of the first kind (regularization method), which has been developed in[1] and in which small changes of $N(E_0)$ correspond to small changes of the sought function $f(\epsilon)$. The application of this method to the analysis of the bremsstrahlung spectrum of the electrons of a plasma heated by electron-cyclotron resonance in a stationary adiabatic trap[2] has made it possible to determine the distribution function $f(\epsilon)$ and to establish the presence of a second local maximum, i.e., to show that the plasma is of the "two-temperature" type (Fig. 1; variants I and II are explained later in the discussion of Fig. 2).

2. If the bremsstrahlung is produced upon scattering of the electrons by the plasma ions, then the radiation-intensity distribution is described sufficiently well by the function

$$I(E) = \int_E^\infty 3 \cdot 10^{15} z^2 N_0 f(\varepsilon) \frac{d\varepsilon}{\sqrt{\varepsilon}} \; [\mathrm{cm}^{-3}\,\mathrm{sec}^{-1}]$$

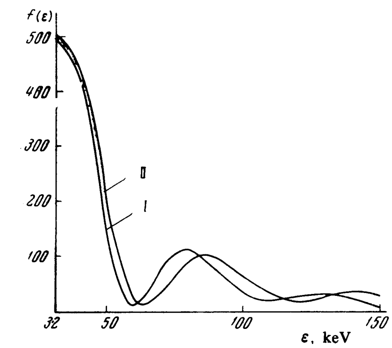

FIG. 1

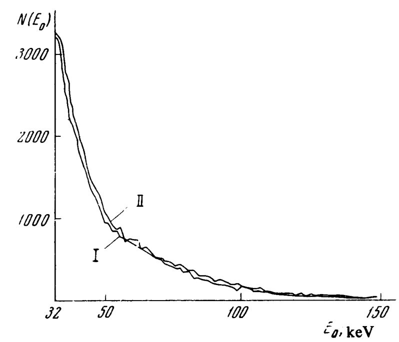

FIG. 2

where $I(E)$ is the radiation intensity of photons with energy E from a unit volume of the plasma, z is the charge of the scattering centers, N_0 is the density of the scattering centers, and $f(\epsilon)$ is the electron energy distribution function (E and ϵ are in keV).

We disregard here relativistic effects, the function $f(\epsilon)$ is assumed isotropic is space[1)], and it is assumed that an electron with energy ϵ radiates energy uniformly in the entire energy interval from 0 to ϵ. The number of quanta with energy E, radiated by a

[1)]The mathematical model assumed here is applicable also in the case of small anistropy. The influence of the possible anisotropy in adiabatic traps is not considered here.

unit volume of the plasma per unit time, is

$$n(E)=\frac{k_0}{E}\int_E^\infty f(\varepsilon)\frac{d\varepsilon}{\sqrt{\varepsilon}},$$

$$k_0=3\cdot10^{15}z^2N_0.$$

Real spectroscopic instruments, as a rule, distort in the spectrum for two reasons.

a) There is always absorption of the radiation in the entrance windows, in the wrappers of the crystals, etc.; this absorption depends on the energy of the radiation quanta. As a result, the number of quanta with energy E registered by the instrument is

$$n'(E)=f_1(E)n(E),$$

where $f_1(E)$ is the ratio of the number of quanta with energy E, incident on the detector per unit time, to the number of registered quanta.

b) As a result of the finite energy resolution of the instrument, the number of quanta registered per unit time by the spectrometer output unit tuned to a quantum energy E_0 is

$$N(E_0)=\int_0^\infty n'(E)f_2(E_0,E)\,dE,$$

where $f_2(E_0, E)$ is a function characterizing the energy resolution of the instrument. Substituting here $n'(E)$ and changing the order of the integration, we obtain

$$N(E_0)=\int_0^\infty K(E_0,\varepsilon)\varphi(\varepsilon)\,d\varepsilon, \tag{1}$$

where

$$K(E_0,\varepsilon)=\frac{k_0}{\varepsilon}\int_0^\varepsilon\frac{f_1(E)}{E}f_2(E_0,E)\,dE,$$

$$\varphi(\varepsilon)=\sqrt{\varepsilon}f(\varepsilon);\ \varepsilon,\ E \text{ and } E_0 \text{ in keV} \tag{2}$$

Thus, when the spectrum of the bremsstrahlung is known, it is possible to determine the electron distribution function by solving the integral equation (1).

3. We analyzed the electron bremsstrahlung spectra of a plasma heated by electron-cyclotron resonance in a stationary adiabatic trap. A detailed description of this setup and the results of the experiment are given in[2]. The heating was in the pulsed mode, after which an exponential decrease of the plasma energy was observed, with a characteristic time ~1 msec.

The spectral distribution of the bremsstrahlung in the energy range (20– 150) keV was measured by a scintillation gamma spectrometer with an NaI(Tl) crystal and an AI-101 pulse-height analyzer. The input of the analyzer could be opened for a short time (~10 μsec) at an arbitrary (and adjustable) instant during the time of the plasma decay.

4. In the energy range of interest to us, the absorption of the quanta in the scintillator occurs mainly as a result of the photoeffect on the K shell of the iodine. At a primary-quantum energy close to the K-radiation energy, some of the quanta of this secondary radiation may leave the crystal[3]. As a result, an additional peak appears in the spectrum, located 28 keV below the energy of the primary radiation; in our case this peak was practically not registered by the spectrometer. The indicated effect is equivalent to an attention of the primary radiation. This attenuation, for $E \geq 32$ keV, was taken into account by introducing the function $f_1(E)$, which was assumed equal to[2)]

$$f_1(E)=\{1+\exp[-(0.124+0.0325E)]\}^{-1} \text{ for } E\geqslant 32 \text{ keV}.$$

$f_1(E)$ has a different form for $E < 32$ keV, for at such energies the photoabsorption occurs primarily on the L-shell of the iodine, and the absorption in the windows and in the wrappers of the crystal becomes appreciable in the employed spectrometer. We shall therefore confine ourself henceforth to the region $E \geq 32$ keV.

For scintillation spectrometers, the function $f_2(E_0, E)$ is described sufficiently well by a Gaussian curve[3] with a dispersion determined experimentally from the reduction of the gamma-radiation spectra of Am^{241} and Cf^{237}:

$$f_2(E_0,E)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(E)}\exp\left\{\frac{-(E_0-E)^2}{2\sigma^2(E)}\right\},$$

$$\sigma(E)=\frac{E}{2.35}\left(0.051+\frac{5,94}{E}\right)^{1/2}.$$

The obtained experimental spectra are shown in Fig. 2. To reduce the time of the experiment, the heating was effected while the chamber was filled with argon. The quanta were registered for ~10 μsec immediately after the end of the heating (variant I) and for the same time interval starting with the instant of time when the diamagnetism of the plasma was decreased by a factor e (variant II). The results were summed by the analyzer over a large number of apparatus cycles (~10^5 cycles).

5. It is practically impossible to calculate the integral over the infinite interval. When $\epsilon > 150$ keV, $f(\epsilon)$ is small, so that we can confine ourselves to integration up to b = 150 keV. When $E < a = 32$ keV, the function $f_1(E)$ has a different form. Therefore the kernel $K(E_0, \epsilon)$ is replaced by the kernel

$$K_1(E_0,\varepsilon)=\frac{k_0}{\varepsilon}\int_a^\varepsilon\frac{f_1(E)}{E}f_2(E_0,E)\,dE.$$

Obviously for $\epsilon \geq a$ we have

$$\Delta K=K(E_0,\varepsilon)-K_1(E_0,\varepsilon)=\frac{k_0}{\varepsilon}\int_0^a\frac{f_1(E)}{E}f_2(E_0,E)\,dE.$$

Inasmuch as the function $f_2(E_0, E) \leq 0.1$ and $f_2(E_0, E)$ is a Gaussian curve when $|E_0 - E| \geq 10$ keV, ΔK is small in the case when $E_0 \geq 42.8$ keV[3)], and the kernel $K(E_0, \epsilon)$ can be replaced by $K_1(E_0, \epsilon)$.

Thus, in place of (1) and (2) we have solved the equation

$$\int_a^b K_1(E_0,\varepsilon)\varphi(\varepsilon)\,d\varepsilon=N(E_0),\quad H=\sqrt{\varepsilon}K_1, \tag{3}$$

$$K_1(E_0,\varepsilon)=\frac{k_0}{\varepsilon}\int_a^\varepsilon\frac{f_1(E)}{E}f_2(E_0,E)\,dE,$$

$$c\leqslant E_0\leqslant d,\quad a=32,\quad c=42.8,\quad b=d=150 \text{(in keV)}. \tag{4}$$

[2)]This function can be obtained by interpolating the graphic dependence given in [3].

[3)]Experimental values of $N(E_0)$ are available for E_0 = 32, 33.7, 35.2, 36.8, 38.2, 39.8, 41.3, 42.8, 44.4, . . .

Inasmuch as the right side of (3) was determined approximately, the problem consisted of finding the approximate solution of this equation. It was constructed by the regularization method.

There exists an infinite family B_δ of functions $\varphi(\epsilon)$ satisfying Eq. (3) with a specified accuracy δ, in the sense that

$$\int_c^d \left\{ \int_a^b K_1(E_0, \varepsilon)\varphi(\varepsilon)\,d\varepsilon - N(E_0) \right\}^2 dE_0 \leqslant \delta^2. \qquad (5)$$

Among the functions of the family B_δ there are functions that differ from one another to an arbitrarily large degree. It is therefore impossible to choose as an approximate solution of (3) the exact solution of this equation (with an approximate right-side part). Let $\tilde{N}(E_0)$ be the exact value of the right hand side of (3), and let $\tilde{\varphi}(\epsilon)$ be the corresponding exact solution of this equation.

To obtain for Eq. (3) an approximate solution $\varphi_\delta(\epsilon)$ satisfying the natural requirements that are imposed in the sense of the approximate solution, namely the conditions $\varphi_\delta(\epsilon) \to \tilde{\varphi}(\epsilon)$ as $\delta \to 0$, it is necessary to choose from among the entire set B_δ of the formal solutions satisfying condition (5) a function $\tilde{\varphi}_\delta(\epsilon)$ that satisfies the condition $\tilde{\varphi}_\delta(\epsilon) \to \tilde{\varphi}(\epsilon)$ as $\delta \to 0$. This can be done if additional information, at least of qualitative character, is available with respect to the sought solution. It is natural to assume for the considered problem that the sought solution is a smooth function. As a measure of the smoothness we assume the quantity

$$\Omega[\varphi] = \int_a^b \left(\frac{d\varphi}{d\varepsilon}\right)^2 d\varepsilon.$$

By way of the aforementioned function $\tilde{\varphi}_\delta(\epsilon)$, we choose from the family B_δ of the approximate solutions of Eq. (3) the function having the greatest smoothness (the minimal fine structure). Mathematically, this reduces to finding that a function from the family B_δ, on which the minimum of the functional $\Omega[\varphi]$ is reached. The solution of such a problem for the conditional extremum reduces to the problem of the unconditional extremum of the functional

$$M^\alpha = \int_c^d \left\{ \int_a^b K_1(E_0, \varepsilon)\varphi(\varepsilon)\,d\varepsilon - N(E_0) \right\}^2 dE_0 + \alpha\Omega[\varphi],$$

where the parameter α (the regularization parameter), can be determined to satisfy the specified accuracy δ, and consequently depends on δ. The indicated solution of the problem of finding the minimum of the functional M^α is called the regularized (approximate) solution $\varphi_\alpha(\epsilon)$ of Eq. (3) and can be found from Euler's equation for the functional M^α.

The parameter α is determined from the condition that the quadratic deviation δ_α^2 over the segment $[c, d]$ of the integral

$$N_\alpha(E_0) = \int_a^b K_1(E_0, \varepsilon)\varphi_\alpha(\varepsilon)\,d\varepsilon$$

from $N(E_0)$ is equal to the quadratic error δ^2 with which the experimental data $N(E_0)$ are known, i.e., from the condition

$$\int_c^d \left\{ N_\alpha(E_0) - N(E_0) \right\}^2 dE_0 = \delta^2.$$

The quantity δ_α^2 characterizes the accuracy with which Eq. (3) is satisfied. If $\alpha \to 0$, then $\delta_\alpha^2 \to 0$. Therefore it is possible to find a value of α such that $\delta_\alpha^2 = \delta^2$.

The quadratic error δ^2 consists of the statistical measurement error δ_{st}^2 and the apparatus error δ_{app}^2 due to the long measurement time (variation of $f(\epsilon)$ from cycle to cycle of the setup, etc). In the experiments, such apparatus errors are difficult to estimate. We assume that $\delta_{app}^2 \approx \delta_{st}^2$, and consequently $\delta^2 = m\delta_{st}^2$, where $m = 2$.

In the described experiments, $\delta_{app}^2 = 4 \times 10^2(b - a)$. From these data it is possible to determine the parameter α algorithmically, with the aid of a computer as part of the determination of the function $\varphi_\alpha(\epsilon)$.

Figure 1 shows a solution of the problem (3) and (4) for two variants of the initial data (Fig. 2).

The random scatter of the measurement results within the limits of the indicated quadratic error leads to changes of the obtained function $f(\epsilon)$ in a range 5–6% (15–20% in the region of the minimum, i.e., at $\epsilon = (65 \pm 3$ keV). Thus, the indicated scatter does not change the qualitative character of the results, namely the "two-temperature character" of the plasma.

If we choose $m = 3, 4, 5, 6$, then the qualitative character of the determined function $f(\epsilon)$ does not change, namely $f(\epsilon)$ retains a local maximum at $\epsilon \approx (80–85)$ keV and a minimum at $\epsilon \approx 65$ keV. The magnitude of the local minimum of the function (at $\epsilon \approx 65$) is sensitive to the accuracy of the initial data $N(E_0)$. To obtain a value more accurately, it is necessary to have more accurate data $N(E_0)$ and a more reliable estimate of their accuracy.

Let us summarize the results of the work.

1) Application of the regularization method to the analysis of the bremsstrahlung spectrum in the described experiment not only makes it possible to determine reliably the distribution function of the electron, in the plasma, but also reveals a clearly pronounced second local maximum of this distribution function ("two-temperature character" of the plasma). The second maximum of the function $f(\epsilon)$ is reached in both variants of the spectrum at $\epsilon \approx (80–85)$ keV.

2) From a comparison of the results for variants I and II it follows that the distribution function does not change significantly during the time that the transverse energy of the plasma is conserved.

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