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THEORY OF FERMI-BOSE QUANTUM LIQUIDS

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A phenomenological theory of a mixture of Fermi and Bose liquids (Fermi-Bose liquid) is formulated in the same spirit as was done by L. D. Landau for Fermi liquids. The energy of the Fermi excitation in a superfluid liquid is defined. An exact system of equations describing the properties of the Fermi-Bose liquid is derived. The acoustic solutions of this system are analyzed.

SOLUTIONS of He^3 in He^4 at low temperatures are examples of quantum liquids in which Fermi and Bose excitations exist simultaneously. At temperatures below the degeneracy temperature the interaction becomes significant for the Fermi excitations, and the Fermi part of the liquid forms in this case a Fermi quantum liquid dissolved in a superfluid liquid.

The diagram of state of solutions of He^3 in He^4 has at a temperature close to 0.8° a critical point, below which the solutions become laminated. The lamination curve joins, as $T \to 0$, the point of the pure substance (He^3) at one end and at the point corresponding to a solution with He^3 concentration of about 6% in He^4 at the other[1]. At concentrations below 6%, the solution does not become laminated, and when the temperature is lowered the region of Fermi degeneracy is always reached. This is confirmed by measurements of the specific heat of the solutions, which changes linearly with temperature[2]. Thus, the indicated solutions represent a sui-generis Fermi-Bose quantum liquid.

We shall use the designation "Fermi-Bose liquid" in all cases when we have a Fermi liquid with superfluidity. An example of a Fermi-Bose liquid should be also pure liquid He^3 in the temperature region where it can become superfluid, owing to the Cooper pairing of the Fermi excitations with nonzero angular momentum[3].

Many properties of solutions of He^3 in He^4 were considered in detail in[4], and also in[5,6]. The purpose of the present paper is to formulate a general phenomenological theory of Fermi-Bose quantum liquids in the same spirit as was done by L. Landau for Fermi liquids[1)][7].

1. EXCITATION ENERGY

Let E' and $\mathbf{P}'$ be the energy and momentum (per unit volume) of the liquid in a reference frame moving at the velocity of the superfluid motion $\mathbf{v}_s$. Then the energy of the liquid in the resting reference system will be

$$E = E' + \mathbf{P}'\mathbf{v}_s + \rho v_s^2/2, \tag{1.1}$$

and the momentum will be

$$\mathbf{J} = \rho\mathbf{v}_s + \mathbf{P}'. \tag{1.2}$$

The same momentum $\mathbf{J}$ can be represented in the form of a sum of the momentum of the superfluid motion of the Bose part of the liquid, equal to $\rho_1\mathbf{v}_s$, and the total momentum of the excitations, equal to $\int \mathbf{p}n_\mathbf{p}d\tau$:

$$\mathbf{J} = \rho_1\mathbf{v}_s + \int \mathbf{p}n_\mathbf{p}\,d\tau. \tag{1.3}$$

The density ρ_1 of the Bose part of the liquid is[2)]

$$\rho_1 = \rho - m\int n_\mathbf{p}\,d\tau \tag{1.4}$$

(m—mass of the Fermi particle). Comparing (1.2) with (1.3) we get

$$\mathbf{P}' = \int(\mathbf{p} - m\mathbf{v}_s)\,n_\mathbf{p}\,d\tau = \int \mathbf{p}n_{\mathbf{p}+m\mathbf{v}_s}\,d\tau. \tag{1.5}$$

Thus, the excitation momentum is expressed in the form of a variational derivative

$$\mathbf{p} = \delta\mathbf{P}'/\delta n_{\mathbf{p}+m\mathbf{v}_s}. \tag{1.6}$$

Analogously, the excitation energy is (in the system where $\mathbf{v}_s = 0$)

$$\varepsilon_\mathbf{p} = \delta E'/\delta n_{\mathbf{p}+m\mathbf{v}_s}. \tag{1.7}$$

The state of our system is described by specifying three functions namely, the density ρ (or ρ_1), the velocity $\mathbf{v}_s$, and the distribution function $n_{\mathbf{p}+m\mathbf{v}_s}$. The excitation energy can be determined as the variational derivatives of the density of the energy E with respect to the distribution function, with the other two functions ρ (or ρ_1) and $\mathbf{v}_s$ constant. For the case of a solution of He^3 in He^4, the natural variable is the density ρ_1, and we define the excitation energy as that additional energy which is acquired by the mixture of the liquids when one atom of He^3 is added (at a specified mass of the Bose liquid He^4). On the other hand, for the case of a Fermi liquid in which, owing to pairing, superfluidity appears, a more natural variable is the total density ρ. Henceforth we shall carry out all the calculations first for the case when the variable is chosen to be the density ρ. The formulas obtained thereby are more compact. At the end we shall summarize the results for both cases.

We now calculate the excitation energy $H_{\mathbf{p}+m\mathbf{v}_s}$ in

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[1)] No limitations are imposed, naturally, on the concentration of the Fermi particles, As is well known, the parameter of the series expansion of the the thermodynamic functions of a degenerate solution is the concentration raised to the 1/3 power. Obviously, at a concentration order of 6% this parameter is not small.

[2)] Just as in the theory of the Fermi liquid, the number of Fermi excitations in the solution He^3 and He^4 is assumed equal to the number of the Fermi particles.

the presence of superfluid motion with velocity $\mathbf{v}_S$ (obviously, the excitation momentum here is equal to $\mathbf{p} + m\mathbf{v}_S$). From (1.1) we get

$$H_{\mathbf{p}+m\mathbf{v}_s} = \frac{\delta E}{\delta n_{\mathbf{p}+m\mathbf{v}_s}} = \varepsilon_{\mathbf{p}} + \mathbf{p}\mathbf{v}_s. \tag{1.8}$$

The energy ϵ_p is a functional of the excitation density, and this dependence can be expressed in the form of the functional derivative

$$f(\mathbf{p}, \mathbf{p}') = \delta\varepsilon_{\mathbf{p}}/\delta n_{\mathbf{p}+m\mathbf{v}_s}. \tag{1.9}$$

Thus, at a small deviation from equilibrium, we have

$$\varepsilon_{\mathbf{p}} = \varepsilon_{\mathbf{p}}^{(0)} + \int f(\mathbf{p}, \mathbf{p}')\, \delta n_{\mathbf{p}+m\mathbf{v}_s}\, d\tau', \tag{1.10}$$

where $\epsilon_p^{(0)}$ is the excitation energy in a system that is in equilibrium, and

$$\delta n_{\mathbf{p}+m\mathbf{v}_s} = n_{\mathbf{p}+m\mathbf{v}_s} - n_{\mathbf{p}}^{(0)} \tag{1.11}$$

($n_p^{(0)}$ is the equilibrium distribution function). The fact that the variation should be defined in this manner follows directly from (1.5), which can be rewritten in the form

$$\mathbf{P}' = \int \mathbf{p}\, (n_{\mathbf{p}+m\mathbf{v}_s} - n_{\mathbf{p}}^{(0)})\, d\tau.$$

We now write down the excitation energy H_p as a function of the momentum p. Using the definition (1.8) and formula (1.9), and separating in explicit form the terms linear in $\mathbf{v}_S$, we get

$$H_{\mathbf{p}} = \varepsilon_{\mathbf{p}-m\mathbf{v}_s} + (\mathbf{p} - m\mathbf{v}_s, \mathbf{v}_s) \cong \varepsilon_{\mathbf{p}}^{(0)} + \left(\mathbf{p} - m\frac{\partial\varepsilon}{\partial\mathbf{p}}, \mathbf{v}_s\right) + \int f(\mathbf{p} - m\mathbf{v}_s, \mathbf{p}')\, \delta n_{\mathbf{p}'+m\mathbf{v}_s}\, d\tau. \tag{1.12}$$

For momentum values close to the Fermi limit, we obtain from this[3)]

$$H_{\mathbf{p}} = \varepsilon_{\mathbf{p}}^{(0)} + \left(\mathbf{p} - m\frac{\partial\varepsilon}{\partial\mathbf{p}}, \mathbf{v}_s\right) - m\left(\frac{\partial\varepsilon}{\partial\mathbf{p}}\mathbf{v}_s\right)\frac{F_1}{3} + \int f(\mathbf{p}, \mathbf{p}')\, (n_{\mathbf{p}'} - n_{\mathbf{p}}^{(0)})\, d\tau'. \tag{1.13}$$

We write the excitation energy $\epsilon_p^{(0)}$ in the form

$$\varepsilon_{\mathbf{p}}^{(0)} = \varepsilon_0 + p^2/2m^*, \tag{1.14}$$

where m* is the total effective mass due to the interaction of the Fermi particle with both the Bose and the Fermi parts of the liquid, and ϵ_0 is the zero-point excitation energy. For this concrete form of the spectrum it follows from (1.13) that

$$H_{\mathbf{p}} = \varepsilon_0 + \frac{p^2}{2m^*} + (\mathbf{p}\mathbf{v}_s)\left(1 - \frac{m}{m^*}\left(1 + \frac{F_1}{3}\right)\right) + \int f(\mathbf{p}, \mathbf{p}')\, (n_{\mathbf{p}'} - n_{\mathbf{p}'}^{(0)})\, d\tau'. \tag{1.15}$$

The obtained formula differs from the analogous formula given by Bardin, Baym, and Pines[4] both in the fact that the term $p^2/2m^*$ contains the total effective mass, and in the coefficient preceding the term $\mathbf{p} \cdot \mathbf{v}_S$. In the presence of superfluid motion, it is impossible to use the Galileo principle to establish the connection between the bare mass (the effective mass that the Fermi excitation situated in the superfluid would have) and the total effective mass m* due also to the interaction of the Fermi excitations. The coefficient of the term $\mathbf{p} \cdot \mathbf{v}_S$ in (1.15) has the following necessary and natural property: in a pure Fermi liquid

$$1 - \frac{m}{m^*}\left(1 + \frac{F_1}{3}\right) = 0$$

and there is no superfluidity.

2. EQUILIBRIUM DISTRIBUTION FUNCTION

The equilibrium distribution function is obtained by minimizing the entropy

$$S = -\int \{(1-n)\ln(1-n) + n\ln n\}\, d\tau \tag{2.1}$$

at specified values of the total energy E, the number of excitations $N = \int n d\tau$, and the relative-motion momentum $\mathbf{P}'$. Thus, it is necessary to minimize the functional

$$\Phi = S - \beta(E - \mu_3 N - \mathbf{v}_n\mathbf{P}') \tag{2.2}$$

(β, μ_3, and $\mathbf{v}_n$ are the Lagrangian factors, which have the usual physical meaning: $\beta = 1/T$, μ_3—chemical potential, $\mathbf{v}_n$—normal velocity). Varying the functional (2.2) with respect to $\delta n_{\mathbf{p}+m\mathbf{v}_S}$, we get

$$n_{\mathbf{p}+m\mathbf{v}_s}^{(0)} = [\exp\{(H_{\mathbf{p}+m\mathbf{v}_s} - \mathbf{p}\mathbf{v}_n - \mu_3)/T\} + 1]^{-1}. \tag{2.3}$$

From (2.2) it follows that the chemical potential

$$\mu_3 = (\partial E' / \partial N)_{S,\mathbf{P}'}, \tag{2.4}$$

and the thermodynamic equation has the following form:

$$dE' = TdS + \mu_3 dN + m_4^{-1}\mu_4 d\rho + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{P}') \tag{2.5}$$

(m_4 is the mass of the He^4 atom).

In the case when the independent variable is not ρ but the density ρ_1, the thermodynamic equation assumes the form

$$dE' = TdS + \mu_3^{(1)} dN + m_4^{-1}\mu_4 d\rho_1 + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{P}'),$$
$$\mu_3^{(1)} = \mu_3 + \frac{m}{m_4}\mu_4. \tag{2.6}$$

3. NORMAL DENSITY AND SPECIFIC HEAT

Let us calculate the momentum of the Fermi excitations in a reference frame in which the superfluid part of the liquid is at rest. We have

$$\mathbf{P}' = \int \mathbf{p}\, n_{\mathbf{p}+m\mathbf{v}_s}\, d\tau. \tag{3.1}$$

We use for the distribution function its equilibrium value (2.3). Let us find the addition to the excitation energy in the presence of superfluid and normal flows. From (1.8) and (1.10) follows the equation

$$\delta H = (\mathbf{p}, \mathbf{v}_s - \mathbf{v}_n) + \int f(\mathbf{p}, \mathbf{p}')\frac{\partial n^{(0)}}{\partial\varepsilon'}\, \delta H' d\tau', \tag{3.2}$$

from which we get

$$\delta H = \frac{(\mathbf{p}, \mathbf{v}_s - \mathbf{v}_n)}{1 + F_1/3}. \tag{3.3}$$

Using this formula, we get from (3.1)

$$\mathbf{P}' = \int \mathbf{p}\frac{\partial n^0}{\partial\varepsilon}\, \delta H d\tau = \frac{m^* N}{1 + F_1/3}(\mathbf{v}_n - \mathbf{v}_s). \tag{3.4}$$

The total momentum of the liquid is equal to

$$\mathbf{J} = \rho\mathbf{v}_s + \mathbf{P}' = \left(\rho - \frac{m^* N}{1 + F_1/3}\right)\mathbf{v}_s + \frac{m^* N}{1 + F_1/3}\mathbf{v}_n. \tag{3.5}$$

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3) We introduce the notation $F(\chi) = (\partial\tau/\partial\epsilon)_\epsilon = \mu_3 f(\chi)$, where $f(\chi)$ is the value of $f(\mathbf{p}, \mathbf{p}')$ at $|\mathbf{p}| = |\mathbf{p}'| = p_F$, which depends only on the angle χ between the vectors $\mathbf{p}$ and $\mathbf{p}'$. We expand the function F in Legendre polynomials. F_1 is the first spherical harmonic

Thus, the density of the normal component is

$$\rho_n = \frac{m^* N}{1 + F_1/3}; \tag{3.6}$$

Naturally, it is not equal to the density of the Fermi part of the liquid mN.

The specific heat of the Fermi excitations is calculated in analogy with the procedure used for the Fermi liquid. It equals

$$c = \frac{C}{N} = \gamma T, \quad \gamma = \frac{\pi^2}{3N}\left(\frac{d\tau}{d\varepsilon}\right)_{\varepsilon=\mu_3} = \left(\frac{\pi}{3N}\right)^{2/3}\frac{m^*}{\hbar^2}. \tag{3.7}$$

The coefficient γ in this formula contains the total effective mass of the excitation m*, and not $m^*/(1 + F_1/3)$, as is the case in formula (3.6) for ρ_n.

4. ENERGY AND MOMENTUM CONSERVATION LAW

Let us write down the kinetic equation for the function $n_{\mathbf{p}+m\mathbf{v}_s}$. We have

$$\frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial t} + \frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{x}}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{p}} - \frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{p}}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{x}} = I(n). \tag{4.1}$$

To derive the complete system of equations describing the Fermi-Bose liquid, we use the following conservation laws: the continuity equation

$$\dot{\rho} + \operatorname{div}\mathbf{J} = 0, \tag{4.2}$$

and the momentum conservation equation

$$\dot{J}_i + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \tag{4.3}$$

where Π_{ik} is the momentum-flux tensor, which is still unknown. We now multiply (4.1) by p_i and integrate over all of phase space (for simplicity we denote the integration operation by a bar). It must be borne in mind here that since $\int I d\tau = 0$, it follows that

$$\int \mathbf{p} I d\tau = \int (\mathbf{p} + m\mathbf{v}_s) I d\tau = 0.$$

We obtain in this manner

$$\frac{\partial P_i'}{\partial t} + \frac{\partial}{\partial x_k}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) + \overline{n_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial x_i}} = 0. \tag{4.4}$$

Further, subtracting (4.4) from (4.3) and using the continuity equation, we obtain

$$\rho\dot{v}_{si} - v_{si}\frac{\partial}{\partial x_k}\left(\rho v_{sk} + \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_k}\right) + \frac{\partial}{\partial x_k}\left(\Pi_{ik} - \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) - \overline{n_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial x_i}} = 0. \tag{4.5}$$

We now transform the obtained equation, taking into account the condition curl $\mathbf{v}_s = 0$. We obtain

$$\rho\left(\dot{v}_{si} + \frac{\partial}{\partial x_i}\frac{v_s^2}{2}\right) + \frac{\partial}{\partial x_k}\left(\Pi_{ik} - \rho v_{si}v_{sk} - \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) - v_{si}\frac{\partial}{\partial x_k}\overline{n_{\mathbf{p}+m\mathbf{v}_s} p_k} - \overline{n_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial x_i}} = 0. \tag{4.6}$$

Transforming the last term of (4.6), we get

$$\begin{aligned}\overline{n_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial x_i}} &= \frac{\partial}{\partial x_i}\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}} - \overline{H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial x_i}} \\ &= \frac{\partial}{\partial x_i}\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}} - \frac{\partial E}{\partial x_i} + \frac{\partial E}{\partial \rho}\frac{\partial \rho}{\partial x_i} + \frac{\partial E}{\partial v_{sk}}\frac{\partial v_{sk}}{\partial x_i} \\ &= \frac{\partial}{\partial x_i}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}} - E + \frac{\partial E}{\partial \rho}\rho\right) \\ &\quad - \rho\frac{\partial}{\partial x_i}\frac{\partial E}{\partial \rho} + \frac{\partial}{\partial x_k}\left(v_{si}\frac{\partial E}{\partial v_{sk}}\right) - v_{si}\frac{\partial}{\partial x_k}\frac{\partial E}{\partial v_{sk}} = 0.\end{aligned} \tag{4.7}$$

Substituting the last expression in (4.6) and taking into account the relation

$$\frac{\partial E}{\partial v_{sk}} = \rho v_{sk} + \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_k}, \tag{4.8}$$

we get

$$\rho\left(\dot{v}_{si} + \frac{\partial}{\partial x_i}\frac{\partial E}{\partial \rho}\right) + \frac{\partial}{\partial x_k}\left(\Pi_{ik} - \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) - \frac{\partial}{\partial x_k}\left(v_{si}\frac{\partial E}{\partial v_{sk}}\right) - \frac{\partial}{\partial x_i}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}} - E + \rho\frac{\partial E}{\partial \rho}\right) = 0. \tag{4.9}$$

From (4.9) we find the equation of superfluid motion and the expression for the momentum flux tensor:

$$\dot{\mathbf{v}}_s + \nabla(\partial E/\partial \rho)_{n,\mathbf{v}_s} = 0, \tag{4.10}$$

$$\Pi_{ik} = \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial \varepsilon}{\partial p_k}} + \overline{n_{\mathbf{p}+m\mathbf{v}_s}(p_i v_{sk} + p_k v_{si})} + \rho v_{si}v_{sk} + \delta_{ik}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s}\varepsilon_p} - E' + \rho\frac{\partial E'}{\partial \rho}\right). \tag{4.11}$$

When account is taken of (1.1), the superfluid-motion equation (4.10) can be rewritten in the form

$$\dot{\mathbf{v}}_s + \nabla\left(\frac{\partial E'}{\partial \rho} + \frac{v_s^2}{2}\right) = 0. \tag{4.12}$$

The pressure is defined by the usual relation

$$P = -E' + \frac{\partial E'}{\partial \rho}\rho + TS. \tag{4.13}$$

Taking this definition into account, we write the expression for the momentum flux tensor in the form

$$\Pi_{ik} = \overline{n_{\mathbf{p}+m\mathbf{v}_s} p_i \frac{\partial \varepsilon}{\partial p_k}} + \overline{n_{\mathbf{p}+m\mathbf{v}_s}(p_i v_{sk} + p_k v_{si})} + \rho v_{si}v_{sk} + \delta_{ik}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s}\varepsilon_p} - TS + P\right). \tag{4.14}$$

To obtain the energy conservation law, we multiply the kinetic equation (4.1) by $H_{\mathbf{p}+m\mathbf{v}_s}$ and integrate over the phase volume. As a result

$$\overline{H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial t}} + \frac{\partial}{\partial x_k}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) = 0. \tag{4.15}$$

Let us calculate the derivative of the total energy with respect to time:

$$\frac{\partial E}{\partial t} = \overline{H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial n_{\mathbf{p}+m\mathbf{v}_s}}{\partial t}} + \frac{\partial E}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial E}{\partial v_{sk}}\frac{\partial v_{sk}}{\partial t} \tag{4.16}$$

and let us use Eq. (4.15), the continuity equation (4.2), and the superfluid motion Eq. (4.10). In this way we get

$$\begin{aligned}\frac{\partial E}{\partial t} &= -\frac{\partial}{\partial x_k}\left(\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial p_k}}\right) - \frac{\partial E}{\partial \rho}\operatorname{div}\mathbf{J} - \mathbf{J}\operatorname{grad}\frac{\partial E}{\partial \rho} \\ &= -\operatorname{div}\left\{\overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{p}}} + \mathbf{J}\frac{\partial E}{\partial \rho}\right\}.\end{aligned} \tag{4.17}$$

The expression under the divergence sign is indeed the energy flux

$$\mathbf{Q} = \mathbf{J}\frac{\partial E}{\partial \rho} + \overline{n_{\mathbf{p}+m\mathbf{v}_s} H_{\mathbf{p}+m\mathbf{v}_s}\frac{\partial H_{\mathbf{p}+m\mathbf{v}_s}}{\partial \mathbf{p}}}. \tag{4.18}$$

5. FIRST AND SECOND SOUNDS

Let us consider low-frequency acoustic oscillations of a Fermi-Bose liquid. In this limiting case, there is a hydrodynamic theory for the entire liquid as a whole, as well as for the Fermi part. However, for the sake of uniformity, we shall use as before the kinetic equation for the Fermi excitations. We denote small deviations of $n_{\mathbf{p}+m\mathbf{v}_s}$ and ρ from the equilibrium values by $n_1 = n_{\mathbf{p}+m\mathbf{v}_s} - n_{\mathbf{p}}^{(0)}$ and $\rho' = \rho - \rho_0$, respectively; the velocity $\mathbf{v}_s$ itself is of the same order of smallness. Assume that all these quantities vary in time and in coordinates in accordance with the periodic law $\exp(i\omega t - i\mathbf{k}\cdot\mathbf{x})$. Then Eqs. (4.1), (4.2), and (4.10) yield in the linear approximation the following system:

$$i(\omega - \mathbf{kv})n_1 + i\mathbf{kv}\frac{\partial n^{(0)}}{\partial \varepsilon}\left(\frac{\partial \varepsilon}{\partial \rho}\rho' + \mathbf{pv}_s + \int f(\mathbf{pp}')n_1'd\tau'\right) = 0, \quad (5.1)$$

$$i\omega\rho' - i\mathbf{k}\left(\rho\mathbf{v}_s + \int n_1\mathbf{p}d\tau\right) = 0, \quad (5.2)$$

$$i\omega\mathbf{v}_s - i\mathbf{k}\left(\frac{s^2}{\rho}\rho' + \int\frac{\partial\varepsilon}{\partial\rho}n_1 d\tau\right) = 0; \quad \mathbf{v} = \frac{\partial\varepsilon}{\partial\mathbf{p}}, \quad \frac{s^2}{\rho} = \frac{\partial^2 E}{\partial\rho^2}. \quad (5.3)$$

We introduce further the following notation and dimensionless variables:

$$n_1 = \frac{\partial n^{(0)}}{\partial\varepsilon}m^* v_F^2 \nu(x), \quad x = \cos\theta,$$

$$F(\chi) = f(\chi)\left(\frac{d\tau}{d\varepsilon}\right)_{\varepsilon=\mu_3} = \sum F_n P_n(\cos\chi),$$

θ is the angle between the excitation momentum $\mathbf{p}$, the wave vector $\mathbf{k}$, and $f(\chi)$ is defined in footnote[3], and

$$v_F = \left(\frac{\partial\varepsilon}{\partial p}\right)_{\varepsilon=\mu_3}, \quad \alpha = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\varepsilon=\mu_3}\frac{\rho}{m^* s^2},$$

$$u = \frac{\omega}{kv_F}, \quad \tilde{\rho}' = \frac{\rho'}{\rho}, \quad \tilde{\mathbf{v}}_s = \frac{\mathbf{v}_s}{v_F}.$$

In terms of the new variables, the system (5.1)–(5.3) assumes the following form:

$$(u - x)\nu(x) + x(\alpha\tilde{\rho}' - \overline{F(\chi)\nu(x')} + x\tilde{v}_s) = I(\nu), \quad (5.4)$$

$$u\tilde{\rho}' - \tilde{v}_s + \nu_1\frac{Nm^*}{\rho} = 0, \quad (5.5)$$

$$u\tilde{v}_s - \frac{s^2}{v_F^2}\tilde{\rho}' + 3\alpha\frac{Nm^*}{\rho}\frac{s^2}{v_F^2}\nu_0 = 0. \quad (5.6)$$

The function $\nu(x)$ can be expanded in Legendre polynomials

$$\nu(x) = \sum \nu_n P_n(x), \quad (5.7)$$

ν_0 and ν_1 in Eqs. (5.4)–(5.6) are respectively the zeroth and first harmonics.

The obtained system of equations is obviously valid for all values of the frequencies. For small values of ω, when the characteristic collision time τ satisfies the condition $\omega\tau \ll 1$, the hydrodynamic theory holds, as indicated above. In this case, the only nonvanishing harmonics of $\nu(x)$ are the zeroth and the first (the higher harmonics have a relative order $\omega\tau$ or higher). Equating the coefficients of $P_0(x)$ and $P_1(x)$ in Eq. (5.4), we obtain two equations for ν_0 and ν_1:

$$u\nu_0 - {}^1\!/_3(1 + F_1/3)\nu_1 + {}^1\!/_3\tilde{v}_s = 0, \quad (5.8)$$

$$-(1 + F_0)\nu_0 + u\nu_1 + \alpha\frac{s^2}{v_F^2}\tilde{\rho}' = 0. \quad (5.9)$$

The condition for the compatibility of the system (5.5), (5.6), (5.8), and (5.9) yields the following dispersion equation, which determines u:

$$u^4 - u^2\left\{\frac{s^2}{v_F^2}\left[1 + \frac{Nm^*}{\rho(1 + F_1/3)}\left(\left(\alpha\left(1 + \frac{F_1}{3}\right) + 1\right)^2 - 1\right)\right]\right.$$

$$\left. + \frac{1}{3}\left(1 + \frac{F_1}{3}\right)(1 + \tilde{F}_0)\right\}$$

$$+ \frac{1}{3}\left(1 + \frac{F_1}{3}\right)(1 + \tilde{F}_0)\frac{s^2}{v_F^2}\left(1 - \frac{Nm^*}{\rho(1 + F_1/3)}\right) = 0, \quad (5.10)$$

where

$$\tilde{F}_0 = F_0 - 3\alpha^2\frac{s^2}{v_F^2}\frac{Nm^*}{\rho}. \quad (5.11)$$

For the case of small Fermi-particle concentrations, Eq. (5.10) has the following two approximate roots:

$$u_1^2 = \frac{s^2}{v_F^2}\left\{1 + \frac{Nm^*}{\rho(1 + F_1/3)}\left[\left(\alpha\left(1 + \frac{F_1}{3}\right) + 1\right)^2 - 1\right]\right\}, \quad (5.12)$$

$$u_2^2 = \frac{1}{3}\left(1 + \frac{F_1}{3}\right)(1 + \tilde{F}_0)\left[1 - \frac{Nm^*}{\rho(1 + F_1/3)}\left(\alpha\left(1 + \frac{F_1}{3}\right) + 1\right)^2\right]. \quad (5.13)$$

Formula (5.12) determines the velocity of ordinary (first) sound, and formula (5.13) the velocity of second sound. Second sound in a Fermi-Bose liquid is a sound propagating through the Fermi part of the liquid. Formula (5.13) coincides in first approximation with the formula that determines the velocity of sound in the Fermi liquid. The only difference is that F_0 is renormalized as a result of the phonon interaction (formula (5.11)).

For the case of small concentrations of the Fermi component, it is more convenient to express the results not in terms of the variables ρ and N, but in terms of the variables ρ_1 and N. We shall write down these results in terms of these more convenient variables somewhat later.

6. ZERO SOUND

We now consider the limiting case of high sound frequencies satisfying the condition $\omega\tau \gg 1$. In this case it is possible to neglect the collision integral in the kinetic equation. To solve the kinetic equation, we expand the distribution function in Legendre polynomials. For simplicity we assume that only the first two spherical harmonics of the function F differ from zero (F_0 and F_1). Then the kinetic equation (5.4) yields two equations for the first two harmonics of the function ν (ν_0 and ν_1):

$$(1 - F_0 w)\nu_0 - \frac{1}{3}F_1 uw\nu_1 + \frac{s^2}{v_F^2}\alpha w\tilde{\rho}' + uw\tilde{v}_s = 0, \quad (6.1)$$

$$-F_0 uw\nu_0 + \frac{1}{3}\left(1 + \frac{F_1}{3} - F_1 u^2 w\right)\nu_1 + \frac{s^2}{v_F^2}\alpha uw\tilde{\rho}' - \left(\frac{1}{3} - u^2 w\right)\tilde{v}_s = 0, \quad (6.2)$$

where

$$w = -1 + \frac{u}{2}\ln\frac{u+1}{u-1}. \quad (6.3)$$

The condition for the compatibility of Eqs. (6.1), (6.2), (5.5), and (5.6) yields the dispersion equation in the high-frequency region. Inasmuch as F_0 and F_1 are small for a solution, the dispersion equation has a root of the order of unity. Near this root, the dispersion equation has the following form:

$$1 - \left(\tilde{F}_0 + \frac{\tilde{F}_1 u^2}{1 + F_1/3}\right)w = 0, \quad (6.4)$$

where

$$\tilde{F}_0 = F_0 - \frac{3Nm^*}{\rho}\alpha^2\frac{s^2}{v_F^2}, \qquad \tilde{F}_1 = F_1 - \frac{Nm^*}{\rho}(\alpha + 1). \quad (6.5)$$

We retain here in F_1, besides the principal terms also the terms of higher order in Nm^*/ρ, since estimates create the impression that they make a noticeable contribution in the case of solutions of He^3 in He^4.

Equation (6.4) determines the velocity of zero sound in the solution; it has undamped solutions only in the case when

$$F_0 + \frac{F_1 u^2}{1 + F_1/3} > 0. \tag{6.6}$$

As to F_0, this quantity is negative and of the order of unity (in absolute magnitude). It is more difficult to estimate the second term, which apparently is smaller than unity when $u^2 \sim 1$.[4)] In this case the dispersion equation has no zero-sound solutions. However, inasmuch as F_1 is not well known, the question of existence of zero sound in the solution remains open (we call attention to the fact that the second term in F_1 is unfortunately negative, which is also unfavorable).

7. TWO DEFINITIONS OF THE EXCITATION ENERGY

We now write down several relations between the parameters of the theory for two different sets of variables. So far we have used the set ρ, $\mathbf{v}_s$, and $\delta n_{\mathbf{p}+m\mathbf{v}_s} = n_{\mathbf{p}+m\mathbf{v}_s} - n_{\mathbf{p}}^{(0)}$. We now take a second set, namely the density of the Bose component $\rho_1 = \rho - m\int n d\tau$, the velocity $\mathbf{v}_s$, and we define the excitation energy as the variational derivative of the energy E with respect to $\delta n_{\mathbf{p}} = n_{\mathbf{p}} - n_{\mathbf{p}-m\mathbf{v}_s}^{(0)}$ at specified values of ρ_1 and $\mathbf{v}_s$. We have

$$H_{\mathbf{p}} = \delta E / \delta n_{\mathbf{p}}. \tag{7.1}$$

We denote by the index 1 all the derivatives calculated at a constant value of ρ_1. Thus, we have for the excitation energy the following connection:

$$H_{\mathbf{p}} = \left(\frac{\delta E}{\delta n_{\mathbf{p}}}\right)_{\rho} = \left(\frac{\delta E}{\delta n_{\mathbf{p}}}\right)_{\rho_1} - m\frac{\partial E}{\partial \rho_1} = H_{\mathbf{p}}^{(1)} - m\frac{\partial E}{\partial \rho_1}. \tag{7.2}$$

The appearance of the second term in (7.2) denotes a change in the point from which the energy is reckoned. Its physical meaning is clear, namely the additions of the excitation at a specified total density ρ and at a specified density of the Bose part ρ_1 are not equivalent. This term equals the chemical potential of the Bose part, multiplied by the mass of the Fermi particle :

$$m\frac{\partial E}{\partial \rho_1} = m\frac{\mu_4}{m_4}.$$

The chemical potentials μ_3 and $\mu_3^{(1)}$ are connected by the relation

$$\mu_3 = \mu_3^{(1)} - \frac{m}{m_4}\mu_4. \tag{7.3}$$

Thus, for the differences $H_{\mathbf{p}} - \mu_3$ we have the connection

$$H_{\mathbf{p}} - \mu_3 = H_{\mathbf{p}}^{(1)} - \mu_3^{(1)}, \tag{7.4}$$

i.e., the excitation energy reckoned from the chemical potential is invariant. We next calculate the function $f(\mathbf{p}, \mathbf{p}')$. From (7.2) we get

$$f(\mathbf{p}, \mathbf{p}') = \frac{\delta H_{\mathbf{p}-m\mathbf{v}_s}}{\delta n_{\mathbf{p}'+m\mathbf{v}_s}} = \frac{\delta \varepsilon_{\mathbf{p}}^{(1)}}{\delta n_{\mathbf{p}'+m\mathbf{v}_s}} - 2m\frac{\partial \varepsilon_{\mathbf{p}}'}{\partial \rho_1} + m^2\frac{\partial^2 E}{\partial \rho_1^2}$$

$$= f^{(1)}(\mathbf{p},\mathbf{p}') - 2m\frac{\partial \varepsilon_{\mathbf{p}}^{(1)}}{\partial \rho_1} + m^2\frac{\partial^2 E}{\partial \rho_1^2}. \tag{7.5}$$

We introduce the notation

$$\frac{\partial^2 E}{\partial \rho_1^2} = \frac{\partial}{\partial \rho_1}\frac{\mu_4}{m_4} = \frac{s_1^2}{\rho_1}, \qquad \alpha_1 = \frac{\rho_1}{m^* s_1^2}\left(\frac{\partial \varepsilon^{(1)}}{\partial \rho_1}\right)_{\varepsilon^{(1)}=\mu_3^{(1)}}. \tag{7.6}$$

Relation (7.3) for the limiting Fermi energies yields

$$F = F^{(1)} + \frac{3Nm}{\rho_1}\frac{s_1^2}{v_F^2}\left(-2\alpha_1 + \frac{m}{m^*}\right). \tag{7.7}$$

The connection between the parameters α and $\alpha^{(1)}$ can also be readily determined:

$$\alpha = \frac{\rho}{m^* s^2}\frac{\partial \varepsilon}{\partial \rho} = \frac{\rho}{m^* s^2}\left(\frac{\partial \varepsilon^{(1)}}{\partial \rho_1} - m\frac{\partial^2 E}{\partial \rho_1^2}\right) = \frac{\rho s_1^2}{\rho_1 s^2}\left(\alpha_1 - \frac{m}{m^*}\right). \tag{7.8}$$

For $\tilde{F}_0$ we have

$$\tilde{F}_0 = F_0 - 3\alpha^2\frac{s^2}{v_F^2}\frac{Nm^*}{\rho} = F_0^{(1)}$$

$$+ \frac{3Nm^*}{\rho_1}\frac{s_1^2}{v_F^2}\left[\frac{m}{m^*}\left(-2\alpha_1 + \frac{m}{m^*}\right) - \frac{\rho_1 s^2}{\rho s_1^2}\left(\alpha_1 - \frac{m}{m^*}\right)^2\right],$$

or, inasmuch as $\partial^2 E/\partial\rho_1^2 = \partial^2 E/\partial\rho^2$, we get $s^2/\rho = s_1^2/\rho_1$ and

$$\tilde{F}_0 = F_0^{(1)} - \frac{3Nm^*}{\rho_1}\frac{s_1^2}{v_F^2}\alpha_1^2 = \tilde{F}_0^{(1)}. \tag{7.9}$$

We express the velocity of sound in a weak solution in terms of its value for the pure solvent:

$$s_1^2 = \rho_1\frac{\partial}{\partial \rho_1}\frac{\mu_4}{m_4} = s_{10}^2 + \rho_1\frac{\partial}{\partial \rho_1}\frac{\delta\mu_4}{m_4}.$$

The quantity $\delta\mu_4$ is determined from the simple relation:

$$\frac{1}{m_4}\delta\mu_4 = \frac{1}{m_4}\frac{\partial \mu_4}{\partial N}N = \frac{\partial \mu_3^{(1)}}{\partial \rho_1}N = \alpha_1\frac{m^* s_1^2}{\rho_1}N. \tag{7.10}$$

We thus find the sought formula

$$s_1^2 \cong s_{10}^2\left(1 + \beta\frac{Nm^*}{\rho_1}\right), \qquad \beta = \frac{\rho_1^2}{m^* s_{10}^2}\left(\frac{\partial^2 \varepsilon}{\partial \rho_1^2}\right)_{\varepsilon=\mu_3^{(1)}}. \tag{7.11}$$

For s^2 we obtain analogously

$$s^2 = \frac{\rho}{\rho_1}s_1^2 = s_{10}^2\left[1 + \left(\beta + \frac{m}{m^*}\right)\frac{Nm^*}{\rho_1}\right]. \tag{7.12}$$

In terms of the new variables, the expressions for the velocities of the first and second sounds respectively are[5)]

$$u_1^2 = \frac{s_{10}^2}{v_F^2}\left\{1 + \frac{Nm^*}{\rho_1(1 + F_1/3)}\left[\left(\alpha_1\left(1 + \frac{F_1}{3}\right) + \frac{\delta m}{m^*}\right)^2 + \beta\left(1 + \frac{F_1}{3}\right) - \frac{\delta m}{m^*}\right]\right\}, \tag{7.13}$$

$$u_2^2 = \frac{1}{3}\left(1 + \frac{F_1}{3}\right)(1 + \tilde{F}_0)\left[1 - \frac{Nm^*}{\rho_1(1 + F_1/3)}\left(\alpha_1\left(1 + \frac{F_1}{3}\right) + \frac{\delta m}{m^*}\right)^2\right],$$

$$\frac{\delta m}{m^*} = 1 - \frac{m}{m^*}\left(1 + \frac{F_1}{3}\right). \tag{7.14}$$

[4)] If $u^2 \gg 1$, then $w = 1/3u^2$ and Eq. (6.4) can have a solution only at very large values of F_1; apparently there are no grounds for this.

[5)] The result (7.13) differs slightly from the result of Baym [5]. In the comparison it is necessary to take into account the fact that m in [5] is defined by the relation $m = m^*(1 + F_1/3)$.

A curious circumstance is the following: We calculate $\tilde{F}_0$. In analogy with the procedure used in[4], we can show that the quantity

$$F_0^{(1)} \approx \frac{3Nm_4}{\rho_1} \frac{s_1^2}{v_F^2} \left(2\alpha_1 - \frac{m_4}{m^*} \right), \tag{7.15}$$

and therefore

$$\tilde{F}_0 = \tilde{F}_0^{(1)} = -\frac{3Nm^*}{\rho_1} \frac{s_1^2}{v_F^2} \left(\alpha_1 - \frac{m_4}{m^*} \right)^2 . \tag{7.16}$$

The parameter $\alpha_1 m^*/m_4$ equals the ratio of the specific volumes of the atoms He^3 and He^4. Indeed,

$$\frac{m^*}{m_4}\alpha_1 = \frac{\partial\mu_3}{\partial\rho_1} \frac{\rho_1}{m_4 s_1^2} = \frac{\partial\mu_3}{\partial\rho_1} \bigg/ \frac{\partial\mu_4}{\partial\rho_1} = \frac{V_3}{V_4}. \tag{7.17}$$

Formula (7.16) with allowance for (7.17) gives the following numerical value:

$$\tilde{F}_0 = -1{,}15c^{1/3}. \tag{7.18}$$

At a concentration c equal to 6% we have $\tilde{F}_0 = -0.45$. As is well known[8], the inequality $1 + \tilde{F}_0 > 0$ as the condition for the stability of the Fermi system relative to the fluctuations of the excitation density. Violation of this inequality would lead to an infinite increase of the fluctuations of the density of the excitations. This is seen from formula (5.13), for when $1 + \tilde{F}_0 < 0$ the velocity of the second sound becomes imaginary. The fact that $1 + \tilde{F}_0$ is small is apparently not an accident and should lead to an intensification of certain effects.

The physical meaning of the function $\tilde{F}_0$ can be easily understood. Unlike F_0, which is the partial derivative $\partial\mu_3/\partial N$ at a constant liquid density ρ, $\tilde{F}_0$ is the derivative of μ_3 with respect to N in the state of equilibrium, i.e., when the function μ_4 is constant. Taking this circumstance into account, we readily obtain

$$\tilde{F}_0 = \left(\frac{\partial\tau}{\partial\varepsilon}\right)_{\varepsilon=\mu_3} \left[\frac{\partial\mu_3}{\partial N} + \frac{\partial\mu_3}{\partial\rho} \left(\frac{\partial\rho}{\partial N}\right)_{\mu_4} \right] = F_0 - \frac{3Nm^*}{\rho} \alpha^2 \frac{s^2}{v_F^2},$$

$$\left(\frac{\partial\rho}{\partial N}\right)_{\mu_4} = -\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\varepsilon=\mu_3} \frac{\rho}{s^2}.$$

CONCLUSION

So far, the temperature was assumed equal to zero or so low that the contribution of the elementary excitations of the Bose type, such as phonons and rotons, could be neglected. In a system such as a solution of He^3 in He^4, at temperatures lower than the degeneracy temperature, i.e., in the region where the Fermi-liquid model takes place, the contribution of the phonons to all the thermodynamic quantities is negligibly small. Thus, the normal density of the phonons at 0.1°K is less than 10^{-8}. In the general case, however, in the presence of various types of excitations, of both the Fermi and the Bose type, it is necessary to sum also over all types of excitations in all the formulas containing integration over the phase volume of the excitations.

So far we have said nothing concerning the spin of the Fermi excitations, since there was no need for it. The spin can be accounted for in the same manner as in the theory of the Fermi liquid. It is necessary to write everywhere $(1/2)\mathrm{Tr}\,\sigma\int d\tau$ (for the case of spin 1/2) in lieu of the integrals of the type $\int d\tau$; in addition, it is necessary to take into account the fact that the function $f_{\sigma\sigma'}(p, p)$ depends also on the spins σ and σ'.

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