BOUND STATES AND RESONANCES IN A MODEL WITH THREE STATIC SOURCES

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Unstable one-particle states corresponding to resonances in the elastic scattering in a quantum field theory model with three fixed sources (a model of the Lee model type) are investigated. It is shown that in this model there exists a large variety of stable and unstable states, which do not have analogs in the Lee model. Nonexponential decay is possible (for appropriate choice of the constants).

THE Lee model long ago served[1,2] and still serves[3] as a most convenient and explicit example of a field theoretic description of bound states and of the decay of unstable one-particle states. It is almost the only model in which it is feasible to trace the phenomenon of radioactive decay in detail (a description of the time evolution of the process, determination of the level half-width). However, the original model is too elementary and does not offer any possibility of studying certain real unstable states, for example, states with close energies. Therefore, unstable states have recently been described by field-theoretic models which are generalizations of the Lee model in various directions. The present article represents one such attempt in which stable and unstable one-particle states are considered in a model which is obtained by the addition of a heavy particle in the $V\theta$ sector of the Lee model.[4] An analogous investigation of the $V\theta$ sector of the Lee model was carried out in[5], and the present work enables one to understand how new phenomena and singularities appear in connection with the addition of one more heavy particle.

1. Four types of particles, A, B, C, and θ, are introduced in the model, and the allowed transitions are:

$$A \rightleftarrows B+\theta, \tag{1}$$
$$B \rightleftarrows C+\theta. \tag{2}$$

The Hamiltonian is taken in the form

$$H = H_0 + H_I, \tag{3}$$

$$H_0 = m_{A0}\int A^*(\mathbf{p})A(\mathbf{p})\,d^3p + m_{B0}\int B^*(\mathbf{p})B(\mathbf{p})\,d^3p + m_C\int C^*(\mathbf{p})C(\mathbf{p})\,d^3p + \int \omega(\mathbf{k})a^*(\mathbf{k})a(\mathbf{k})\,d^3k, \tag{4}$$

$$H_I = \lambda_{01}\int\frac{d^3kf(k)}{\sqrt{2\omega_k}}\int d^3p\,[A^*(\mathbf{p})B(\mathbf{p}-\mathbf{k})a(\mathbf{k}) + \text{h.c.}] + \lambda_{02}\int\frac{d^3kf(k)}{\sqrt{2\omega_k}}\int d^3p\,[B^*(\mathbf{p})C(\mathbf{p}-\mathbf{k})a(\mathbf{k}) + \text{h.c.}], \tag{5}$$

where A(p) (A*(p)), B(p) (B*(p)), and C(p) (C*(p)) are unrenormalized fermion operators for the annihilation (creation) of heavy particles A, B, and C satisfying the usual anticommutation relations; a(k) (a*(k)) is the boson annihilation (creation) operator for the relativistic θ-particle, satisfying the usual commutation relations; $\omega(\mathbf{k}) \equiv \omega_{\mathbf{k}} = (\mathbf{k}^2+\mu^2)^{1/2}$, μ is the mass of the θ-particle, $f(\omega)$ is a real cutoff function; λ_{01} and λ_{02} are the unrenormalized coupling constants for interactions (1) and (2). The physical one-particle states $|A)$ and $|B)$ of the A and B particles do not coincide with the "bare" states and are solutions of the Schrödinger equation

$$H|A(\mathbf{p})) = m_A|A(\mathbf{p})), \tag{6}$$
$$H|B(\mathbf{p})) = m_B|B(\mathbf{p})). \tag{7}$$

Solutions of these equations are sought in the form

$$|B(\mathbf{p})) = Z_B^{1/2}\left\{|B(\mathbf{p})\rangle + \int d^3k\varphi(\mathbf{k})C^*(\mathbf{p}-\mathbf{k})a^*(\mathbf{k})|0\rangle\right\}, \tag{8}$$

$$|A(\mathbf{p})) = Z_A^{1/2}\left\{|A(\mathbf{p})\rangle + \int d^3k\varphi_1(\mathbf{k})B^*(\mathbf{p}-\mathbf{k})a^*(\mathbf{k})|0\rangle + \frac{1}{\sqrt{2!}}\int d^3kd^3q\varphi_2(\mathbf{k},\mathbf{q})C^*(\mathbf{p}-\mathbf{k}-\mathbf{q})a^*(\mathbf{k})a^*(\mathbf{q})|0\rangle\right\} \tag{9}$$

with the normalization condition

$$(A(\mathbf{p}')|A(\mathbf{p})) = (B(\mathbf{p}')|B(\mathbf{p})) = \delta(\mathbf{p}-\mathbf{p}').$$

The solution of the Schrödinger equation in the sector (C, θ) coincides with the solution of the ordinary Lee model in the $N\theta$ sector, whereupon

$$\varphi(\mathbf{k}) = -\lambda_{02}\frac{u_k}{\omega_k - b}, \tag{10}$$
$$u_k = f(k)(2\omega_k)^{-1/2}, \quad b = m_B - m_C,$$

where

$$m_B = m_{B0} + \delta m_B, \tag{11}$$

$$\delta m_B = -\gamma_{02}\int\frac{d^3ku_k^2}{\omega_k - b}, \quad \gamma_{02} = \lambda_{02}^2, \tag{12}$$

$$Z_B^{-1} = 1 + \gamma_{02}\int\frac{d^3ku_k^2}{(\omega_k - b)^2}. \tag{13}$$

In what follows we shall everywhere assume $b < \mu$ (the B particle is stable).

Consideration of the (B, θ) sector leads to the following relations for $\varphi_1(\mathbf{k})$ and $\varphi_2(\mathbf{k},\mathbf{q})$:

$$m_A = m_{A0} + \lambda_{01}\int d^3ku_k\varphi_1(\mathbf{k}), \tag{14}$$

$$m_A\varphi_1(\mathbf{k}) = (m_{B0}+\omega_k)\varphi_1(\mathbf{k}) + \lambda_{01}u_k + \sqrt{2}\lambda_{02}\int d^3qu_q\varphi_2(\mathbf{k},\mathbf{q}), \tag{15}$$

$$m_A\varphi_2(\mathbf{k},\mathbf{q}) = (m_C+\omega_k+\omega_q)\varphi_2(\mathbf{k},\mathbf{q}) + \frac{1}{\sqrt{2}}\lambda_{02}[u_k\varphi_1(\mathbf{q}) + u_q\varphi_1(\mathbf{k})]. \tag{16}$$

2. The integral equation for $\varphi_1(\mathbf{k})$ resulting from Eqs. (14)–(16) is easiest to analyze by replacing the kernel in this equation by a degenerate kernel.[6] As a result an equation of the following form is obtained:

$$\varphi_1(\mathbf{k}) = \frac{u_k}{(\omega_k-\omega_0)[1+\gamma_{02}P(\omega_k,\omega_0)]}\Big[-\lambda_{01} + \gamma_{02}\frac{1}{\mu+\omega_k-\omega_0-b}\int d^3q\, u_q\varphi_1(\mathbf{q})\Big], \qquad (17)$$

where

$$\omega_0 = m_A - m_B,$$

$$P(\omega_k,\omega_0) = \int \frac{u_q^2 d^3q}{(\omega_q-b)(\omega_k+\omega_q-\omega_0-b)}. \qquad (18)$$

The Fredholm equation (17) has the following solution:

$$\varphi_1(\mathbf{k}) = -\lambda_{01}\frac{u_k}{(\omega_k-\omega_0)[1+\gamma_{02}P(\omega_k,\omega_0)]} \times\Big[1+\gamma_{02}\frac{G(\omega_0)}{\mu+\omega_k-\omega_0-b}\Big], \qquad (19)$$

where

$$G(\omega_0) = \frac{F(\omega_0)}{1-\gamma_{02}T(\omega_0)}, \qquad (20)$$

$$F(\omega_0) = \int \frac{u_k^2 d^3k}{(\omega_k-\omega_0)[1+\gamma_{02}P(\omega_k,\omega_0)]}, \qquad (21)$$

$$T(\omega_0) = \int \frac{u_k^2 d^3k}{(\omega_k+\mu-\omega_0-b)(\omega_k-\omega_0)[1+\gamma_{02}P(\omega_k,\omega_0)]}. \qquad (22)$$

It is easy to see that for $\gamma_{02} = 0$ the function $\varphi_1(\mathbf{k})$ coincides with the corresponding function in the Lee model.[7] Substituting the solution (19)–(22) of the Fredholm equation into relation (14), we obtain the mass equation

$$m_{A0} - m_A = \gamma_{02}G(\omega_0), \qquad (23)$$

which we shall consider too. We are interested in the physical one-particle states whose energies (masses) are the roots of Eq. (23).

As a first step, we shall use the methods developed in[5,8], consisting of the use of a specific form factor

$$f(k) = M^2/(M^2+k^2), \qquad (24)$$

which according to[8] guarantees the convergence of all integrals. In addition, in order to simplify the calculations without limiting the generality (see[8,9]) we shall regard the θ particles as nonrelativistic, i.e., in H_0 we put

$$\omega_k = k^2/2\mu + \mu \qquad (25)$$

and in H_I we replace the factor $(2\omega_k)^{1/2}$ by $(2\mu)^{1/2}$. We note that the function $G(\omega_0)$ is real for $\omega_0 < \mu$. The point $\omega_0 = \mu$ is a branch point in the complex ω_0 plane, and the function $G(\omega_0)$ has a cut along the real axis for $\omega_0 > \mu$. Using the assumptions that have been made, now it is not difficult to show that

$$F(\omega_0) = \int_0^d \Big\{\omega_k-\omega_0+\gamma_{02}\Big[\frac{\pi^2M^3}{(M+a)^2} - \frac{\pi^2M^3}{(M+i\sqrt{d^2-k^2})^2}\Big]\Big\}^{-1} u_k^2 d^3k + \int_d^\infty \Big\{\omega_k-\omega_0+\gamma_{02}\Big[\frac{\pi^2M^3}{(M+a)^2} - \frac{\pi^2M^3}{(M+\sqrt{k^2-d^2})^2}\Big]\Big\}^{-1} u_k^2 d^3k, \qquad (26)$$

where $a^2/2 = \mu - b > 0$, $d^2 = a^2 + k_0^2$. The behavior of the function $F(\omega_0)$ is depicted in Fig. 1.

Now let us consider the denominator of the function $G(\omega_0)$

$$\Phi(\omega_0) = 1 - \gamma_{02}T(\omega_0). \qquad (27)$$

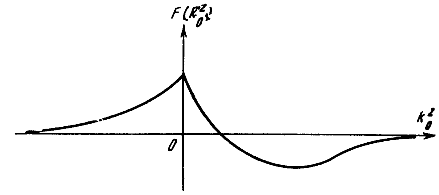

FIG. 1. Behavior of the function $F(\omega_0)$. For $k_0^2 > 0$ ($\omega_0 > \mu$) the corresponding integrals are to be evaluated in the sense of a principal value. The behavior of the function $F(\omega_0)$ is analogous to the behavior of the propagator for the $N\theta$ sector of the Lee model.[8]

One can represent the function $T(\omega_0)$ in the form

$$T(\omega_0) = \int \frac{u_k^2 d^3k}{(\omega_k+\mu-\omega_0-b)(\omega_k-\omega_0)} \times[1-\gamma_{02}P(\omega_k,\omega_0)+\gamma_{02}^2P^2(\omega_k,\omega_0)-\ldots]. \qquad (28)$$

For a small value of γ_{02} this integral can be evaluated explicitly but, because of the cumbersome nature of the resulting expressions, we do not cite the results here; in this case $\Phi(\omega_0)$ is represented by the dotted curve shown in Fig. 2. For arbitrary values of γ_{02} the analysis of the function $T(\omega_0)$ is somewhat more complicated, but it turns out that the value of γ_{02} in principle does not change the behavior of the function $T(\omega_0)$. A graph of the function $\Phi(\omega_0)$ for arbitrary γ_{02} is given in Fig. 2.

Thus, for γ_{02} very small the behavior of the function $G(\omega_0)$ practically coincides with the behavior of the function $F(\omega_0)$, and as $\gamma_{02} \to 0$ the mass equation agrees in appearance with the mass equation for the $N\theta$ sector of the Lee model (see[8]). With increase of γ_{02} from zero up to a certain value γ'_{02} no essential differences from case[8] were observed, although the value of the function $G(\omega_0)$ increases without limit with increase of γ_{02} ($\gamma_{02} < \gamma'_{02}$) at the point $k_0^2 = 0$ ($\omega_0 = \mu$). In this connection, there is one bound state for $\omega_0 < \mu$ and zero, one, or two resonances (depending on the values of the bare mass m_{A_0} and of the parameter M) for $\omega_0 > \mu$. We note that we have defined the resonance energies as zeros of Re $G(\omega_0)$, in accordance with article[10] where it is shown that such a definition does not contradict the conventional definition.

But a major distinction appears for $\gamma_{02} > \gamma'_{02}$: Poles of Re $G(\omega_0)$ appear in the neighborhood of the point $\omega_0 = \mu$; with a further increase in the value of γ_{02} one of these poles moves to the left along the energy axis (the stable region), and the second pole moves to the right (the resonance region) (see Fig. 3). We note that in contrast to[8] where changing the value of m_{A_0} may

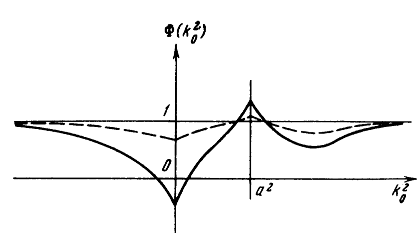

FIG. 2. Graph of the function $\Phi(\omega_0)$ for arbitrary γ_{02}. The dotted curve corresponds to a small value for γ_{02}.

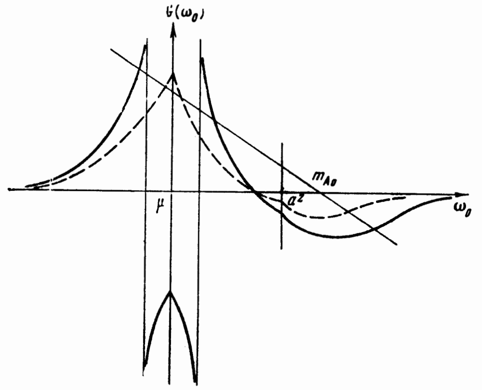

FIG. 3. Behavior of the function $G(\omega_0)$. For small values of γ_{02}, $G(\omega_0)$ does not have any poles (the dotted curve). Starting at $\gamma_{02} = \gamma'_{02}$ two poles appear at the point $\omega_0 = \mu$, moving away from this point in opposite directions as the value of γ_{02} increases.

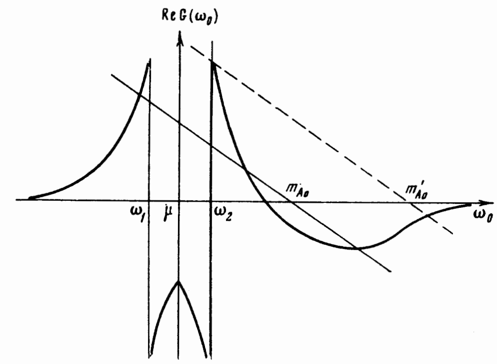

FIG. 4. Shape of the function $G(\omega_0)$ with Im $T(\omega_0)$ and Im $F(\omega_0)$ taken into account [see Eq (30)]. The points of intersection between the sloping straight lines and the smooth curve correspond to the roots of Eq. (23)). The value m'_{A0} corresponds to a double pole [see Eq. (31)].

not only shift but even annihilate the stable state and the low-energy resonance (having taken, for example, $m_{A_0} > m'_{A_0}$ in Fig. 3), now both of these states become nonremovable (for a given coupling constant $\gamma_{02} > \gamma'_{02}$). And what is more, the low-energy resonance becomes asymptotically stationary, and its position depends only on the coupling constant γ_{02}. This is associated with the fact that for $\omega_0 > \mu$ we have actually written not Re $G(\omega_0)$ but the function

$$g(\omega_0) = \frac{\operatorname{Re} F(\omega_0)}{1 - \gamma_{02} \operatorname{Re} T(\omega_0)}. \tag{29}$$

The quantity Re $G(\omega_0)$ has the form

$$\operatorname{Re} G(\omega_0) = \frac{\operatorname{Re} F(\omega_0)[1 - \gamma_{02} \operatorname{Re} T(\omega_0)]}{[1 - \gamma_{02} \operatorname{Re} T(\omega_0)]^2 + \gamma_{02}^2 [\operatorname{Im} T(\omega_0)]^2} - \gamma_{02} \frac{\operatorname{Im} F(\omega_0) \operatorname{Im} T(\omega_0)}{[1 - \gamma_{02} \operatorname{Re} T(\omega_0)]^2 + \gamma_{02}^2 [\operatorname{Im} T(\omega_0)]^2}. \tag{30}$$

The influence of Im $T(\omega_0)$ and Im $F(\omega_0)$ gives a correction which is taken into account in Fig. 4. The presence of several low-energy resonances is clearly evident from Fig. 4, and also the energy (ω_2) of one of these resonances only depends on the interaction constant γ_{02}.

It should also be noted that for a definite value of the bare mass m'_{A_0} (see the dotted straight line in Fig. 4) both low-energy resonances may merge into a single resonance which is characterized by a nonexponential decay law: A double pole is obtained, the time-dependence of whose decay is expressed by the formula[11,12]

$$(A|A(t)) \sim te^{-\Gamma t}, \tag{31}$$

where $\Gamma = \operatorname{Im} G(\omega_0)$ (see, for example, [10]).

Thus, one can classify the number of roots of the mass equation and represent them in the form of a Table.

One can carry out a more detailed examination of the resulting states, but we shall not do this here. The methods for such a classification are based on the behavior of the renormalization constants and are investigated in[8,13].

3. Analysis of the exact solution of the integral equation for $\varphi_1(\mathbf{k})$, resulting from relations (14)–(16), indicates that all conclusions reached for the approximate solution with a degenerate kernel are also valid in the case of the exact solution. The exact solution was obtained in[14] and gives the following mass equation:

$$g(\omega_0) = \omega_0 + i\varepsilon - b - \gamma_{02} \mathcal{G}(\omega_0 + i\varepsilon) = 0, \tag{32}$$

where

$$\mathcal{G}(\omega_0 + i\varepsilon) = \frac{1}{\pi\gamma} \int_\mu^\infty \frac{j(\omega, \omega_0) \operatorname{Im} h(\omega) \, d\omega}{\omega_0 - \omega + i\varepsilon}, \tag{33}$$

$$j(\omega, \omega_0) = \frac{1 + h(\omega_0) L_0(\omega, \omega_0)}{1 - h(\omega_0) L_1(\omega_0)}, \tag{34}$$

$$L_1(\omega_0) = \frac{1}{\pi} \int_\mu^\infty \frac{d\omega'}{h(\omega_0 + b - \omega')} \operatorname{Im} [h(\omega')]^{-1}, \tag{35}$$

$$L_0(\omega, \omega_0) = \frac{1}{\pi} \int_\mu^\infty \frac{d\omega'}{(\omega' + \omega - \omega_0 - b) h(\omega_0 + b - \omega')} \operatorname{Im} \frac{1}{h(\omega')}, \tag{36}$$

$$h(\omega) = (\omega - b) \left[1 + 4\pi g^2 (\omega - b) \int_\mu^\infty \frac{f^2(\omega') \sqrt{\omega'^2 - \mu^2} \omega' d\omega'}{(\omega' - b)(\omega' - \omega - i\varepsilon)} \right]. \tag{37}$$

Thus, the problem reduces to finding the points of intersection

$$\omega_0 - b = \gamma_{02} \mathcal{G}(\omega_0) \tag{38}$$

(compare with Eq. (23) for the degenerate kernel). One can write the function $\mathcal{G}(\omega_0)$ in the form

$$\mathcal{G}(\omega_0) = A(\omega_0) \left\{ \int_\mu^\infty \frac{\operatorname{Im} h(\omega) \, d\omega}{\omega_0 - \omega + i\varepsilon} + h(\omega_0) \int_\mu^\infty \frac{L_0(\omega, \omega_0) \operatorname{Im} h(\omega) \, d\omega}{\omega_0 - \omega + i\varepsilon} \right\}. \tag{39}$$

γ_{02}	m_{A0}	Number of stable states	Number of resonances*
$0 < \gamma_{02} < \gamma_{02}'$	$m_{A0} > m'_{A0}$	0	1
	$\mu < m_{A0} < m'_{A0}$	1	2 (0)
	$m_{A0} < \mu$	1	0 (2)
$\gamma_{02}' < \gamma_{02} < \infty$	$m_{A0} > m'_{A0}$	1	1
	$\mu < m_{A0} < m'_{A0}$	1	3 (1)
	$m_{A0} < \mu$	2(1)	4 (2)

*Number inside parentheses indicate the presence of removable states.

where

$$A(\omega_0) = \frac{1}{\pi\gamma} \frac{1}{1 - h(\omega_0) L_1(\omega_0)}. \tag{40}$$

It is not difficult to verify that the behavior of the function $\mathcal{G}(\omega_0)$ and its singularities coincide with the behavior and singularities of the function $A(\omega_0)$. But $A(\omega_0)$, as was shown in[5] is, to within a factor, the propagator for the $V\theta$ sector of the Lee model. Using the results of article[5] and having considered the second factor in formula (39) in detail, one can convince oneself that the behavior of the function $\mathcal{G}(\omega_0)$ repeats all fundamental singularities of $G(\omega_0)$ given by Eq. (20).

Thus, the approximate solution with a degenerate kernel describes the present model sufficiently accurately, and all basic conclusions pertaining to the one-particle physical states remain in force.

In addition, the example itself of the model under consideration shows that even a small complication of the Lee model gives two stable and four unstable states corresponding all together to a single, bare, one-particle state, that is, it appreciably enriches the content of the model.

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