# FIELD EQUATIONS OF THE GENERAL THEORY OF RELATIVITY

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Submitted June 28, 1968

Zh. Eksp. Teor. Fiz. 55, 2334-2344 (December, 1968)

A variational method, which is a generalization of the method based on the Palatini Lagrangian, is used to derive equations in terms of fourth rank tensors which relate geometrical quantities characterizing space-time to local properties of matter and of vacuum. The equations so obtained do not lead outside the framework of the general theory of relativity and are related in a natural manner to its logical structure. They are field equations of the general theory of relativity which are more general than the Einstein equations in that they are compatible with the possibility of a local interaction between matter and the gravitational field (vacuum). These equations can be utilized for providing a basis for the theory of gravitational radiation and in those astrophysical problems where the local interaction with the gravitational field affects the properties and the structure of matter. In contrast to the Einstein equations the equations obtained here enable one to postulate a substantial nature for vacuum and to treat it (together with matter) as possessing space-time properties.

1. The relation between geometrical properties of space-time (ST) and the physical properties of matter is expressed by the field equations of the general theory of relativity (GTR). The characteristic feature of the Einstein field equations is that the Riemann tensor which contains the most complete information concerning the local properties of ST does not appear in them. This suggests that the Einstein equations possibly are not the most general equations relating the macroscopic properties of ST to those of matter. Indeed, it turns out to be possible to obtain more general equations containing the Riemann tensor by means of a variational method which is a generalization of the method of deriving the Einstein equations based on the Palatini Lagrangian.

It is remarkable that the equations so obtained do not lead outside the framework of the GTR and this emphasizes the perfection and "rigidity" of its logical structure. In comparison with the Einstein equations obtained here consists of the possibility of directly associating with them also such subsidiary equations ("equations of state") which cannot be formulated as a relation between the components of the energy-momentum tensor.

Subsidiary equations of this type must unavoidably appear in the theory when the local structure of the gravitational field (vacuum) appears in the local internal (quantum) interactions between elements of matter - a situation which is very probable in the case of high density of matter and an intense gravitational field. Therefore, superdense states of matter are a possible domain for the application of the equations obtained here in astrophysics. Another domain is the theory of gravitational radiation where they are, in fact, already used (cf. Sec. 4). The equations obtained here are also related to the problem in which Einstein was interested[1,2] from the moment of creation of GTR: does geometry, i.e., ST, appear in GTR as an independent entity similar to the Newtonian absolute ST even though it may be subject to deformation under the action of material fields, or does it express only certain relations between material fields. The equations found here, in contrast to the Einstein equations, are compatible with the latter possibility (cf. Sec. 7) which corresponds to Einstein's prognostication[1,2], but which, possibly, does not correspond to modern views[3-5]. In principle, this enables one to associate the appearance of forces of inertia with a change in the state of the system "matter-vacuum" (Sec. 7), thereby treating the macroscopically manifested interaction between matter and vacuum as a universal one.

2. Following Palatini we adopt the coefficients of affine connectivity $\Gamma^j_{kl} = \Gamma^j_{lk}$ as dynamic variables. It is possible to construct from the quantities $\Gamma^j_{kl}$ two tensors: the Riemann tensor $R^j_{klm} = \Gamma^j_{k[m,l]} + \Gamma^j_{a[l}\Gamma^a_{m]k}$ [1] and the Ricci tensor $R_{kl} = R^a_{kla}$, which depend linearly on the derivatives of $\Gamma^j_{kl}$. It is not possible to construct directly from the $\Gamma^j_k$ a scalar density (Lagrangian) satisfying the last condition, i.e., the theory needs another tensor which does not depend on the derivatives of $\Gamma^j_k$. Palatini utilizes the Ricci tensor $R_{kl}$ and introduces a new tensor which we denote by $a^{kl}$. Its components are regarded as dynamic variables which are added to $\Gamma^j_{kl}$; their geometrical meaning is established later from the equations which we obtain. The assumption that the tensor $a^{kl}$ is symmetric leads to GTR. Then from $R_{kl}$ and $a^{kl}$ one can construct a single invariant $a^{bc}R_{bc}$ which contains only the symmetric part of the tensor $R_{kl}$, and which therefore can be regarded as symmetric. The assumption that the con-

[1] Antisymmetrization is carried out with respect to subscripts in square brackets, i.e., $A_{[kl]} = ½(A_{kl} - A_{lk})$, and symmetrization is carried out with respect to subscripts in curved brackets, i.e., $A_{(kl)} = ½(A_{kl} + A_{lk})$. A comma preceding a subscript indicates differentiation with respect to the coordinate corresponding to this subscript.

nectivity of $\Gamma^j_{kl}$ is Riemannian is equivalent to the above; in this case without loss of generality one can assume that $a^{kl} = a^{lk}$ (the geometrical meaning of $a^{kl}$ as before is elucidated in subsequent discussion). Let a be the determinant constructed from the components of the tensor $a_{kl}$ which is reciprocal to $a^{kl}$ (i.e., $a_{ka}a^{al} = \delta^l_k$). The scalar density which contains derivatives of the dynamic variables $\Gamma^j_{kl}$, $a^{kl}$ of order not higher than the first (the Palatini Lagrangian) will be of the form ${}^{(1)}\Lambda = a^{bc}R_{bc}\sqrt{-a}$.

The Einstein equations can now be obtained by assuming that the Lagrangian ${}^{(m)}\Lambda$ for matter depends on the tensor $a^{kl}$ and on its derivatives, but is independent of the coefficients of affine connectivity $\Gamma^j_{kl}$. As a result of this the tensor $a^{kl}$ is interpreted as a quantity which determines the manifestations of geometry in processes associated with matter (which exhausts the geometrical properties that can be established by experiments on matter). Subsequently this is confirmed by the fact that $\Gamma^j_{kl}$ are uniquely expressed in terms of $a^{kl}$. Defining action as the integral over four-volume of the sum ${}^{(1)}\Lambda + \kappa\,{}^{(m)}\Lambda$, where $\kappa$ is a dimensional constant (which enables one to establish independently the units of measurement for quantities appearing in nongravitational phenomena), and setting equal to zero the variational derivatives of the action with respect to the dynamic variables $\Gamma^j_{kl}$ and $a^{kl}$ we obtain

$$(a^{kl}\sqrt{-a})_{;j} = 0, \tag{1}$$

$$R_{kl} - {}^1/_2 a_{kl}a^{bc}R_{bc} = -\varkappa T_{kl}, \tag{2}$$

where the semicolon denotes covariant differentiation with respect to the affine connectivity $\Gamma^j_{kl}$, while $T_{kl}$ is the variational derivative with respect to $a^{kl}$ of the action ${}^{(m)}J = \int {}^{(m)}\Lambda d^4x$ for matter. Equations (1) can be easily solved with respect to the variables $\Gamma^j_{kl}$, and this (cf. for example,[6]) leads to the relation between $\Gamma^j_{kl}$ and the tensor $a^{kl}$ which is exactly the same as the relation between $\Gamma^j_{kl}$ and the components of the metric tensor $g^{kl}$ in Riemannian geometry. In this way the geometry which can be discovered (observed) by experiments on matter having a Lagrangian with the indicated properties is a Riemannian geometry with the metric tensor $g^{kl} = a^{kl}$. The tensor $a^{kl}$ acquires a clear geometric interpretation of the metric tensor, and the Riemannian geometry acquires the interpretation of a geometry consistent with the theory based on the Lagrangian ${}^{(1)}\Lambda + \kappa\,{}^{(m)}\Lambda$, which, in fact, is the GTR.

In the method presented above based on the Palatini Lagrangian, just as in other variational methods of deriving the field equations of GTR, use is made essentially without proper motivation of the Ricci tensor and not of the Riemann tensor which contains a greater amount of geometric information and which is more closely related to gravitation in the sense that it differs from zero when, and only when, space-time is curved - a property which is not possessed by the Ricci tensor. Following the idea of the Palatini method we try to base our discussion on the invariant which directly involves the Riemann tensor.

For the dynamic variable appearing in the Lagrangian ${}^{(m)}\Lambda$ for matter we adopt the fourth rank tensor which we denote by $a_j^{klm}$. Its contraction with the Riemann tensor yields the invariant $\overset{*}{R} = a_a^{bcd}R^a_{bcd}$. We assume such symmetry for the tensor $a_j^{klm}$ so that, just as above, there would follow from it without further assumptions the possibility of a geometry with the symmetric Ricci tensor $R_{kl} = R_{lk}$. The latter presupposes the validity of the identities for the Riemann tensor:

$$R^j{}_{k(lm)} = 0; \quad R^j{}_{klm} + R^j{}_{lmk} + R^j{}_{mkl} = 0; \quad R^a{}_{akl} = 0, \tag{3}$$

and also of the identities $\Gamma^j_{[kl]} = 0$, $\Gamma^a_{a[k,l]} = 0$, which corresponds to a special case of affine connectivity without torsion. Apparently, it is sufficient to assume that the tensor $a_j^{klm}$ satisfies identities analogous to (3). In this case the contracted tensor $a^{kl} = a_a^{kla}$ and the tensor reciprocal to it $a_{kl}$ ($a_{ka}a^{al} = \delta^l_k$) are symmetric. If a is the determinant constructed from the components of $a_{kl}$, then $\Lambda = \overset{*}{R}\sqrt{-a}$ is a scalar density and one can construct the action integral

$$J = \int (\Lambda + \varkappa\,{}^{(m)}\Lambda)\, d^4x.$$

Noting that

$$\delta R^j{}_{klm} = -2(\delta\Gamma^j{}_{k[l]})_{;\,m]}, \qquad \delta\sqrt{-a} = -{}^1/_2\sqrt{-a}\, a_{b[c}\delta^a_{d]}\,\delta a_a{}^{bcd},$$

we obtain

$$\delta J = \int \Big[ \Big( R^a{}_{bcd} - \frac{1}{2}\overset{*}{R} a_{b[c}\delta^a{}_{d]} + \varkappa \check{T}^a{}_{bcd} \Big) \delta a_a{}^{bcd} + \frac{2}{\sqrt{-a}} (a_a^{(bc)d}\sqrt{-a})_{;d}\,\delta\Gamma_{bc}{}^a \Big] \sqrt{-a}\, d^4x. \tag{4}$$

Here $\check{T}^j_{klm}$ is the tensor obtained by varying ${}^{(m)}\Lambda$ and having the same symmetry as $a_j^{klm}$. If, for example, ${}^{(m)}\Lambda$ does not contain derivatives of $a_j^{klm}$ of order higher than the first, then

$$\sqrt{-a}\,\check{T}^j{}_{klm} = \frac{\partial^{(m)}\Lambda}{\partial a_j{}^{klm}} - \partial_a \frac{\partial^{(m)}\Lambda}{\partial a_j{}^{klm}{}_{,a}}.$$

The factors preceding the variations of the dynamic variables in (4) have the same symmetry as the variations. Therefore, the field equations will be the following:

$$(a_j{}^{(kl)d}\sqrt{-a})_{;d} = 0, \tag{5}$$

$$R^j{}_{klm} - {}^1/_2 \overset{*}{R} a_{k[l}\delta^j_{m]} = -\varkappa \check{T}^j{}_{klm}. \tag{6}$$

Equations (5) do not contain quantities characterizing matter, and, just as (1), they are connective equations which determine the dependence of the variables $\Gamma^j_{kl}$ on $a_j^{klm}$. Altogether there are 16 independent equations (5) and it is possible that in the general case they do not suffice for an unambiguous determination of $\Gamma^j_{kl}$ in terms of $a_j^{klm}$. However, one can easily indicate a solution if the tensor $a_j^{klm}$ degenerates into the multiplicative tensor:

$$a_j^{klm} = {}^2/_3 a^{k[l}\delta_j^{m]}. \tag{7}$$

(The factor $^2/_3$ is introduced in order to have $a_a^{kla} = a^{kl}$.) It turns out that it is just this case that leads to the GTR. Indeed, substitution of (7) into (5) after simple calculations again yields (1). This means that the geometry discovered by means of experiments on matter must be Riemannian, and the tensor $a^{kl}$ should be identified with the metric tensor $g^{kl}$. Substitution of (7) into (6) with $a_{kl} = g_{kl}$ transforms the field equations (6) into the form

$$\check{G}^j_{klm} = R^j_{klm} - {}^1/_3 g_{k[l}\delta^j_{m]}R = -\varkappa \check{T}^j_{klm}, \tag{8}$$

where $R = g^{bc}R_{bc}$. Contracting (8) with respect to j and m we obtain

$$G_{kl} = R_{kl} - {}^1/_2 g_{kl}R = -\varkappa T_{kl}, \tag{9}$$

where $T_{kl} = \check{T}^a_{kla}$. If $T_{kl}$ is identified with the energy-momentum tensor for matter, as we shall assume in the following, then Eqs. (9) are the Einstein equations, i.e., the connection between the Riemannian metric and the energy-momentum tensor is found to be the same as in GTR.

It is convenient to bring equations (8) into a different form. We introduce the tensor

$$G^j_{klm} = R^j_{klm} - {}^1/_3 g_{k[l}\delta^j_{m]}R - 2C^j_{klm},$$

where

$$C^j_{klm} = R^j_{klm} - g_{k[l}R^j_{m]} - R_{k[l}\delta^j_{m]} + {}^1/_3 g_{k[l}\delta^j_{m]}R$$

is the well-known conformally-invariant Weyl tensor. It may be easily seen that the covariant divergence of the tensor $G^j_{klm}$ is identically equal to zero[7]. We further assume

$$Q^j_{klm} = \check{T}^j_{klm} - g_{k[l}T^j_{m]} - T_{k[l}\delta^j_{m]} + {}^1/_3 g_{k[l}\delta^j_{m]}T,$$

and

$$T^j_{klm} = \check{T}^j_{klm} - 2Q^j_{klm}.$$

The field equations (8) can now be represented in the form indicated in the preliminary communication[7];

$$G^j_{klm} = -R^j_{klm} + 2g_{k[l}R^j_{m]} + 2R_{k[l}\delta^j_{m]} - g_{k[l}\delta^j_{m]}R = -\varkappa T^j_{klm}. \tag{10}$$

Since the contracted tensors $G^a_{kla}$ and $T^a_{kla}$ are respectively equal to the Einstein tensor $G_{kl}$ and the energy-momentum tensor $T_{kl}$, then contraction of equations (10) with respect to the indices j and m again yields the Einstein equations. In view of $G^a_{klm;a} \equiv 0$ Eqs. (10) can also be put in the form of "conservation equations";

$$T^a_{klm;a} = 0. \tag{11}$$

(This form of the field equations in terms of fourth rank tensors is to some degree analogous to the representation of the Einstein equations proposed by Pagels[8] in the form $T^a_{j;a} = 0$, $T^a_{[j;k]a} = 0$.

3. From (10) it follows formally that the tensor $T^j_{klm}$ describes sources of gravitational field in the same sense as the energy-momentum tensor does in the Einstein equations. Starting with this analogy we discuss the possibility of the situation that the tensor $T^j_{klm}$ describes physical objects—sources for the curvature of ST which differ from ST itself. In this sense we shall refer to the tensor $T^j_{klm}$ as being substantial. We shall relate the classification of the objects described by it to its contracted forms: we shall give the name matter to the medium with the energy-momentum tensor $T^a_{klm} = T_{kl} \neq \lambda g_{kl}$, where $\lambda$ is a constant, and in the opposite case we shall speak of vacuum. The latter term is justified by the fact that the velocity of a freely moving test particle with respect to an object with $T_{kl} = \lambda g_{kl}$ is unobservable: it has the macroscopic properties of vacuum[9] (c.f. also Sec. 6). The value $\lambda = 0$ corresponds to vacuum in the ordinary sense. We emphasize that in the interpretation of the substantial tensor adopted above the concepts of vacuum and of ST are not identical.

In contrast to the tensor $a_j^{klm}$ which in accordance with (7) is algebraically related to the whole contraction $a^{kl} = g^{kl}$ the tensor $T^j_{klm}$ can be expressed in terms of the energy-momentum tensor $T_{kl} = T^a_{kla}$ only in special cases. Indeed, when this is possible, by substituting into (10) the expression for $T_{kl}$ from (9) the Riemann tensor can be expressed algebraically in terms of the tensors $R_{kl}$ and $g_{kl}R$, which cannot be done in the general case. Consequently, the physical meaning of the tensor $T^j_{klm}$ is not exhausted by its connection with the energy-momentum tensor. This can also be seen from the fact that in virtue of (10) the tensor $T^j_{klm}$ must have the same number of algebraically independent components as the Riemann tensor, i.e., in the general case it must have 20 components.

In the case of vacuum the field equations (10) take on the form

$$R^j_{klm} = \varkappa T^j_{klm} \quad (T^a_{kla} = \lambda g_{kl},\ \lambda = \text{const}). \tag{12}$$

Since the Riemann tensor gives an exhaustive local description of the geometric properties of ST, then the tensor $T^j_{klm}$ gives the most complete possible local macroscopic description of vacuum as a physical object in the sense of its effect on the metric. In view of (12) the classification developed by Petrov[10-12] for the Riemann tensor can be carried over to the tensor $T^j_{klm}$. In particular, the local macroscopic state of the vacuum can be defined with the aid of not more than ten quantities—we shall refer to them as the Petrov parameters—which define a) a nonholonomic orthonormal set of basis vectors in terms of which the tensor $T^j_{klm}$ assumes a certain canonical form, and b) the components of $T^j_{klm}$ in terms of this set of orthonormal basis vectors. Such a description of vacuum recalls the local macroscopic description of matter by means of an energy-momentum tensor which can also be defined by not more than ten quantities— parameters of the state of matter—defining a nonholonomic orthonormal set of basis vectors which puts the energy-momentum tensor into canonical form and the components of the latter in terms of this set of basis vectors.

In the presence of matter we set

$$T^{j}{}_{klm} = {}^{0}T^{j}{}_{klm} + g_{k[l}T^{j}_{m]} + T_{k[l}\delta^{j}_{m]} - {}^{1}/{}_{3}g_{k[l}\delta^{j}_{m]}T. \tag{13}$$

The last three terms on the right-hand side are defined by the parameters of the state of matter, and the contraction ${}^{0}T_{kl}$ of the tensor ${}^{0}T^{j}_{klm}$, as can be easily verified, is equal to zero, so that it possesses the properties of the tensor $T^{j}_{klm}$ for vacuum and is defined by not more than ten parameters which we shall also call Petrov parameters. In virtue of the field equations (10) the totality of the parameters describing the state of matter and of the Petrov parameters completely determines the local properties of ST, i.e., the substantial tensor $T^{j}_{klm}$ gives the most complete possible macroscopic description of the medium in the sense of its effect on geometry. With respect to a model of the medium in the form of a system of particles moving in vacuum without a macroscopically essential interaction with it one can say that the tensor $T^{j}_{klm}$ describes both the state of matter and the state of the vacuum. Of course, in the general case the separation of physical reality into two components (vacuum and matter) is arbitrary.

Since the number of algebraically independent components of the tensor $T^{j}_{klm}$ is equal to 20, then the field equations (10), and also (11), each contain 20 independent equations for 30 unknowns: the 20 components of the tensor $T^{j}_{klm}$ and the 10 components of the tensor $g_{kl}$. In view of the general covariance of GTR four components of $g_{kl}$ can be specified arbitrarily, i.e., the total number of physically essential unknown functions in (10) and (11) is equal to 26. Thus, just as in the case of the system of Einstein equations, the system of field equations in terms of fourth rank tensors is underdetermined, i.e., it presupposes the existence of subsidiary equations interrelating the components of $T^{j}_{klm}$, but not derivable from GTR. Just as for the system of Einstein equations, the number of subsidiary equations must exceed by six the number of new unknown functions appearing in them.

Any solution of the total system of equations (10) and the subsidiary equations interrelating the components of $T^{j}_{klm}$ evidently identically satisfies the Einstein equations since the latter are a consequence of (10). Conversely, we consider a certain solution of the total system consisting of the Einstein equations and the subsidiary equations interconnecting the components of the energy-momentum tensor. This solution is always compatible with (10), since the system (10) is underdetermined to just such a degree that with respect to any such solution it can be considered as a definition of the tensor $T^{j}_{klm}$. Indeed, knowing the field $g_{kl}$ one can evaluate the tensor $G^{j}_{klm}$, and from (10) the tensor $T^{j}_{klm}$. In this case the values of $T^{j}_{klm}$ automatically satisfy the single required condition: $T^{a}_{kla} = T_{kl}$ since $G^{a}_{kla} \equiv G_{kl}$, and the field $g_{kl}$ by hypothesis satisfies the Einstein equations.

Thus, from the point of view of physical consequences, Eqs. (10) differ from the Einstein equations by the fact, and only by the fact, that to them one can directly add such subsidiary equations which depend on the Petrov parameters and, therefore, cannot be represented as relations interconnecting only the parameters of the state of matter (components of the energy-momentum tensor). Since the subsidiary equations are not set up within the framework of the GTR, then the transition to the field equations in terms of fourth rank tensors does not alter its physical content. Therefore equations (10) and (11) are also field equations of GTR, but, in the sense indicated above more general than the Einstein equations. There can be no still more general field equation within the framework of GTR since the substantial tensor completely determines the local geometric properties of ST (the Riemann tensor). The fact that this property is not possessed by the energy-momentum tensor is what enables one to generalize the Einstein equations.

4. We consider the case when the system of field equations of GTR is complete, so that the addition of subsidiary equations is not required (and not possible). This will be the case if $T_{kl} = 0$, and this situation is the subject of investigation in the modern theory of gravitational radiation. The field equations (10) for $T_{kl} = 0$ have the form (12).

The theory of gravitational radiation which does not assume the weakness of the gravitational field is based on an analysis of the algebraic structure of the Riemann tensor carried out by Petrov. The algebraic approach at first led to the fact that the theory had a static nature: it did not contain equations containing the characteristics of radiation at the point under discussion and in its neighborhood. Pirani[13] proposed to obtain the required equations by introducing the Petrov parameters into the Bianca identities which for $R_{kl} = 0$ assume the form $R^{a}_{klm;a} = 0$. This idea, which for the Petrov parameters treated as unknowns leads to equations in terms of fourth rank tensors, was developed in a number of papers (c.f. for example[14,15]). However it remained unclear as to what was the connection of such an apparently purely geometrical approach with the GTR, and this led to the problem which has not been completely solved until the present time[15] of the derivation of all the results of the theory of gravitational radiation directly from the Einstein equations.

The field equations in the form (10) and (11) which, as has been shown, are in the absence of subsidiary equations completely equivalent to the Einstein equations, essentially provide a direct basis for the theory of gravitational radiation. From (12) and (11) follows the identity $R^a_{klm;a} = 0$ used by Pirani. Conversely, substitution into it of the Petrov parameters leads to field equations in the form (11), which, consequently, were proposed by Pirani for the particular case under consideration without indicating their relation to the GTR. The analysis of the structure of the Riemann tensor in the theory of gravitational radiation from the point of view arising from the field equations in the form (10) and (11) does not have a geometric nature, but, in fact, is an analysis of the substantial tensor $T^j_{klm}$.

In this sense the use of the field equations in terms of fourth rank tensors (10) and (11) provides the finishing touches for the theory of gravitational radiation regarded as a subdivision of the GTR.

5. We now discuss the possibility of and the meaning of the dependence of the subsidiary equations on the Petrov parameters.

Subsidiary equations in GTR take into account local interactions which manifest themselves not in macroscopically distinguishable motion, but macroscopically locally without being related to the equations of motion and to geometry. We shall call such interactions which are macroscopically indistinguishable from point interactions as local interactions. The laws governing them are not contained within GTR and this is naturally related to the fact that the basic concept of geometry in GTR is nonlocal since it expresses relations appearing in macroscopic motion. Mathematically this is reflected in the fact that from the values of the components of the metric tensor at a fixed point it is not possible, generally speaking, to deduce any consequences: the behavior of this tensor in the neighborhood of the point, i.e., its derivatives, is essential. The invariant characteristic of the geometry is given by a tensor which has no direct geometrical interpretation for a given point, but which characterizes geometric relations in its neighborhood, and in this sense it is nonlocal. This is the Riemann tensor, and it, naturally, is constructed from the derivatives of the metric tensor. Correspondingly in GTR there are no methods for a direct measurement of the Riemann tensor at a given point: measurements which do not go outside the framework of GTR, i.e., geometrical measurements, are always nonlocal. They are based either on an observation of the motion of bodies, or on a geometrical interpretation of a stressed state of an extended crystal, etc.

The field equations of GTR relate quantities which are measured purely geometrically and quantities which are measured locally. Thus in the Einstein equations $G_{kl} = -\kappa T_{kl}$ the Einstein tensor $G_{kl}$ is determined by the field of the metric tensor. A measurement of $G_{kl}$ is therefore equivalent to a measurement of the metric tensor over a certain region which can be carried out purely geometrically (for example, by the *observation* of the free motion of test bodies) without using forces of nongravitational nature, but it is nonlocal. We measure the tensor $T_{kl}$ (energy and momentum densities of matter and their fluxes) locally and, therefore, independently of a measurement of the metric, but with the aid of forces of nongravitational nature. Consequently the tensors $G_{kl}$ and $T_{kl}$ describe independent measurements or, generally speaking, independent phenomena. Therefore the relation between them expressed by the Einstein equations is physically meaningful.

Utilizing the Einstein equations one can determine from local measurements of the energy-momentum tensor at a point only certain geometrical characteristics in its neighborhood, i.e., specifically those which are expressed by the Ricci tensor. In this respect the structure of the Einstein equations is logically not quite consistent. This inconsistency is not present in the structure of the field equations in terms of fourth rank tensors: a local measurement of the substantial tensor in view of (10) would permit one to determine the Riemann tensor and, consequently, to know all the geometric relations in the neighborhood of a (nonsingular) point of measurement. However, physically this is possible if not only the state parameters but also the Petrov parameters are measureable locally, i.e., if there exists a local interaction between vacuum and matter (or if there is no separation into these two components, i.e., the medium is unique).

In quantum field theory the interaction between vacuum and matter is reflected in the procedure of renormalization of masses, charges, etc. For a low particle density and a weak gravitational field it is strictly localized in the macroscopic sense to regions of the order of the dimensions of the particles. The effect of the structure of the vacuum on the renormalized characteristics of the particles is not essential in this case, and there arises a picture of "dressed particles" in a vacuum which locally practically does not affect their structure, and, moreover, the inhomogeneities of which can be neglected (the gravitational field is weak). The macroscopic equations of state for matter in this case cannot contain the Petrov parameters. They must appear in the subsidiary equations when the picture of two separated components "clothed particles—vacuum" is violated and the local structure of vacuum (in the linear approximation the quantum interaction with gravitons) affects the state of matter. Such a state of affairs appears to be quite probable in certain situations in astrophysics (high matter density, "prestellar" form of matter proposed by Ambartsumyan). In such a case, due to the presence of Petrov parameters in the subsidiary equations one has to use field equations in terms of fourth rank tensors.

6. The form of the subsidiary equations relating the parameters of the state of matter with the Petrov parameters could be determined in some at all consistent way only after the creation of the quantum theory of gravitation. Therefore we restrict ourselves to a demonstration of the formal possibility of an invariant formulation of such equations in the special case of "ordinary matter" ($T_{kl} \neq 0$, $\overset{*}{T} \neq 0$) "in a vacuum" with $T_{kl} = \lambda g_{kl}$ of the type $\mathrm{T}_1$ (non-degenerate), in accordance with Petrov's classification (reference[12], p. 116).

Matter and vacuum (gravitational field) are under ordinary conditions macroscopically sharply distinguishable as different physical objects, and therefore the substantial tensor $T^{j}_{klm}$ can be naturally written in the form (13) having separated out the tensor ${}^{0}T^{j}_{klm}$ with the contraction ${}^{0}T^{a}_{kla} = \lambda g_{kl}$ and assuming that this tensor macroscopically describes a vacuum. The fact that vacuum belongs to type $\overset{*}{T}_1$ means that in a certain nonholonomic orthonormal set of basis vectors $R_0$ the matrix of the tensor ${}^{0}T_{jklm}$ written in terms of collective indices ($14 \to 1$, $24 \to 2$, $34 \to 3$, $23 \to 4$, $31 \to 5$, $12 \to 6$) has the form

$$({}^{0}T_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix},$$

where M and N are diagonal matrices: $M = \mathrm{diag}(\alpha_1\alpha_2\alpha_3)$, $N = \mathrm{diag}(\beta_1\beta_2\beta_3)$, with

$$\sum_1^3 \alpha_s = -\lambda, \quad \sum_1^3 \beta_s = 0,$$

so that of the six parameters $\alpha_s$, $\beta_s$, there are only four independent ones. Neglecting the tangential stresses, the energy-momentum tensor for matter is assumed to be of the form $T_{kl} = (p + \mu) u_k u_l + p g_{kl}$, where $u_k$ is the four-velocity of matter in the orthonormal set of basis vectors ${}^{0}R$, and p and $\mu$ are formally the pressure and the density of matter. Then there are 21 unknown functions (6 Petrov parameters defining the orthonormal set of basis vectors ${}^{0}R$; 4 independent Petrov parameters $\alpha_s$, $\beta_s$; 3 independent components $u_k$; the parameters p and $\mu$) and, consequently, one subsidiary equation must be added to (10).

Since the quantities $\alpha_s$, $\beta_s$, $u_k$, p, are defined with respect to specified sets of basis vectors they are specified invariantly, i.e., they are scalars. Instead of the scalars $u_k$ we introduce three scalars $v_1$, $v_2$, $v_3$—the components of the three-velocity of matter with respect to the orthonormal set of basis vectors ${}^{0}R$. It can be easily shown that in the orthonormal set of basis vectors ${}^{0}R$ the matrix of the tensor $T_{jklm}$ written in terms of collective indices has the form

$$(T_{ab}) = \begin{pmatrix} M+U+B & N+A \\ N-A & -M+U-B \end{pmatrix} \tag{14}$$

where

$$B = {}^{1}/_{12} T\, \mathrm{diag}(111),$$

$$U = -\frac{1}{4}\frac{p+\mu}{1-v^2} \times \begin{pmatrix} 1+v_2^2+v_3^2-v_1^2 & -2v_1v_2 & -2v_1v_3 \\ -2v_1v_2 & 1+v_3^2+v_1^2-v_2^2 & -2v_2v_3 \\ -2v_1v_3 & -2v_2v_3 & 1+v_1^2+v_2^2-v_3^2 \end{pmatrix},$$

$$A = -\frac{1}{2}\frac{p+\mu}{1-v^2}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \quad v^2 = v_1^2 + v_2^2 + v_3^2.$$

The quantities appearing in (14) separate into two groups: the parameters $\alpha_s$, $\beta_s$, $v_s$, which roughly speaking, characterize the nonisotropic nature of the state of the matter-vacuum system, and the scalars $T = 3p - \mu$ and $\rho = (p + \mu)/(1 - v^2)$ which characterize the state of the system at a point. The role of the scalars T and $\rho$ is also formally expressed by the structure of (14): they are the common factors of all the components of the matrices B, U and A characterizing the contribution of matter to the substantial tensor.

The "usual" subsidiary equation which does not include the local interaction with vacuum relates p and $\mu$. In view of the linear independence of the quantities $T = 3p - \mu$ and $\overset{*}{\rho} = p + \mu$ it can be put into the form $\overset{*}{F}(T, \overset{*}{\rho}) = 0$. In (14) the quantity $p + \mu$ appears only in $(p + \mu)/(1 - v^2) = \rho$. This suggests that the subsidiary equation should be written in the form $F(T, \rho) = 0$, assuming that it significantly deviates from $\overset{*}{F} = 0$ for $v^2 \sim 1$. The equation $F = 0$ is an example of an invariant relation relating the parameters of the state of matter with the Petrov parameters ($v^2$ depends on the latter).

We note that from $T = \overset{*}{\rho}$ there follows the ultrarelativistic equation of state due to Zel'dovich $p = \mu$. From $T = \rho$ we obtain $p = (2 - v)\mu/(2 - 3v)$ which for $v = 0$ gives the Zel'dovich equation of state, while for $v \to 1$ it gives the equation of state $p = -\mu$ proposed in reference[9] as a limiting ultrarelativistic equation of state (c.f.[16] for a justification on the basis of quantum concepts of the possibility of equations of state of such a type).

For $p = -\mu$ the energy-momentum tensor for matter is $T_{kl} = -\mu g_{kl}$ and it has canonical form (is diagonal) for any orientation of the orthonormal set of basis vectors. This "degenerate state" of matter is vacuumlike[9] in the sense that a) for matter in this state any reference system is comoving, so that with respect to it the velocity of a test particle is unobservable (the principle of relativity); b) if matter is absent in states with $T_{kl} \neq \lambda g_{kl}$, then $\mu$ = const and the algebraic structure of the tensor $T^{j}_{klm}$ corresponds to vacuum. The attitude to media with $T_{kl} \sim g_{kl}$ as to a natural element of physical reality is quite widely accepted since the publication of[9] (c.f. for example[17]). The so-called cosmological constant also finds a natural interpretation in this approach.

7. We consider certain consequences of the possibility of a local interaction between matter and vacuum.

The principle of equivalence becomes narrower (the special theory of relativity is valid in the small). It holds only to the extent to which one can neglect the influence of the structure of vacuum on the properties of matter.

The GTR has retained such a feature of Newtonian theory as the special role of geodesics as world lines for free motion. This circumstance also preserved the absolute nature of acceleration (forces of inertia) which is foreign to the assumptions of the theory[4]. The situation is different[9] if matter locally interacts with vacuum. Then vacuum becomes a system of reference and acceleration, and also the forces of inertia, can be associated with the relative rotation of the two sets of orthonormal basis vectors characterizing the vacuum and comoving with the matter which changes the state of the "matter-vacuum" system. Such a picture, in essence, is similar to other macroscopic quantum processes, say, superconductivity; the superconducting current is conserved and corresponds to a stationary state of the system; a change in the current requires work to be done. The scalar $\rho = (p + \mu)/(1 - v^2)$ gives an example of an invariantly defined macroscopic characteristic of the type of specific density a change in which could be associated with a change in the state of the "matter-vacuum" system and with the appearance of forces of inertia.

Finally we shall touch upon a problem formulated by Einstein: does ST appear in GTR as an independent entity. Since in GTR the forces of inertia are defined with respect to a system of geodesics and, as a result, their existence, observed on a certain body, does not depend on the presence or absence of other bodies in the Universe, then the answer to this question depends on whether the distinctive role of geodesics reflects the existence of a certain, necessarily vacuum, system of reference. The existence of the latter presupposes the substantial nature of vacuum in the sense of the existence of local interaction between it and matter. In this case vacuum (together with matter) can be regarded as the carrier of the properties of ST. But if the vacuum locally does not interact with matter, i.e., if its only macroscopic manifestations are geometric relations, then it essentially becomes identified with ST and loses attributes which enable one to speak of it as of a carrier of space-time relations. In this case ST unavoidably appears in the GTR as an independent object, but with strange properties which manifest themselves macroscopically only nonlocally.

Thus, if in the GTR one takes for field equations equations in terms of fourth rank tensors as is presupposed by the possibility of local interactions between matter and vacuum, then the GTR is consistent with the Einstein hypothesis that ST is not an independent entity. But if the Einstein equations are regarded as the most general field equations of the GTR, then this is not so. Indeed, the necessary condition for the manifestation of vacuum not only in geometric relations (measurability of the Petrov parameters) means that the Petrov parameters appear in the subsidiary equations. In this case in order for the system of equations to be complete, one must add to it equations containing geometric tensors of the fourth rank defining the Petrov parameters. These equations interrelating geometric and locally measurable quantities will, by this very fact, be field equations and, therefore, the tensor rank of the field equations must be equal to four.

The author is deeply grateful to L. É. Gurevich, A. Z. Dolginov, Ya. B. Zel'dovich, V. A. Ruban, A. D. Sakharov, A. D. Chernin and all those giving reports at the seminars in the A. F. Ioffe Physico-technical Institute for detailed discussion of this work and for critical comments.

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Translated by G. Volkoff
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