TURBULENT HEATING OF PLASMA IONS BY AN ELECTRON-ACOUSTIC INSTABILITY

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An investigation is made of weak turbulence in a plasma with hot ions and cold electrons in which a small-scale, high-frequency, electron-acoustic instability is excited. This instability is due to the motion of the ions with respect to the electrons in the electric field associated with the circularly polarized electromagnetic wave at a frequency of the order of the ion-cyclotron frequency ω_{Hi}. In a time interval $1/\omega_{Hi}$, in the reference system that moves with the wave there is established a stationary electron-acoustic spectrum which exhibits in wave-vector space **k** a sharp peak that is several orders of magnitude larger than the thermal noise. The scattering of ions by electron-acoustic waves leads to an increase in the transverse thermal energy of the ions (transverse with respect to the external magnetic field). Expressions are obtained for the spectral intensity of the electron-acoustic waves, taking account of the nonlinear interaction between waves and the quasilinear equation for the background ion distribution function, which determines the turbulent heating of the ions; the heating time is estimated. A number of experiments on plasma heating by ion-cyclotron waves and fast magneto-acoustic waves are discussed. These experiments have shown anomalies in the absorption of the waves and it is found that these anomalies can be explained by the excitation of a two-stream instability in the electric field of the waves in particular, the electron-acoustic instability.

1. INTRODUCTION

IT has been shown earlier [1] that the motion of plasma ions with respect to electrons across an external magnetic field, caused by the electric field of a low-frequency electromagnetic wave (for example the ion-cyclotron wave or the fast magnetoacoustic wave), leads to the excitation of high-frequency, small-scale instabilities. In a highly nonisothermal plasma with hot ions and cold electrons the instability is the so-called electron-acoustic instability. The frequency ω_k and the growth rate γ_k for the electron-acoustic instability are given by

$$\omega_{\mathbf{k}} = \Omega_\varkappa + \mathbf{k}\mathbf{u}_e, \tag{1.1}$$

where $\mathbf{u}_e$ is the electron velocity caused by the electric field of the low-frequency wave:

$$\Omega_\varkappa = \operatorname{sign} \mathbf{k} \frac{k_\parallel v}{\sqrt{Z_i + k^2\rho^2}},$$

$$\gamma_{\mathbf{k}} = \sqrt{\frac{\pi m_i}{8 m_e}} \frac{Z_i \cos\vartheta (\mathbf{k}\mathbf{u} - \Omega_\varkappa)}{(Z_i + k^2\rho^2)^{3/2}} \tag{1.2}$$

Here, eZ_i is the ion charge, $k_\parallel = k\cos\vartheta$ is the projection of the wave vector **k** in the direction of the external magnetic field H_0, $v = \sqrt{T_i/m_e}$ is the electron-acoustic velocity, $\rho = v/\omega_{He}$ and **u** is the relative velocity of the ions with respect to the electrons. Equations (1.1) and (1.2) apply for waves that propagate almost perpendicularly to the magnetic field ($\vartheta \approx \pi/2$) and whose wavelength is much smaller than the ion-Larmor radius ($kv_{Ti}/\omega_{Hi} \gg 1$) and much larger than the electron-Larmor radius ($kv_{Te}/\omega_{He} \ll 1$). The growth rate (1.2) and the frequency (1.1) are much higher than the ion-cyclotron frequency ω_{Hi} and the frequency of the low-frequency electromagnetic wave Ω but much smaller than the electron-cyclotron frequency ω_{He}. The phase velocity of the electron-acoustic wave along the magnetic field $\omega_k/k_\parallel$ is much greater than the electron thermal velocity v_{Te} so that the electron gas can be regarded as being cold; the phase velocity ω_k/k is much smaller than the ion-thermal velocity v_{Ti}. For these waves the ion gas is essentially not effected by the magnetic field. If we assume as an approximation that $k\rho \sim 1$, $\cos\vartheta \sim \sqrt{m_e/m_i}$ and $ku \sim \Omega_\varkappa$ we find that $\omega_k \sim \sqrt{\omega_{He}\omega_{Hi}}$ and $\gamma_k \sim 0.1\,\omega_k$.

It will be shown in the present work that the development of the electron-acoustic instability in a plasma subject to a circularly polarized low-frequency electromagnetic wave that propagates along the external magnetic field leads to rapid turbulent heating of the ion component of the plasma.[1)] One feature of the case being considered is the fact that the limitation on the amplitude of the turbulent waves is not due to quasilinear effects associated with the formation of a plateau on the distribution function for the resonant ions. Rather the limiting is due to a nonstationary feature of the situation: Growing waves, which at a given time satisfy the condition $\mathbf{k}\cdot\mathbf{u} > \Omega_\varkappa$, will, after a time $\Delta t \sim \pi/\Omega$, become damped because at this time $\mathbf{k}\cdot\mathbf{u}$ changes sign so that the quantity γ_k becomes smaller than zero, in which case the magnitude of the turbulent waves must drop to the level of the thermal noise. As a result, a stationary wave spectrum is established in the reference system that rotates with the vector **u**.

In the present work we shall find the spectrum for the steady-state electron-acoustic waves and the total energy of these waves, both neglecting and taking account of the nonlinear interaction between waves. The

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1) this effect has been pointed out by the present authors together with Teichmann [2] and independently by Aref'ev, Kovan and Rudakov [3].

wave spectrum exhibits a sharp peak in the wave-vector space **k** which is several orders of magnitude greater than the level of the thermal noise. The total intensity of the waves is a strong function of $|u|$: when $u > u_{cr} \sim v_{Ti}$ the level of the turbulent waves is proportional to u^2 (relatively weak growth) and when $u < u_{cr}$ the turbulent waves fall off much more rapidly (exponentially) with diminishing u.

The scattering of ions on the turbulent waves leads to a rapid randomization of the ion velocity distribution function in the plane perpendicular to $\mathbf{H}_0$ (the ion velocity distribution along $\mathbf{H}_0$ is not changed) and to a subsequent diffusion in $v_\perp$, that is to say, to an increase in the transverse thermal energy of the ions. In the present work we obtain a diffusion equation for the function $f_0(v_\perp)$ and we derive an expression for the diffusion coefficient, which is found to be proportional to the energy of the turbulent waves; the ion heating rate is also estimated.

The feature of the spectrum noted above implies that the ion heating is of a threshold nature: the ion temperature increases as long as the ion thermal velocity does not become comparable with the relative velocity u; when this occurs the ion heating is terminated sharply. This conclusion is related to the assumption that the electron-acoustic waves grow from thermal noise which, in the region being considered (low-frequencies $\omega \ll \omega_{pe}$ and large wavelengths $k \ll \omega_{pi}/v_{Ti}$) are of very low intensity. If the electron-acoustic waves are maintained by virtue of coupling (for example, due to nonlinear effects or plasma inhomogeneities) to other plasma wave branches and are maintained at a sufficiently high level, then the threshold value u can be reduced and the heating mechanism being considered here becomes operative even when $u < v_{Ti}$.

2. BASIC EQUATIONS

We consider a circularly polarized electromagnetic wave that propagates along the uniform fixed magnetic field $\mathbf{H}_0$:

$$\mathbf{E} = (E\cos(Kz-\Omega t),\quad E\sin(Kz-\Omega t), 0),$$
$$\mathbf{H} = (-H\sin(Kz-\Omega t),\quad H\cos(Kz-\Omega t), 0), \qquad (2.1)$$

where K and Ω are the wave vector and frequency and where the peak magnetic field of the wave $H = cKE/\Omega$. When $K > 0$ and $\Omega > 0$ the electric vector of the wave (2.1) rotates in the ion gyration direction in the field $\mathbf{H}_0$. As is known from linear theory, in a dense plasma this wave (the Alfvén wave or the ion-cyclotron wave) can propagate at frequencies below the ion-cyclotron frequency. When $K > 0$ and $\Omega < 0$ (2.1) represents a wave that propagates in the direction opposite to $\mathbf{H}_0$ with the electric vector rotating opposite to the ion gyration direction. This is the magneto-acoustic wave, which can propagate at frequencies below the electron-cyclotron frequency in a dense plasma. Below, in making estimates, in order to be definite we shall assume that the frequency Ω is of order ω_{Hi}.

The ion motion with respect to the electrons in the wave field (2.1) leads to the electron-acoustic instability. The electron-acoustic waves modify the ion distribution function and the wave amplitude (2.1). In the present section we derive an equation for the ion distribution function and also obtain the equation of motion in the presence of turbulent electron-acoustic waves. The kinetic equation for the ion distribution function F is of the form

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{r}} + \frac{Z_i e}{m_i}\left(\mathbf{E} - \nabla\varphi + \frac{1}{c}[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}]\right)\frac{\partial F}{\partial \mathbf{v}} = 0, \qquad (2.2)$$

where φ is the potential associated with the electron-acoustic waves.

We shall assume, in the usual way for quasilinear theory, that $F = f + f'$ where f' is the oscillating portion of the distribution function that arises because of the electron-acoustic waves. We assume that $\langle\varphi\rangle = \langle f'\rangle = 0$ so that $\langle F\rangle = f$, where the average is taken over the ensemble of random phases of the potential or in a time of the order of several oscillation periods and over a distance much smaller than the wavelength (2.1) $\Lambda = 1/K$ but much larger than the dimensions of the wave packet associated with the electron-acoustic waves $1/\Delta k_{\|}$ ($\Delta k_{\|}$ is the width of the wave packet in the direction of $\mathbf{H}_0$ in wave-vector space). We then find

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{Z_i e}{m_i}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}]\right)\frac{\partial f}{\partial \mathbf{v}} = \frac{Z_i e}{m_i}\frac{\partial}{\partial \mathbf{v}}\langle\nabla\varphi f'\rangle, \qquad (2.3)$$

$$\frac{\partial f'}{\partial t} + \mathbf{v}\frac{\partial f'}{\partial \mathbf{r}} + \frac{Z_i e}{m_j}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}]\right)\frac{\partial f'}{\partial \mathbf{v}} - \frac{Z_i e}{m_i}\nabla\varphi\frac{\partial f}{\partial \mathbf{v}}$$
$$= \frac{Z_i e}{mi}\frac{\partial}{\partial \mathbf{v}}(\nabla\varphi f' - \langle\nabla\varphi f'\rangle). \qquad (2.4)$$

We shall consider waves in a low-pressure plasma, in which case $\beta = 8\pi n_0 T_i/H_0^2 \ll 1$. Under these conditions, we can neglect the effect of the magnetic field of the wave on the ions in Eqs. (2.3) and (2.4). Since the frequency and growth rate of the electron-acoustic waves are much greater than the wave frequency (2.1) and the ion cyclotron frequency, in Eq. (2.4) we can neglect terms that are proportional to $\mathbf{E} + (\mathbf{v}\times\mathbf{H}_0)/c$ compared with $\partial f'/\partial t + \mathbf{v}\partial f'/\partial \mathbf{r}$. Furthermore, we can neglect small effects associated with the nonlinear wave interaction in treating f' (as shown in[4] the nonlinear interaction between electron-acoustic waves is primarily associated with electrons and the ion contribution in the decay and induced scattering is a factor $\sqrt{m_i/m_e}$ times smaller than the electron contribution). Then, substituting φ and f' in the following form in Eq. (2.3):

$$\varphi = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} \exp[i(\mathbf{kr} - \omega_{\mathbf{k}} t)], \quad f' = \sum_{\mathbf{k}} f_{\mathbf{k}} \exp[i(\mathbf{kr} - \omega_{\mathbf{k}} t)] \qquad (2.5)$$

and neglecting the weak dependence of f on z and t in Eq. (2.4), we find

$$f_{\mathbf{k}} = -\frac{Z_i e \varphi_{\mathbf{k}}}{m_i(\omega - \mathbf{kv})}\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right), \qquad (2.6)$$

where $\omega \equiv \omega_k + i\gamma$ and $\gamma = \partial \ln\varphi_k/\partial t$.

Substituting (2.6) in (2.3) and making use of the inequality $\beta \ll 1$ so that we can neglect terms $u\partial f/\partial r \sim K v_{Ti} f$, we have

$$\frac{\partial f}{\partial t} + \frac{Z_i e}{m_i}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}_0]\right)\frac{\partial f}{\partial \mathbf{v}}$$
$$= \pi\left(\frac{Z_i e}{m_i}\right)^2 \frac{\partial}{\partial \mathbf{v}} \sum_{\mathbf{k}} \mathbf{k} I_{\mathbf{k}}\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega_{\mathbf{k}} - \mathbf{kv}), \qquad (2.7)$$

where $I_k = |\varphi_k|^2$. This equation determines the behavior

*$[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}] \equiv \mathbf{v} \times (\mathbf{H}_0 + \mathbf{H})$.

of the ion distribution function in the wave field (2.1) in the presence of electron-acoustic waves.

We now introduce the mean ion velocity $\mathbf{u}_i(z, t) = (1/n_0)\int \mathbf{v} f d\mathbf{v}$, where n_0 is the equilibrium ion density (we shall neglect possible small deviations of the density from the equilibrium value which can arise because of the small nonlinearity in the wave (2.1) and the electron-acoustic waves). Multiplying Eq. (2.7) by $\mathbf{v}$ and integrating over velocity space we obtain the equation of motion

$$\frac{\partial \mathbf{u}_i}{\partial t} = \frac{Z_i e}{m_i}\left(\mathbf{E} + \frac{1}{c}[\mathbf{u}_i \mathbf{H}_0]\right) + \frac{1}{m_i}\mathbf{F}_{\text{turb}}, \tag{2.8}$$

where F_{turb} is the force exerted on the ion gas by virtue of the electron-acoustic oscillations:

$$\mathbf{F}_{\text{turb}} = -\frac{\pi Z_i^2 e^2}{n_0 m_i}\sum_{\mathbf{k}} \mathbf{k} I_{\mathbf{k}} \int d\mathbf{v}\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega_{\mathbf{k}} - \mathbf{k}\mathbf{v}). \tag{2.9}$$

In a high density plasma we have $n_e = Z_i n_i$ for the electron-acoustic waves so that precisely the same force as in (2.9), but with opposite sign, acts on the electrons. Thus, the turbulent fluctuations do not change the mean value of the total momenta of the electrons and ions per unit volume of plasma. Although the force of turbulent friction in (2.8) is much smaller than the Lorentz force (a factor m_i/m_e) even in the case of strong turbulence, nonetheless, it can lead (for free waves) to a strong modification of the amplitude $\mathbf{u}_0$ of the ion velocity $\mathbf{u}_i \approx \mathbf{u}_0(t)\cos(Kz - \Omega t + \psi)$ in which case the amplitude change $\delta \mathbf{u}_0$ will be of order $\mathbf{u}_0(0)$ in a time interval $t \sim m_i/m_e\omega_{Hi}$.

In what follows it will be convenient to convert to a coordinate system in which the mean directed ion velocity vanishes. Making the substitutions

$$\mathbf{v} \to \mathbf{u}_i + \mathbf{v}', \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{\partial \mathbf{u}_i}{\partial t}\frac{\partial}{\partial \mathbf{v}'}, \quad \omega_{\mathbf{k}} - \mathbf{k}\mathbf{v} \to \Omega_\varkappa - \mathbf{k}\mathbf{u} - \mathbf{k}\mathbf{v}'$$

we now write Eq. (2.7) in the form

$$\frac{\partial f}{\partial t} + \frac{1}{m_i}\left(-\mathbf{F}_{\text{turb}} + \frac{Z_i e}{c}[\mathbf{v}'\mathbf{H}_0]\right)\frac{\partial f}{\partial \mathbf{v}'} = \frac{\partial}{\partial v_\alpha'} D_{\alpha\beta}\frac{\partial f}{\partial v_\beta'}, \tag{2.10}$$

where

$$D_{\alpha\beta} = \pi\left(\frac{Z_i e}{m_i}\right)^2 \sum_{\mathbf{k}} k_\alpha k_\beta I_{\mathbf{k}}\delta(\Omega_\varkappa - \mathbf{k}\mathbf{u} - \mathbf{k}\mathbf{v}'). \tag{2.11}$$

If we neglect the nonlinear wave interactions, the equation for the intensity of the electron-acoustic waves becomes

$$\partial I_{\mathbf{k}} / \partial t = 2\gamma I_{\mathbf{k}} + \eta, \tag{2.12}$$

where γ is the linear growth rate:[1]

$$\gamma = \frac{\pi Z_i \Omega_\varkappa^2}{2k_\parallel^2(m_i/m_e)}\int d\mathbf{v}'\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}'}\right)\delta(\Omega_\varkappa - \mathbf{k}\mathbf{u} - \mathbf{k}\mathbf{v}'), \tag{2.13}$$

η is the radiance, which is given by[5]

$$\eta = \frac{Z_i^2 e^2 n_0 \Omega_\varkappa^6}{\sqrt{2\pi}\,\omega_{pe}^4 k_\parallel^4 k v_{Ti}},$$

and ω_{pe} is the electron plasma frequency.

The motion of the electrons, which are assumed to be cold, is described by the equations

$$\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e\nabla)\mathbf{v}_e = -\frac{e}{m_e}\left(\mathbf{E} - \nabla\varphi + \frac{1}{c}[\mathbf{v}_e\mathbf{H}_0]\right), \quad \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_e = 0, \tag{2.14}$$

Now, writing $\mathbf{v}_e = \mathbf{u}_e + \mathbf{v}_e'$ and $n = n_e + n_e'$ where $\mathbf{u}_e = \langle \mathbf{v}_e \rangle$ and $n_e = \langle n \rangle$ we have

$$\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e\nabla)\mathbf{u}_e + \langle(\mathbf{v}_e'\nabla)\mathbf{v}_e'\rangle = -\frac{e}{m_e}\left(\mathbf{E} + \frac{1}{c}[\mathbf{u}_e\mathbf{H}_0]\right), \tag{2.15}$$

$$\frac{\partial \mathbf{v}_e'}{\partial t} + (\mathbf{u}_e\nabla)\mathbf{v}_e' + (\mathbf{v}_e'\nabla)\mathbf{v}_e' - \langle(\mathbf{v}_e'\nabla)\mathbf{v}_e'\rangle = -\frac{e}{m_e}\left(-\nabla\varphi + \frac{1}{c}[\mathbf{v}_e'\mathbf{H}_0]\right), \tag{2.16}$$

$$\frac{\partial n_e'}{\partial t} + \operatorname{div}(n_e\mathbf{v}_e' + n_e'\mathbf{u}_e + n_e'\mathbf{v}_e' - \langle n_e'\mathbf{v}_e'\rangle) = 0, \tag{2.17}$$

$$\frac{\partial n_e}{\partial t} + \operatorname{div}(n_e\mathbf{u}_e + \langle n_e'\mathbf{v}_e'\rangle) = 0. \tag{2.18}$$

In Eq. (2.15) the term $\langle(\mathbf{v}_e'\nabla)\mathbf{v}_e'\rangle$ takes account of the modification of the mean velocity caused by the electron-acoustic waves. In Eq. (2.15) we can neglect the small terms $|\partial \mathbf{u}_e/\partial t| \sim \Omega \mathbf{u}_e$ and $|(\mathbf{u}_e\nabla)\mathbf{u}_e| \sim K u_e^2$ as compared with the right side of (2.15) since $\Omega \ll \omega_{He}$. We then find that $\mathbf{u}_e = \mathbf{u}_{e0} + \delta\mathbf{u}_e$ where $\mathbf{u}_{e0} = c(\mathbf{E} \times \mathbf{H}_0)/H_0^2$ is the electron drift velocity in the crossed fields $\mathbf{E}$, $\mathbf{H}_0$ and

$$\delta\mathbf{u}_e = (cm_e / eH_0^2)\,[\langle(\mathbf{v}_e'\nabla)\mathbf{v}_e'\rangle, \mathbf{H}_0]$$

is the change in the electron drift velocity caused by the electron-acoustic waves. It will be shown below, that $|\mathbf{v}_e'| \sim \omega_k/k \sim v_{Ti}$ so that $\delta u_e/u_{e0} \sim \sqrt{m_e/m_i}$ and in practice it can be assumed that the electron-acoustic waves have essentially no effect on the mean electron velocity.

Equations (2.17) and (2.17) describe the nonlinear interaction between electron-acoustic waves (cf. Sec. 5).

The system of equations (2.7) and (2.13) together with the Maxwell equations have the energy integral

$$n_0 T_i + \frac{1}{2}n_0 m_i u_i^2 + \frac{1}{8\pi}\langle E^2 + H^2\rangle + W = \text{const}, \tag{2.19}$$

where $n_0 T_i$ is the ion thermal energy and W is the energy associated with the electron-acoustic waves:

$$n_0 T_i = \frac{1}{2}m_i\int v'^2 f\,d\mathbf{v}', \quad W = \frac{Z_i n_0 e^2}{T_i}\sum_{\mathbf{k}} I_{\mathbf{k}}(Z_i + k^2\rho^2). \tag{2.20}$$

3. INTENSITY OF THE ELECTRON-ACOUSTIC WAVES

We shall first find the spectral intensity of the electron-acoustic wave without taking account of nonlinear wave interactions. We shall show that in a time t, of order several times $1/\omega_{Hi}$, in the reference system that rotates with the vector $\mathbf{u}$ there is established a stationary distribution I_k which depends on $k_\perp$, φ and $k_\parallel$, where φ is the azimuthal angle in wave-vector space computed from the vector $\mathbf{u} = \mathbf{u}_i - \mathbf{u}_e$. In Eq. (2.12) the time dependence appears in the form $\varphi + \Omega t$, that is to say $\gamma_k = \gamma(k_\perp, \varphi + \Omega t, k_\parallel)$. In this case the mean value of γ_k is negative:

$$\bar{\gamma} = \frac{1}{2\pi}\int_0^{2\pi}\gamma(\varphi + \Omega t)\,d\varphi < 0. \tag{3.1}$$

Integrating Eq. (2.12) we find

$$I(\varphi, t) = I(\varphi, 0)\exp\left[\frac{2}{\Omega}\int_0^{\Omega t}\gamma(\varphi + \varphi')\,d\varphi'\right] + \frac{\eta}{\Omega}\int_0^\infty d\varphi' \exp\left[\frac{2}{\Omega}\int_0^{\varphi'}\gamma(\varphi + \Omega t - \varphi'')\,d\varphi''\right] - \frac{\eta}{\Omega}\int_{\Omega t}^\infty d\varphi' \exp\left[\frac{2}{\Omega}\int_0^{\varphi'}\gamma(\varphi + \Omega t - \varphi'')\,d\varphi''\right]. \tag{3.2}$$

Since $|\bar{\gamma}/\Omega| \gg 1$ then even when $\Omega t \sim 1$ the first term in Eq. (3.2), which depends only on the initial perturbation $I(\varphi, 0)$, and the third term become exponentially small so that $I(\varphi, t)$ assumes the form

$$I(k_{\|}, \varphi', k_{\perp}) = \frac{\eta}{\Omega} \int_0^\infty d\psi \exp\left[\frac{2}{\Omega} \int_0^\psi \gamma(\varphi' - \psi') d\psi'\right], \tag{3.3}$$

where $\varphi' = \varphi + \Omega t$. In the reference system that rotates with the vector **u**, Eq. (2.12) assumes the form

$$\frac{\partial I_{\mathbf{k}}}{\partial t} + \Omega \frac{\partial I_{\mathbf{k}}}{\partial \varphi'} = 2\gamma_{\mathbf{k}}(\varphi') I_{\mathbf{k}} + \eta, \tag{3.4}$$

where $\gamma_k(\varphi')$ is independent of time. It is evident that Eq. (3.3) is a solution of Eq. (3.4) with $\partial I_k/\partial t = 0$. The relation in (3.3) can also be regarded as the stationary solution, periodic in φ', of Eq. (3.4):

$$\frac{\partial I_{\mathbf{k}}}{\partial \varphi'} = 2\frac{\gamma(\varphi')}{\Omega} I_{\mathbf{k}} + \frac{\eta}{\Omega}, \tag{3.5}$$

which is of the form

$$I(\varphi') = \frac{\eta}{\Omega} e^{y(\varphi')} \left[(e^{-y(2\pi)} - 1)^{-1} \int_0^{2\pi} d\varphi\, e^{-y(\varphi)} + \int_0^{\varphi'} d\varphi\, e^{-y(\varphi)} \right], \tag{3.6}$$

where

$$y(\varphi) = \frac{2}{\Omega} \int_0^\varphi \gamma(\varphi') d\varphi'.$$

The linear growth rate can be written in the form

$$\gamma(\varphi') = \gamma_0(\cos\varphi' - \cos\varphi_0), \tag{3.7}$$

where

$$\cos\varphi_0 = \frac{\Omega_\varkappa}{k_\perp u_\perp}, \quad \gamma_0 = \sqrt{\frac{\pi m_i}{8 m_e}} \frac{Z_i |k_{\|}| u_\perp}{(Z_i + k^2\rho^2)^{3/2}}.$$

Since $|y| \gg 1$ the integration over φ in Eq. (3.6) can be carried out by the method of the steepest descent. The function $I(\varphi')$ has a sharp maximum near the value $\varphi = \varphi_0$:

$$I(\varphi') = \frac{\eta}{\Omega} \left[\frac{1}{2\pi} |y''(2\pi - \varphi_0)|\right]^{-1/2} \exp[y(\varphi') + y(2\pi) - y(2\pi - \varphi_0)], \tag{3.8}$$

in particular, when $\varphi \approx \varphi_0$

$$I(\varphi') = \frac{\eta}{\Omega} \left[\frac{1}{2\pi} |y''(2\pi - \varphi_0)|\right]^{-1/2}$$
$$\times \exp[y(\varphi_0) + y(2\pi) - y(2\pi - \varphi_0)] \exp\left[-\frac{1}{2} |y''(\varphi_0)| (\varphi' - \varphi_0)^2\right]. \tag{3.9}$$

Far from the angle φ_0 the intensity $I(\varphi')$ is much smaller than in (3.8), being of the order of the thermal noise: $I(\varphi') \sim \eta/\Omega$. Thus, in the problem at hand the limitation on the growth $I_k(t)$ and the establishment of a stationary spectrum are due to the fact that the problem is not stationary: the growing waves become damped waves because of the rotation of the vector **u** (the damping is more rapid than the growth); on the other hand, the damped waves become growing waves, so that the spectrum I_k follows (with some deviation with respect to the angle φ') the vector **u** (the intensity I_k reaches a peak when $\varphi' = \varphi_0$ where the growth rate vanishes, rather than $\varphi' = 0$, where the growth rate is a maximum). The stationary level (3.6) is maintained by thermal fluctuations.

In concluding this section we find the total intensity of the electron-acoustic waves W. The expression for W given by Eq. (2.20) is a summation (integral) of the form

$$U = \sum_{\mathbf{k}} \psi_{\mathbf{k}} I_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_{\|} \int_0^{2\pi} d\varphi' \psi_{\mathbf{k}} I_{\mathbf{k}}, \tag{3.10}$$

where I_k has a sharp peak at $\mathbf{k} = \mathbf{k}_0$ while ψ_k is a smoothly varying function of **k**. The integral in (3.10) can be computed by the method of steepest descent. In the integration over φ' we need only consider the region $\varphi' \approx \varphi_0(k_{\|}, k_\perp)$ in which I_k can be written in (3.10) in the form given by (3.9) while the quantity ψ_k is taken out from under the integral over φ' at the point $\varphi' = \varphi_0$. The subsequent integration with respect to $k_{\|}$ is also carried out by the method of steepest descent near the saddle point $k_{\|} = k_{\|m} = k_\perp(u/v)\sqrt{Z_i + k^2\rho^2}\cos\varphi_m$ at which $\varphi_0 = \varphi_m \approx 66°$, where $2\varphi_m = \tan\varphi_m$. The remaining integration over $k_\perp$ is also carried out by the method of steepest descent near the point $k_\perp = \sqrt{Z_i}/\rho$. As a result we have

$$U = \left(\frac{32}{\pi}\right)^{1/4} \frac{T_i \rho^2 Z_i^{-15/4} \operatorname{ctg}\varphi_m}{(2\pi)^3} \left(\frac{|\Omega|}{\omega_{Hi}}\right)^{1/2} \left(\frac{\omega_{He}}{\omega_{pe}}\right)^2$$
$$\times \left(\frac{\omega_{Hi}}{v_{Ti}}\right)^3 \left(\frac{m_i}{m_e}\right)^{3/4} \frac{\psi_{\mathbf{k}}}{\sqrt{N}} e^N, \tag{3.11}$$

where ψ_k is taken at the point $k_\perp = \sqrt{Z_i}/\rho$, $\varphi = \varphi_m$, and $k_{\|} = \sqrt{2Z_i}(u/v\rho)\cos\varphi_m$

$$N = 0{,}22 \frac{\omega_{Hi}}{|\Omega|} \left(\frac{u}{v_{Ti}}\right)^2 \left(\frac{m_i}{Z_i m_e}\right)^{1/2}.$$

Then, taking $\psi_k = Z_i n_0 e^2 (Z_i + k^2\rho^2)/T_i$ we find that W is given by

$$W / n_0 T_i = w e^N, \tag{3.12}$$

where

$$w = \left(\frac{2}{\pi}\right)^{1/4} \frac{\operatorname{ctg}\varphi_m}{8\pi^4} \left(\frac{|\Omega|}{\omega_{Hi}}\right)^{1/2} \left(\frac{\omega_{Hi}}{v_{Ti}}\right)^3 \left(\frac{m_i}{m_e}\right)^{3/4} \frac{Z_i^{-7/4}}{n_0\sqrt{N}}. \tag{3.13}$$

Since the level of thermal noise is extremely small in the region being considered, the quantity w is also very small. For example, with $Z = 1$, $|\Omega| \approx \omega_{Hi}$, $n_0 \sim 10^{13}\ \mathrm{cm}^{-3}$, $H_0 \sim 10^4$ g, $v_{Ti} \sim 10^7$ cm/sec and $m_i/m_e = 4 \times 10^3$ we have $w \sim 10^{-14}$. Hence the level of turbulent waves W will be appreciable if N is large, since $N \gg 1$ and $N \sim u^2/v_{Ti}^2$ the quantity W is very sensitive to the magnitude of u/v_{Ti} (in the present example W becomes comparable with $n_0 T_i$ when $u = 1.5\, v_{Ti}$ and even when $u = v_{Ti}$ we have $W/n_0T_i \sim 10^{-10}$). Hence, effects associated with electron-acoustic turbulence are found to be important only when $u \gtrsim v_{Ti}$ although the electron-acoustic instability can develop when $u \ll v_{Ti}$.

4. QUASILINEAR RELAXATION AND TURBULENT ION HEATING

We now consider the change in the background ion distribution function $f(v_\perp, \Phi, v_{\|}, t)$ under the effect of the waves (3.11). In this case it will be convenient to convert to a rotating coordinate system ($\Phi' = \Phi + \Omega t$) in which the spectrum is essentially stationary (in the case of the free waves the spectrum (3.12) changes slowly since the energy of the turbulent waves goes into ion heating). In this reference system Eq. (2.10) assumes the form

$$\frac{\partial f}{\partial t} + (\Omega - \omega_{Hi}) \frac{\partial f}{\partial \Phi'} = \frac{\partial}{\partial v_{\|}} D_{z\alpha} \frac{\partial f}{\partial v_\alpha'} + \frac{1}{v_\perp^2} \frac{\partial}{\partial \Phi'} \left(A \frac{\partial f}{\partial \Phi'} - B \frac{\partial f}{\partial v_\perp}\right)$$
$$+ \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \frac{1}{v_\perp} \left(D \frac{\partial f}{\partial v_\perp} - B \frac{\partial f}{\partial \Phi'}\right) + \frac{1}{m_i} \mathbf{F}_{\mathrm{turb}} \frac{\partial f}{\partial \mathbf{v}'}, \tag{4.1}$$

where

$$A = \pi\left(\frac{Z_i e}{m_i}\right)^2 \sum_{\mathbf{k}} k_\perp^2 I_{\mathbf{k}}(\varphi')\sin^2(\Phi'-\varphi')\,\delta(\Omega_\varkappa - \mathbf{ku} - \mathbf{kv}'),$$

$$B = -\pi\left(\frac{Z_i e}{m_i}\right)^2 \sum_{\mathbf{k}} k_\perp I_{\mathbf{k}}(\varphi')(\Omega_\varkappa - \mathbf{ku})\sin(\Phi'-\varphi')\,\delta(\Omega_\varkappa - \mathbf{ku} - \mathbf{kv}'),$$

$$D = \pi\left(\frac{Z_i e}{m_i}\right)^2 \sum_{\mathbf{k}} I_{\mathbf{k}}(\varphi')(\Omega_\varkappa - \mathbf{ku})^2\,\delta(\Omega_\varkappa - \mathbf{ku} - \mathbf{kv}'). \quad (4.2)$$

Since $D_{z\alpha}$ is proportional to $k_\parallel \sim k_\perp(m_e/m_i)^{1/2}$, on the right side of Eq. (4.1) we can neglect the first term, which takes account of ion diffusion (due to the waves) in velocity space in the direction of the magnetic fields; for the same reason we can neglect the term $k_\parallel v'_\parallel$ compared with $\mathbf{k}_\perp \cdot \mathbf{v}'_\perp$ under the sign of the δ-function in (4.2). Under these conditions the ion distribution with respect to $v_\parallel$ remains constant in time.

In Eq. (4.2) the quantity $I_k(\varphi')$ has a sharp peak in the region $\varphi' \cong \varphi_0$ with width $|\Delta k/k| \sim 1/\sqrt{N}$; thus, $\Omega_K \approx ku$ and for values $|v'| \approx v_{Ti}$ in Eq. (4.2), under the sign of the δ-function we can neglect $\Omega_K - \mathbf{k}\cdot\mathbf{u}$ as compared with $\mathbf{k}_\perp \mathbf{v}'_\perp$. The presence of the quantity $\delta(\mathbf{k}_\perp \mathbf{v}'_\perp)$ in Eq. (4.2) means that the ion scattering on electron-acoustic waves is essentially elastic. This leads to a rapid randomization of the distribution function over the angle[2)] Φ'. Formally this means that in Eq. (4.1) the diffusion coefficient with respect to Φ', the quantity A, is much larger than the diffusion coefficients B and D [in Eq. (4.2) the quantities B and D contain the small factors $(\Omega_K - \mathbf{k}\cdot\mathbf{u})$ and $(\Omega_K - \mathbf{k}\cdot\mathbf{u})^2$]. Calculating the coefficients in (4.2) we have

$$A = A_0 \sum_{n=1,2} \exp(N\xi_n),$$

$$B = A_0 u \sum_{n=1,2} (\cos\psi_n - \cos\alpha_n)\exp(N\xi_n),$$

$$D = A_0 u^2 \sum_{n=1,2} (\cos\psi_n - \cos\alpha_n)^2 \exp(N\xi_n), \quad (4.3)$$

where

$$A_0 \approx \frac{1}{30}\sqrt{\frac{|\Omega|}{\omega_{Hi}}}\left(\frac{m_i}{m_e}\right)^{5/4}\frac{\omega_{Hi}^4}{n_0 v_\perp}$$

$$\xi_n = \cos\alpha_n(\sin\psi_n + \sin\alpha_n)/\sin 2\varphi_m,$$

$$\sin\psi_n + \sin\alpha_n = 2(\psi_n + \alpha_n)\cos\alpha_n, \qquad \psi_{1,2} = \Phi' \pm \pi/2.$$

It is then evident that the diffusion coefficients exhibit sharp maxima in the regions $|\psi_{1,2} - \varphi_m| \equiv \Delta\Phi' \lesssim 1/\sqrt{N}$ in which case the following approximate relations hold:

$$D \sim Bu/\sqrt{N} \sim Au^2/N \sim \omega_{Hi}(W/n_0T_i)(m_i/m_e)^{1/4}(v_i^5/u),$$

Thus, in fact the diffusion with respect to Φ' occurs $\sqrt{N}$ times faster than the diffusion in $v_\perp$. If the distribution function f′ is not isotropic with respect to the angle Φ' at the initial time, then in the course of a time interval $\tau_\varphi \sim (1/\omega_{Hi})(m_e/m_i)(n_0T_i/W)$ the diffusion terms and the term $(\Omega - \omega_{Hi})\partial f/\partial\Phi'$ cause the function f to become isotropic over the entire region of variation of Φ' so that in the subsequent relaxation stages we can assume

$$f = f_0(v_\perp, t) + f_1(v_\perp, \Phi', t), \quad (4.4)$$

where $|f_1| \ll f_0$. Then, from Eq. (4.1) we have

$$\frac{\partial f_0}{\partial t} = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp}\left(\langle D\rangle_{\Phi t}\frac{\partial f_0}{\partial v_\perp} - \left\langle B\frac{\partial f_1}{\partial\Phi'}\right\rangle_{\Phi t}\right) + \frac{1}{m_i}\left\langle \mathbf{F}_{\text{turb}}\frac{\partial f_0}{\partial \mathbf{v}'}\right\rangle_{\Phi t}, \quad (4.5)$$

$$\frac{\partial f_1}{\partial t} + (\Omega - \omega_{Hi})\frac{\partial f_1}{\partial\Phi'} = \frac{1}{v_\perp^2}\frac{\partial}{\partial\Phi'}\left(A\frac{\partial f_1}{\partial\Phi'} - B\frac{\partial f_0}{\partial v_\perp}\right)$$
$$+ \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp}\left[(D - \langle D\rangle_{\Phi t})\frac{\partial f_0}{\partial v_\perp} - B\frac{\partial f_1}{\partial\Phi'} + \left\langle B\frac{\partial f_1}{\partial\Phi'}\right\rangle_{\Phi t}\right]$$
$$+ \frac{1}{m_i}\left(\mathbf{F}_{\text{turb}}\frac{\partial f_0}{\partial\mathbf{v}'} - \left\langle\mathbf{F}_{\text{turb}}\frac{\partial f_0}{\partial\mathbf{v}'}\right\rangle_{\Phi t}\right), \quad (4.6)$$

where $\langle\ldots\rangle_{\Phi t}$ denotes an average over Φ' and over the time interval $\Delta t = 2\pi/|\Omega - \omega_{Hi}|$.

If the following inequality is satisfied:

$$\left|(\Omega - \omega_{Hi})\frac{\partial f_1}{\partial\Phi'}\right| \gg \left|\frac{1}{v_\perp^2}\frac{\partial}{\partial\Phi'}A\frac{\partial f_1}{\partial\Phi'}\right|, \quad (4.7)$$

this being equivalent to the condition $\Delta t \ll 2\pi\tau_\Phi$, then the solution of Eq. (4.6) can be written in the form $f_1(\Phi', t) = f_1^0(\Phi') + f_1^1(\Phi' - [\Omega - \omega_{Hi}]t)$ where f_1^1 is an arbitrary periodic function of the variable Φ' (and consequently the variable t) while f_1^0 is determined from

$$(\Omega - \omega_{Hi})\frac{\partial f_1^0}{\partial\Phi'} = -\frac{1}{v_\perp^2}\frac{\partial f_0}{\partial v_\perp}\frac{\partial B}{\partial\Phi'}$$
$$+ \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp}\left[(D - \langle D\rangle_{\Phi t})\frac{\partial f_0}{\partial v_\perp}\right] + \frac{1}{m_i}\mathrm{F}_{\text{туpб}}\frac{\partial f_0}{\partial\mathbf{v}'}. \quad (4.8)$$

Assuming that $\langle B(\Phi')\partial/\partial\Phi' f_1^1(\Phi' - [\Omega - \omega_{Hi}]t)\rangle_{\Phi t} = 0$, from Eq. (4.8) we have

$$\left\langle B\frac{\partial f_1}{\partial\Phi'}\right\rangle_{\Phi t} = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp}\left[(\langle DB\rangle_{\Phi t} - \langle B\rangle_{\Phi t}\langle D\rangle_{\Phi t})\frac{\partial f_0}{\partial v_\perp}\frac{1}{(\Omega - \omega_{Hi})}\right]$$
$$+ \left\langle B\frac{1}{m_i}\mathbf{F}_{\text{turb}}\frac{\partial f_0}{\partial\mathbf{v}'}\frac{1}{(\Omega - \omega_{Hi})}\right\rangle_{\Phi t}. \quad (4.9)$$

Using the expression that has been obtained (4.9) we see that if (4.7) is satisfied, then in Eq. (4.5) we can neglect terms like $\langle B\partial f_1/\partial\Phi'\rangle_{\Phi t}$ compared with $\langle D\rangle_{\Phi t}\partial f_0/\partial v_\perp$. It is also evident that the last term in Eq. (4.5) vanishes, that is to say, the frictional force $\mathbf{F}_{\text{turb.}}$ does not tend to increase the temperature in the distribution function, but only retards the ions. Finally, we find that the isotropic part of the distribution function is described by a diffusion equation[2]

$$\frac{\partial f_0}{\partial t} = \frac{D_0}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp^2}\frac{\partial f_0}{\partial v_\perp}, \quad (4.10)$$

where

$$D_0 = \left(\frac{Z_i e}{m_i}\right)^2 \sum_{\mathbf{k}} \frac{1}{k} I_{\mathbf{k}}(\varphi')(\Omega_\varkappa - \mathbf{ku})^2. \quad (4.11)$$

The integration in Eq. (4.1) can be carried out assuming that $\Phi_k(\varphi')$ has a maximum at $\Omega_K = \mathbf{k}\cdot\mathbf{u}$. In this case

$$D_0 = \sqrt{\frac{8}{\pi}} Z_i \Omega \varphi_m v_{Ti}^5 \frac{W}{n_0 T_i}. \quad (4.12)$$

It follows from Eq. (4.10) that the distribution function $f_0(v_\perp)$ becomes smeared out as t increases, corresponding to ion heating. The characteristic heating time is of order

$$\tau \sim \frac{v_{Ti}^5}{D_0} \sim \frac{1}{\omega_{Hi}}\frac{n_0 T_i}{W}. \quad (4.13)$$

If the inverse inequality to (4.7) is satisfied

$$\left|(\Omega - \omega_{Hi})\frac{\partial f_1}{\partial\Phi'}\right| \ll \left|\frac{1}{v_\perp^2}\frac{\partial}{\partial\Phi'}A\frac{\partial f_1}{\partial\Phi'}\right|, \quad (4.14)$$

---

2) A similar situation occurs in turbulent heating of a plasma with hot electrons and cold ions in a strong electric field in which the scattering of ions on ion-acoustic waves is essentially elastic and leads to a randomization of the electron distribution function [6].

it is not difficult to find the steady-state solution $f_1(\Phi')$ from Eq. (4.6); this solution satisfies the equation

$$\frac{\partial f_1}{\partial \Phi'} = \frac{B}{A}\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp^2(\Omega - \omega_{Hi})}{A} f_1. \qquad (4.15)$$

The second term on the right side of Eq. (4.15) is smaller than the first by a factor $A/v_{Ti}^2\Delta\Phi'|\Omega - \omega_{Hi}|$. Substituting the solution of Eq. (4.15) in Eq. (4.5) we find that for the case in (4.14) the isotropic part of the distribution function satisfies the equation

$$\frac{\partial f_0}{\partial t} = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp}\left[\left\langle D - \frac{B^2}{A}\right\rangle_{\Phi t}\frac{\partial f_0}{\partial v_\perp} + v_\perp^2(\omega_{Hi} - \Omega)\left\langle \frac{B}{A} f_1\right\rangle_{\Phi t}\right]. \qquad (4.16)$$

Substitution of the diffusion coefficients A, B and D in the form given in (4.3) means that $\langle D - B^2/A\rangle_{\Phi t} = 0$. Consequently in the case given by (4.14) the change in the function $f_0(v_\perp, t)$ (ion heating) occurs at a rate at least $A/v_{Ti}^2\Delta\Phi'|\Omega - \omega_{Hi}|$ times slower than in the case in (4.7). Thus, in plasma heating by the ion-cyclotron wave when $\Omega \to \omega$ the heating rate is reduced for a fixed value of u.

We note that in deriving Eqs. (4.10) and (4.16) we have only made use of the fact that there is a sharp peak in the functions $I_k(\varphi')$ so that these relations hold not only for the spectrum in (3.8), but for the spectral density that is obtained taking account of nonlinear wave interactions.

We emphasize that since W is extremely small when $u < v_{Ti}$ the heating process is essentially terminated when the ion thermal velocity reaches some critical value $(v_{Ti})_{max} \sim u$.

We note that since $f(v, t) \approx f_0(v_\perp, t)$, the growth rate in (2.13) is formally of the same form as in the case of a Maxwellian distribution, Eq. (3.7), where now

$$\gamma_0 = -\frac{\pi m_e u Z_i \Omega_\varkappa^3}{m_i k_\parallel^2}\int_0 \frac{1}{v_\perp}\frac{\partial f_0}{\partial v_\perp} dv_\perp. \qquad (4.17)$$

5. NONLINEAR WAVE INTERACTIONS

Above, in determining $I_k(\varphi')$ we have not taken account of nonlinear wave interactions. This effect can be estimated in order-of-magnitude terms by making a comparison, in the electron equations of motion, between the linear term $(\mathbf{u}\nabla)\mathbf{v}'$ and the nonlinear term $(\mathbf{v}'\nabla)\mathbf{v}'$. These terms are of the same order of magnitude when $v' \sim u$. In this case the energy of the electron-acoustic waves (per unit volume) is

$$W \sim {}^1/_2 n_0 m_e u^2 \qquad (5.1)$$

and the oscillations of the electron gas become highly nonlinear (strong turbulence).

In order to obtain a more accurate estimate of the nonlinear interaction of electron-acoustic waves between themselves and of the wave intensity in the weak turbulence region ($W < n_0 m_e u^2$) we must start from Eq. (2.12) where γ is the growth rate computed with the nonlinear interaction taken into account. The nonlinear dispersion equation for I_κ, to accuracy of order I_κ^2, is of the form

$$\varepsilon_\varkappa I_\varkappa = \frac{1}{2(\varepsilon_\varkappa^+)'}\sum_{\varkappa'}|v_{\varkappa,\varkappa'}|^2 I_{\varkappa'} I_{\varkappa-\varkappa'} + \sum_{\varkappa'}\frac{v_{\varkappa,\varkappa'}v_{\varkappa-\varkappa',\varkappa}}{\varepsilon^+_{\varkappa-\varkappa'}} I_{\varkappa'} I_\varkappa + \sum_{\varkappa'} u_{\varkappa,\varkappa'} I_{\varkappa'} I_\varkappa, \qquad (5.2)$$

where $\kappa = (\mathbf{k}, \omega)$, ϵ_κ is the longitudinal dielectric constant multiplied by k^2, $\epsilon_\kappa^+ \equiv \epsilon_\kappa$ for $\omega = \omega_k + i0$. The explicit expressions for the matrix elements $v_{\kappa,\kappa'}$ and $u_{\kappa,\kappa'}$ are given in[4].

In the zeroth approximation, neglecting the right side of Eq. (5.2) we find that $I_\kappa = I_k\delta(\omega - \omega_k)$. Since the spectral density $I_k(\varphi')$ has a sharp peak at $k = k_m$ the decay terms in Eq. (5.2) can be neglected since it is impossible to satisfy simultaneously the conditions $k_\parallel = k_m$, $k_\parallel - k_\parallel' = k_{\parallel m}$ and $k_\parallel' = \pm k_m$. The matrix elements $v_{\kappa,\kappa'}$ and $v_{\kappa-\kappa',\kappa}$ that appear in the second term contain a factor of the form

$$\frac{\Delta_{\varkappa-\varkappa'}}{\Omega_\varkappa - \Omega_{\varkappa'}}\left(\frac{k_\parallel'}{\Omega_{\varkappa'}} - \frac{k_\parallel}{\Omega_\varkappa}\right)\left(\frac{k_\parallel - k_\parallel'}{\Omega_\varkappa - \Omega_{\varkappa'}} - \frac{k_\parallel'}{\Omega_{\varkappa'}}\right), \qquad (5.3)$$

where $\Delta_{\kappa-\kappa'} = 1$ when $|k_\parallel - k_\parallel'| \gg K$ and $\Delta_{\kappa-\kappa'} = 0$ when $|k_\parallel - k_\parallel'| \lesssim K$; $\Omega_\kappa = k_\parallel v/\sqrt{Z_i + k^2\rho^2} + i\gamma$. Since $|k_\parallel - k_\parallel'| \sim k_\parallel/\sqrt{N}$ in accordance with Eq. (3.8), then (5.3) is small because $\Delta_{\kappa-\kappa'}$ vanishes if $k_\parallel/\sqrt{N} \lesssim K$. In this case the contribution comes from $k_\parallel' \approx -k_\parallel$ but then (5.3) is proportional to the small parameter $(k_\parallel'/\Omega_{\kappa'} - k_\parallel/\Omega_\kappa)^2$ and the second term in Eq. (5.2) can be neglected compared with the first. The condition $k_\parallel/\sqrt{N} \lesssim K$ limits our analysis to the case in which[3)]

$$\beta \equiv \frac{8\pi n_0 T_i}{H_0^2} \gtrsim 3\sqrt{\frac{m_e}{m_i}}. \qquad (5.4)$$

Equation (5.2) now assumes the form

$$\varepsilon_{\mathbf{k}} - \sum_{k'>0} U_{\mathbf{k},\mathbf{k}'} I_{\mathbf{k}'} = 0, \qquad (5.5)$$

where

$$\varepsilon_{\mathbf{k}} = k_\perp^2\left[\frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{\omega_{pe}^2}{\Omega_\varkappa^2}\cos^2\vartheta + \frac{\omega_{pi}^2}{k^2 v_{Ti}^2}\left(1 + i\sqrt{\pi}\frac{\Omega_\varkappa - \mathbf{ku}}{\sqrt{2}kv_{Ti}}\right)\right],$$

$$U_{\mathbf{k},\mathbf{k}'} = \left(\frac{e}{m_e}\right)^2\frac{\omega_{pe}^2}{\omega_{He}^2}\frac{2k_\parallel}{\Omega_\varkappa^3}\left(\frac{k_\parallel}{\Omega_\varkappa} - \frac{k_\parallel'}{\Omega_{\varkappa'}}\right)[\mathbf{kk}']_z^2.$$

Taking the imaginary part in Eq. (5.5) we have

$$\gamma = \gamma_{\mathbf{k}}(\varphi')[1 + p^2\sin^2(\varphi' - \varphi_m)]^{-1}, \qquad (5.6)$$

where $\gamma_k(\varphi')$ is the linear growth rate and

$$p^2(k_\parallel, k_\perp) = \frac{\pi}{2}\left(\frac{e}{m_e}\right)^2\frac{W}{\omega_{pe}^2}\frac{k_\perp^2}{k_\parallel(\Omega_\varkappa + \Omega_{\varkappa m})}\left(\frac{k_\parallel + k_{\parallel m}}{\Omega_\varkappa + \Omega_{\varkappa m}} + \frac{2k_\parallel}{\Omega_\varkappa} + \frac{k_{\parallel m}}{\Omega_{\varkappa m}}\right). \qquad (5.7)$$

Here, $k_{\parallel m}$, φ_m and $k_{\perp m}$ are the values of $k_\parallel$, φ', and $k_\perp$ for which the intensity $I_k(\varphi')$ is a maximum and $\Omega_\kappa \equiv \mathrm{Re}\,\Omega_\kappa$.

As before, the stationary intensity as determined from Eq. (3.5) [in which $\gamma(\varphi')$ is determined from Eq. (5.6)] is of the form in (3.6) where now, however, $y(\varphi')$ is given by

$$y(\varphi') = \frac{2\gamma_0}{|\Omega|}\left\{\frac{\cos\varphi_m}{p}\,\mathrm{arctg}\,\frac{p[\sin(\varphi' - \varphi_m) + \sin\varphi_m]}{1 - p^2\sin\varphi_m\sin(\varphi' - \varphi_m)}\right.$$

$$+ \frac{\sin\varphi_m}{2p\sqrt{1+p^2}}\ln\frac{[1 + (p/\sqrt{1+p^2})\cos(\varphi' - \varphi_m)][1 - (p/\sqrt{1+p^2})\cos\varphi_m]}{[1 - (p/\sqrt{1+p^2})\cos(\varphi' - \varphi_m)][1 + (p/\sqrt{1+p^2})\cos\varphi_m]}$$

$$\left. - \frac{\cos\varphi_0}{\sqrt{1+p^2}}\,\mathrm{arctg}\,\frac{\sqrt{1+p^2}[\mathrm{tg}(\varphi' - \varphi_m) + \mathrm{tg}\,\varphi_m]}{1 - (1+p^2)\,\mathrm{tg}\,\varphi_m\,\mathrm{tg}(\varphi' - \varphi_m)}\right\}. \qquad (5.8)$$

---

3) If the inequality in (5.4) is not satisfied use can be made of the estimate in (5.1).

The quantity (5.8) [and consequently $I_k(\varphi')$] has a peak at $\varphi' = \varphi_0$. Taking account of this feature we can carry out the integration in integrals of the form in (3.10) as in the case in which there is no nonlinear wave interactions. Omitting the rather lengthy intermediate calculations we present the final system of equations for the determination of W(u), $k_{\perp m}$, $k_{\parallel m}$ and φ_m:

$$W = 4n_0 m_i u^2 p_m^2 \cos^2\varphi_m (m_e/m_i), \tag{5.9}$$

$$W = w_{\text{nonlin}} N_{\text{nonlin}}^{-2} \exp N_{\text{nonlin}}, \tag{5.10}$$

$$N_{\text{nonlin}} = 1{,}25 \frac{\omega_{Hi}}{|\Omega|} \sqrt{\frac{m_i}{m_e}} \left(\frac{u}{v_{Ti}}\right)^2 \cos\varphi_m Z(p_m^2) \frac{k_{\perp m}\rho}{Z_i + k_{\perp m}^2\rho^2}. \tag{5.11}$$

$$w_{\text{nonlin}} \sim 10^{-3} \cos^3\varphi_m (k_{\perp m}\rho)^4 (Z_i + k^2\rho^2)^{-1/2} \left(\frac{u}{v_{Ti}}\right)^3 \left(\frac{m_i}{m_e}\right)^{3/2} \left(\frac{\omega_{Hi}}{v_{Ti}}\right)^3 T_i, \tag{5.12}$$

$$Z(p_m^2) \equiv \int_{-\varphi_m}^{\varphi_m} \frac{(\cos\varphi - \cos\varphi_m)\, d\varphi}{1 + p_m^2 \sin^2(\varphi - \varphi_m)}, \tag{5.13}$$

$$Z(p_m^2) = \frac{2\cos\varphi_m}{\sqrt{1+p_m^2}} \operatorname{arctg}\left(\sqrt{1+p_m^2}\operatorname{tg}\varphi_m\right) + \frac{3}{2} p_m^2 \frac{\partial Z(p_m^2)}{\partial p_m^2}, \tag{5.14}$$

$$k_{\parallel m} = k_{\perp m} \cos\varphi_m (u/v) \sqrt{Z_i + k_{\perp m}^2 \rho^2}, \tag{5.15}$$

$$k_{\perp m}^2 \rho^2 = Z_i \frac{1-4x}{1-3x}, \tag{5.16}$$

where

$$x \equiv -p_m^2 \frac{\partial Z(p_m^2)}{\partial p_m^2} \left[\frac{4\cos\varphi_m}{\sqrt{1+p_m^2}} \operatorname{arctg}\left(\sqrt{1+p_m^2}\operatorname{tg}\varphi_m\right)\right]^{-1} > 0. \tag{5.17}$$

The behavior of $W(\frac{1}{2} n_0 m_e u^2)$, which is described by Eqs. (5.9)–(5.17), is shown schematically as a function of u in the figure. In the region $u < u_{cr} \sim v_{Ti}$ (in which $p_m^2 \ll 1$) we can neglect the nonlinear wave interactions and Eq. (5.10) coincides with Eq. (3.12) ($N_{\text{nonlin}} \to N$, $w_{\text{nonlin}}/N^2_{\text{nonlin}} \to n_0 T_i w$). In this region W falls off exponentially with u. In the region $u > u_{cr}$, as u increases the quantity $k_{\perp m}\rho$, as is evident from Eq. (5.16), approaches zero ($x \to 1/4$). The reduction in $k_{\perp m}\rho$ leads to a slow growth of N_{nonlin} (and consequently W) so that when $u > u_{cr}$ (formally when $u \gg v_{Ti}$) we have[4)]

$$W = {}^1/_2 \varkappa m_e u^2, \tag{5.18}$$

where

$$\varkappa \equiv \lim_{u\to\infty} (8p_m^2 \cos^2\varphi_m) \sim 1 \div 5,$$

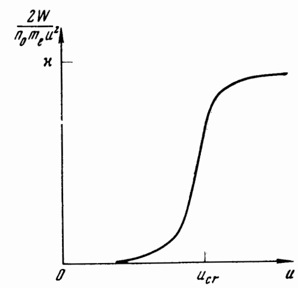
$$k_{\perp m}\rho \approx \frac{|\Omega|}{\omega_{Hi}} \sqrt{\frac{m_e}{m_i}} \left(\frac{v_{Ti}}{u}\right)^2 \cdot \frac{Z_i}{Z(p_m^2)\cos\varphi_m} \ln\frac{W}{w_{\text{nonlin}}}, \tag{5.19}$$

while the quantities p_m^2 and φ_m are essentially independent of u and are determined from the equations

$$p_m^2 \frac{\partial Z(p_m^2)}{\partial p_m^2} = \frac{\cos\varphi_m}{\sqrt{1+p_m^2}} \operatorname{arctg}\left(\sqrt{1+p_m^2}\operatorname{tg}\varphi_m\right), \tag{5.20}$$

$$Z(p_m^2) - \frac{3}{2} \frac{\partial Z(p_m^2)}{\partial p_m^2} p_m^2 = \frac{2\cos\varphi_m}{\sqrt{1+p_m^2}} \operatorname{arctg}\left(\sqrt{1+p_m^2}\operatorname{tg}\varphi_m\right). \tag{5.21}$$

We note that at very large values of u/v_{Ti} it is possible for a hydrodynamic instability to appear; for this instability $\gamma_k \sim \omega_k \sim \sqrt{\omega_{He}\omega_{Hi}}$.[7] The development of



this instability is bounded by the nonlinear wave interaction and as a result these waves exhibit a broad spectrum. The interaction of these waves with the electron-acoustic waves, which are responsible for ion heating, can lead to a strong expansion of the spectrum of electron-acoustic waves so that at large values of u/v_{Ti} Eqs. (5.9)–(5.21) no longer apply. However, as the plasma heating continues v_{Ti} approaches u, in which case the hydrodynamic instability is no longer important. In this stage the formulas in the present section apply. With further heating the quantity u/v_{Ti} is diminished so much that we are considering a region in which the level of the noise is exponentially small compared with (5.18) although it is still significantly larger than the thermal noise level. In this stage the heating rate falls off sharply.

Using Eq. (5.18) and Eqs. (4.10)–(4.12) we can obtain the estimate

$$\frac{\partial T_i}{\partial t} \sim \frac{1}{\tau_i} T_i \sim \frac{1}{2} \varkappa \omega_{Hi} m_e u^2. \tag{5.22}$$

From this equation we find that

$$T_i \sim \frac{1}{2} m_i u^2 \varkappa \frac{m_e}{m_i} \omega_{Hi} t, \tag{5.23}$$

and the thermal velocity v_{Ti} reaches a value of order u in a time

$$t \sim \frac{1}{\omega_{Hi}} \frac{m_i}{\varkappa m_e} \tag{5.24}$$

starting from the initiation of heating.

In concluding this section we present an estimate of the applicability of Eq. (4.10) for the description of ion heating in the nonlinear regime (5.18). Using the characteristic plasma parameters $n_0 \sim 10^{13}$ cm^{-3}, $H_0 \sim 10^4$ g, $m_i/m_e = 4 \times 10^3$ and $v_{Ti} \sim 10^7$ cm/sec we make use of Eqs. (5.9)–(5.21); the quantity $A/v_{Ti}^2 \Delta\Phi' |\Omega - \omega_{Hi}|$, which determines the applicability of (4.7) or (4.14), is found to be of order

$$\left|\frac{A}{v_{Ti}^2 \Delta\Phi' (\Omega - \omega_{Hi})}\right| \sim \mu \frac{\omega_{Hi}}{|\Omega - \omega_{Hi}|}, \tag{5.25}$$

where $u \sim 0.1$–0.5.

It follows from Eq. (5.25) that turbulent heating of the ions in the plasma is weak for propagation, in the plasma, of an ion-cyclotron wave of low amplitude (so that $u \sim v_{Ti}$) with frequency very close to ω_{Hi}, ($|\Omega - \omega_{Hi}| \ll 0.1\,\omega_{Hi}$ but $|\Omega - \omega_{Hi}| \gg K v_{Ti}$). Heating is realized effectively [cf. Eqs. (4.10) and (5.23)], on the other hand, for propagation of a fast magneto-acoustic wave with frequency $|\Omega| \sim \omega_{Hi}$ or for an ion-cyclotron wave at a frequency that is not very close to ω_{Hi}, with the same $|u| \sim v_{Ti}$.

4)Equations (5.18) – (5.21) may not apply if the matrix elements in the higher order terms of order I_κ^n ($n \geqslant 3$) in Eq. (5.2) that have been neglected do not contain small factors such as (5.3). In this case Eq. (5.18) gives an estimate of Π which is of the same order as (5.1).

6. DISCUSSION OF RESULTS

The general behavior of a plasma in the field of a low-frequency electromagnetic wave with $\Omega - \omega_{Hi}$ can be described as follows when the electron-acoustic instability develops. Assume that we switch on an electromagnetic field of large amplitude so that u is several times larger than the initial mean thermal velocity of the ions. Under these conditions the electron-acoustic wave will be excited and in the course of a time $1/\omega_{Hi}$ a stationary spectrum will be established with a noise level given by (5.18). This then leads to randomization of the ion velocity distribution function in the plane perpendicular to H_0 in the reference system in which the mean ion velocity is zero (if the initial distribution is not isotropic). There then follows a slower ion heating process given by the diffusion equation (4.10) which occurs in the time given by (4.13) and (5.24). As a result the transverse thermal velocity of the ions increases to a value of order u; subsequently, since v_{Ti} reaches a value such that $u < u_{cr} \sim v_{Ti}$, the noise level falls off sharply and the ion heating process is essentially terminated.

We note that the interpretation regarding the threshold nature of heating is related to the fact that the electron-acoustic waves start from thermal noise which, in the region of low frequencies and large (compared with the Debye radius) wavelengths, is very low in intensity. If the electron-acoustic waves are driven, for example, by a nonlinear interaction of the electron-acoustic waves with some other wave branch, being maintained at a rather high level, then the ion heating mechanism being considered will also operate when $u < u_{cr} < v_{Ti}$ so that the ion heating occurs even when $u < v_{Ti}$. This question will require special investigation.

In the case at hand, the electron-acoustic instability has essentially no effect on the ion velocity distribution along the magnetic field. The anisotropy in the ion distribution that arises in heating ($T_\perp > T_\parallel$) can lead to the excitation of various instabilities associated with the anisotropy.[8] The feedback effect of these oscillations on the ions can lead to a conversion of transverse ion energy into longitudinal energy.

In a number of experiments[9-14] on plasma heating by the ion-cyclotron waves or magnetoacoustic waves of large amplitude there have been observed strong wave absorption and rapid plasma heating. These results cannot be explained on the basis of the linear theory of cyclotron and Cerenkov absorption nor by collisions. It is our opinion that these effects are due to the development of a two-stream instability in the plasma in the electric field associated with the electromagnetic wave.[1]

In experiments[9] on plasma heating at a density $n_0 \sim 10^{13}$ cm^{-3} by the ion-cyclotron wave the ion temperature increased in a time of order $\tau \sim 5-10$ μsec to values $T_i = 1-2$ keV, after which the temperature remained unchanged although the rf power applied to the plasma was absorbed in magnetic beaches (the applied energy was lost from the plasma by charge exchange). The electrons were found to remain cold ($T_e \sim 20$ eV). The heating process can be explained by the electron-acoustic instability. Supporting evidence for this assumption is the linear dependence of T_i on the square of the voltage applied to the excitation coil, that is to say, on u^2. The final value of the thermal velocity for the hydrogen ions $v_{Ti} \sim 3 \times 10^7$ cm/sec was close to the directed velocity of the ions u in the ion-cyclotron wave $u \approx eE/m_i(\omega_{Hi} - \Omega) \sim 2 \times 10^7$ cm/sec ($E \approx 150$ V/cm, $\Omega \approx 6 \times 10^7$ sec^{-1}, $|\omega_{Hi} - \Omega|/\Omega \approx 0.15$ and the heating time [as estimated from (4.13) and (5.24)] $\tau \sim (30/\kappa)$ μsec ≈ 6 μsec coincides with the experimentally observed values for $\kappa \sim 5$.

In experiments[11,12] on ion-cyclotron resonance carried out in a cold plasma ($T_i \sim T_e \sim 0.2$ eV) with a density $n_0 \sim 5 \times 10^{12}$ cm^{-3} using an electromagnetic wave of low amplitude $E \sim 1$ V/cm for some critical value of the current in the excitation coil $j = j_{cr}$ there is observed a sharp break in the effective width of the resonance absorption curve γ_{eff}; when $j > j_{cr}$ there is essentially no change in γ_{eff}. The value of the current $j = j_{cr}$ corresponds to the value u being exactly equal to the thermal velocity of the ions. Under these conditions a two-stream instability should arise.[1]

In experiments[13] on plasma heating with hot electrons and cold ions ($T_e \sim 1$ keV, initial ion temperature $T_i \sim 10^2$ eV) using the ion-cyclotron wave, the curve showing the dependence of the absorption for the ion-cyclotron wave on the amplitude of the magnetic field of the ion-cyclotron wave exhibits a break when $H = H_{cr} \approx 300$ g. In this case $H_0 \approx 2 \times 10^3$ g, $\Omega = 1.4 \times 10^7$ sec^{-1}, $(\omega_{Hi} - \Omega)/\Omega \approx 0.2$, $E \approx 70$ V/cm and the directed velocity $u \sim 2 \times 10^7$ cm/sec while the ion-acoustic velocity $v_s = \sqrt{T_e/m_i} \approx 2 \times 10^7$ cm/sec. Under these conditions the ion-acoustic instability should develop.[1]

Thus, the experiments[9,11-13] show explicitly the features of wave absorption under conditions for which a two-stream instability should arise, this instability being due to the low-frequency wave. However, direct verification of an instability (measurements of the noise spectra, frequency and growth rate etc.) are not available at the present time. Such experiments would be extremely important in connection with the question of plasma heating to high temperatures.

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