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POISSON DISTRIBUTION OF BREMSSTRAHLUNG AT HIGH ENERGIES

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It is shown that the cross section for the emission of n photons with small perpendicular momenta is described in the doubly-logarithmic approximation by the Poisson formula.

1. Gribov[1] has shown that at high energies the regions of applicability of the known formulas for the accompanying radiation become much simpler. Namely, the amplitude of the accompanying radiation, given by the formula

$$F = \sqrt{\frac{\alpha_0}{2k_0}}\, e_\mu \left(\frac{p_{2\mu}}{p_2 k} - \frac{p_{1\mu}}{p_1 k}\right) f(s, t), \qquad \alpha_0 = \frac{1}{137} \tag{1}$$

(k_0—fourth component of the photon momentum) is valid under the conditions

$$\frac{2p_1k}{s} \ll 1, \quad \frac{2p_2k}{s} \ll 1, \quad -k_\perp{}^2 = \frac{4(p_1k)(p_2k)}{s} \ll \mu^2,$$
$$s = (p_1 + p_2)^2 \gg \mu^2, \tag{2}$$

where f(s, t) is the amplitude of the main process without radiation, e_μ and k are the polarization and momentum of the photon, $k_\perp$ is the photon momentum component perpendicular to the momenta of the charged particles p_1 and p_2, and μ is the characteristic mass (the pion mass in the case of hadrons).

Formula (1) is determined graphically by the photons emitted from the free charged ends, the internal amplitude being taken on the mass shell. The cross section corresponding to formula (1), integrated over the photon momenta k subject to the limitations (2), contains a term proportional to $\alpha_0 \ln^2 s$. The approximation under which the remaining terms are assumed to be small and discarded is usually called the doubly-logarithmic (d.l.) approximation. This approximation is valid if

$$\ln\frac{s}{\mu^2} \gg 1, \qquad \alpha_0 \ln\frac{s}{\mu^2} \ll 1. \tag{3}$$

It is shown in this paper that in the d. l. approximation the cross section for the emission of n photons under the condition (2) is described by the Poisson formula

$$d\sigma_n = d\sigma_0 \frac{a_1{}^n}{n!} e^{-a}, \tag{4}$$

$$a = a_1 = \frac{-ie^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \left(\frac{p_2}{p_2k} - \frac{p_1}{p_1k}\right)^2 \Big| \mathscr{R} =$$
$$= \frac{ie^2}{(2\pi)^4} 4 \int \frac{d^4k}{k^2 k_\perp{}^2} = \frac{\alpha_0}{2\pi}\left\{\ln^2\frac{s}{\mu^2} + 4\ln\frac{s}{\mu^2}\ln\frac{\mu}{\varepsilon}\right\}, \tag{5}$$

where the symbol $\mathscr{R}$ denotes that the integral is taken at $p_1k > \mu\epsilon$ and $p_2k > \mu\epsilon$, and $-k_\perp^2 \ll \mu^2$; a_1^n is the contribution of the real bremsstrahlung photons, and e^{-a} is the contribution of the virtual bremsstrahlung photons; ϵ is the experimental resolution[1)]; $d\sigma_0$ is the cross section of the main process without radiation, containing no d. l. contribution from the virtual bremsstrahlung photons limited to the region (2). Generally speaking, $d\sigma_0$ contains a d. l. contribution from the virtual photons limited by conditions opposite to (2):

$$k_\perp{}^2 \gg \mu^2. \tag{6}$$

In the case of certain lepton reactions, the proof of formula (4) for photons (2) and the calculation of the contribution of the photons (6) is contained in[2,3]. However, the photons (6) make no contribution to hadron reactions whose amplitude at high energies is large only at small momentum transfers $t \lesssim \mu^2$ and is small when $t \gg \mu^2$. This is connected with the fact that the appearance of large perpendicular photons (6) is equivalent to the appearance of large momentum transfers t[1]. In this case $d\sigma_0$ does not contain a dependence on the electromagnetic interactions in the d. l. approximation.

Formula (4) is valid also in the case when the cutoff used in the experiment for the real bremsstrahlung photons differs from (2). In this case a_1 is determined as before by the integral (5), but now taken between the appropriate limits. The values of a_1 in different experimental situations are given in the Appendix. We present below also a generalization of formulas (4) and (5) to include an arbitrary number of charged particles.

2. To prove formula (4), let us consider the simplest case of production of a hadron scalar charged particle in the decay of a superheavy mass (Fig. 1a). The amplitude f(s) of this process enters as a component part in the amplitude for the production of a hadron pair in high-energy electron-positron collisions (Fig. 1b).

Let us separate all diagrams with one virtual photon that satisfies (2) from the block of Fig. 1a. The corresponding amplitude (Fig. 2) can be represented in the form

$$\frac{-ie^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \delta_{\mu\nu} F_{\mu\nu}(s, p_1k, p_2k, k^2) \Big| \mathscr{R}, \tag{7}$$

where $\mathscr{R}$ denotes: $p_1k > \mu\epsilon$, $p_2k > \mu\epsilon$, $-k_\perp^2 \ll \mu^2$.

The amplitude of the five-point diagram $F_{\mu\nu}$, by virtue of the equality of the momenta of the two photons, depends only on the four variables indicated in the parenthesis in (7). It satisfies the condition for gauge invariance

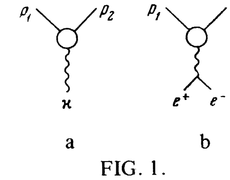

FIG. 1.

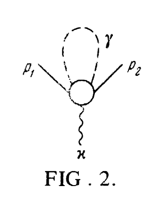

FIG. 2.

[1)]The energy resolution ϵ is the smallest admissible photon energy in the rest systems of both charged particles (see the Appendix). We shall assume that $\epsilon \sim \mu$. It is easy to see that a change of ϵ by several times does not change the d.l. contribution (5).

$$k_\mu F_{\mu\nu} = k_\nu F_{\mu\nu} = 0. \tag{8}$$

The function $F_{\mu\nu}(s, p_1k, p_2k, k^2)$ can be expanded in powers of k^2 near $k^2 = 0$, since it has singularities up to $k^2 \sim \mu^2$ only on the multiphoton sections. Such sections contain closed loops α_0 and can be discarded in the d. l. approximation. The term proportional to k^2 and the next terms in the expansion of $F_{\mu\nu}$ in k^2 cancel out k^2 in the denominator of the integral (7) and make no d. l. contribution. We can therefore put $k^2 = 0$ in the function $F_{\mu\nu}$, i.e., we can assume that all photons are real.

We shall henceforth use the method of Low[4] and Gribov[1]. The amplitude $F_{\mu\nu}$ with $k^2 = 0$ can be represented in the form of one tensor invariant

$$F_{\mu\nu} = \left(\frac{p_{1\mu}}{p_1k} - \frac{p_{2\mu}}{p_2k}\right)\left(\frac{p_{1\nu}}{p_1k} - \frac{p_{2\nu}}{p_2k}\right)\{f_0(s) + F(s, p_1k, p_2k)\}. \tag{9}$$

In formula (9), the first term contains simultaneously poles in the variables p_1k and p_2k at $p_1k = 0$ and $p_2k = 0$. This term corresponds to the diagram of Fig. 3a, in which both photons are emitted from free charged lines, and the internal amplitude is taken on the mass shell. It is easy to see that this amplitude coincides with the amplitude $f_0(s)$ of the main process with the electrodynamics disconnected, for photons satisfying the condition (2). The second term of (9) is the contribution of the singularities which do not contain simultaneously poles at $p_1k = 0$ and $p_2k = 0$, i.e.,

$$F(s, 0, 0) = 0. \tag{9$'$}$$

This term contains contributions of the diagrams 3b, 3c, and 3d, and also part of diagram 3a in which the internal amplitude is taken off the mass shell.

When the first term of (9) is substituted in (7), we obtain an integral of the form (5), containing a d. l. contribution. We shall show that the second term of (9) is of the order of $(k_\perp^2/\mu^2)f(s)$, i.e., it is small if the condition (2) is satisfied. This, generally speaking, is not obvious and does not follow from (9′). Thus, for example, the equality (9′) is satisfied by the quantity $(p_1k)(p_2k)/(p_1k - M^2)(p_2k - M^2)$, which is of the order of unity when p_1k, $p_2k \sim M^2$, and does not contain any small quantity. Similar terms making a d. l. contribution are actually present in the individual diagrams of Fig. 3 (see[2]), but no such terms are contained in the total sum of the diagrams, determining the second term of (9).

For the proof, we consider the imaginary part of $F_{\mu\nu}$ with respect to the variable p_1k at the two-particle division (Fig. 4). By virtue of the gauge invariance (8), this imaginary part is given by

$$F_{1\mu\nu} = \int d\Gamma_l \left(\frac{p_{1\mu}}{p_1k} - \frac{l_\mu}{lk}\right) f_1(p_1k, lp_1)\Phi_\nu(s, p_1k, p_2k, lp_2), \tag{10}$$

where l is the momentum of the intermediate charged particle. We resolve the momenta k and l in longitudinal and transverse components

$$k = p_1\beta + p_2\alpha + k_\perp,\ k^2 = 0;$$
$$s\alpha = 2p_1k,\ s\beta = 2p_2k,\ -k_\perp^2 = s\alpha\beta,$$
$$l = p_1b + p_2a + k_\perp. \tag{11}$$

Recognizing that the only perpendicular left after averaging over the angles is $k_\perp$, we have replaced $l_\perp$ in (11) by $k_\perp$. With the aid of (11), we can rewrite the round bracket in (10) in the form

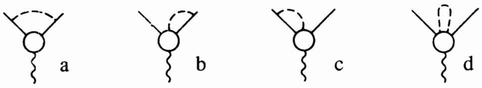

FIG. 3.

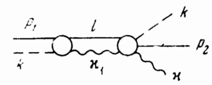

FIG. 4.

$$\frac{p_{1\mu}}{p_1k} - \frac{l_\mu}{lk} = 2\left(\frac{p_{1\mu}}{s\alpha} - \frac{p_{2\mu}}{s\beta}\right) s\alpha\beta \frac{1}{lk}\left(\frac{sa}{s\alpha} + 1\right). \tag{12}$$

It is seen from (12) that the usual vector variant arises (see (9)), multiplied by the small quantity $s\alpha\beta = -k_\perp^2$, and the first term $p_{1\mu}/p_1k$ which leads to the d. l. contribution, cancels out[1].

Since the amplitude f_1 enters in (10) in the physical region, the square of the momentum transfer does not exceed the square of the energy $2p_1l = sa \lesssim 2p_1k = s\alpha$. By virtue of this, the sum in the last brackets of (12) turns out to be of the order of unity. Substituting (12) in (10) and averaging over the angles, we obtain

$$F_{1\mu\nu} = \left(\frac{p_{1\mu}}{p_1k} - \frac{p_{2\mu}}{p_2k}\right)|k_\perp^2|\overline{\left[\frac{f_1(s\alpha; sa)}{lk}\right]\overline{\Phi}_\nu(p_1k, p_2k, s; sb)\,\overline{d\Gamma_l}}. \tag{13}$$

The amplitude $f_1 \lesssim s\alpha/\mu^2$ at small ($p_1l \sim \mu^2$, $lk \sim s\alpha$) and large ($p_1l \sim s\alpha$, $lk \sim \mu^2$) angles, when $\overline{d\Gamma_l} \sim 1/s\alpha$. At intermediate angles ($p_1l \sim s\alpha$, $lk \sim s\alpha$), where $\overline{d\Gamma_l} \sim 1$, f_1 is small. It can therefore be assumed that

$$\overline{f_1(s\alpha, sa)/lk} \lesssim 1/\mu^2.$$

The function $\overline{\Phi}_\nu(p_1k, p_2k, s; sb)$ contains a contribution from the pole at $p_2k = 0$ as well as a non-pole term. The contribution from the pole, by virtue of (8), can be written in the form

$$\left(\frac{p_{1\mu}}{p_1k} - \frac{p_{2\mu}}{p_2k}\right)\Phi_1(s\alpha, s),$$

where the amplitude Φ_1 is represented by the diagram of Fig. 5. When the mass of the compound particle $l\kappa_1 = s\alpha \sim \mu^2$, the amplitude Φ_1 will obviously be of the order of the amplitude f(s), Fig. 1. We assume that when $s\alpha$ increases the function Φ_1 does not increase2). It can therefore be assumed that $\Phi_1(s\alpha, s) \lesssim f(s)$. Thus, the pole term with respect to the variable p_2k makes the following contribution to (13)

$$F_{1\mu\nu} = \left(\frac{p_{1\mu}}{p_1k} - \frac{p_{2\mu}}{p_2k}\right)\left(\frac{p_{1\nu}}{p_1k} - \frac{p_{2\nu}}{p_2k}\right)F_1, \tag{14}$$

$$F_1 \lesssim \frac{|k_\perp^2|}{\mu^2} f(s). \tag{15}$$

To estimate the non-pole part of (13) it is necessary to consider the imaginary part of (13) with respect to

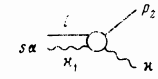

FIG. 5.

2)This may not be the case if the main process is forbidden by virtue of an approximate conservation law, and the emission of the photon lists the forbiddeness [1]. In the latter case our analysis is not valid.

the variable p_2k. We then arrive, using a similar reasoning, at the inequality (15), which is enhanced by the appearance of a second factor $|k_\perp^2|$.

In the presence of several charged particles in the intermediate states, the quantity in the round brackets in (10) is replaced by

$$\frac{p_{1\mu}}{p_1k} - \sum_i \varepsilon_i \frac{l_{i\mu}}{l_ik},$$

which, by virtue of the charge conservation $\Sigma_i \epsilon_i = 1$, leads to the same results.

We see thus that the imaginary part of F in (9) turns out to be, by virtue of (2), much smaller than f(s). We note that the integral (7) with respect to $s\alpha$ will contain only the imaginary part of the function F with respect to the variable $s\alpha$, since the integration contour piles up around the pole and the cut of the function F with respect to this variable. It is easy to verify that substitution in (5) of a factor proportional to $|k_\perp^2|$ leads to a vanishing of the d. l. contribution.

We have considered only one virtual photon. In exactly the same manner we can show in succession that for an arbitrary number of virtual phonons the d.l. terms arise only upon emission and absorption of photons on the free ends, with the internal amplitude taken on the mass shell. All the photons are emitted and absorbed independently of one another and of the charged particles. Taking into account the identity of the photons, it is necessary to divide the contribution of n photons by n!, after which the summation over n leads immediately to the result

$$f(s) = f_0(s)\, e^{-a/2}, \tag{16}$$

where $f_0(s)$ is the amplitude of the Fig. 1, which does not contain the d. l. contribution of the virtual photons with $k_\perp^2 \ll \mu^2$, and the value of a is determined by the integral (5).

The analysis of real photons does not differ in any way from that of virtual photons, since the latter contribute only when $k^2 = 0$. In the determination of the cross section it is only necessary to take into account the fact that the real photons are emitted in one amplitude and are absorbed in another amplitude which is the complex conjugate of the first (it can also be assumed that the amplitudes are identical and the photons are emitted at their two ends), making the sign of the contribution of the real photon opposite that of the virtual photon.

For the cross section of a process in which n real quanta are emitted, we obtain an expression in the form

$$d\sigma_n = |f_0(\varkappa^2)|^2 e^{-a} \prod_{i=1}^{n} \frac{1}{n!} a(\alpha_i, \beta_i, k_{\perp i})$$
$$\times \frac{s}{2} \delta\Big(b - 1 - \sum_i \beta_i\Big) \delta\Big(a - 1 - \sum_i \alpha_i\Big) \delta^2\Big(\varkappa_\perp - \sum_i k_{\perp i}\Big) d^3p_1\, d^3p_2,$$
$$\varkappa = p_1a + p_2b + \varkappa_\perp, \quad \varkappa^2 = sab + \varkappa_\perp^2. \tag{17}$$

In (17), the functions $a(\alpha_i, \beta_i, k_{\perp i})$ are the integrals (5) with respect to the photon variables (11); the δ-functions of the parallel and perpendicular components of the momenta are separated in the integration over the phase volume.

The Poisson distribution (4) corresponding to independent photon emission is obtained from (17) after discarding the photon variables in the δ-function, if account is taken of the fact that

$$|f_0(\varkappa^2)|^2 \frac{s}{2} \delta(b-1)\delta(a-1)\delta^2(\varkappa_\perp)\, d^3p_1\, d^3p_2$$
$$= |f_0(s)|^2 \delta^4(p_1 + p_2 - \varkappa)\, d^3p_1\, d^3p_2 = d\sigma_0.$$

Let us see what limitation this imposes on the photons. From (4) we find that the significant n making the main contribution to the total cross section are of the order of a:

$$n_{\text{eff}} \sim \bar{n} = a.$$

Expanding the δ-function in (17) in a series in the parallel components of the photon variables, we obtain

$$f(\varkappa^2)\delta\Big(a - 1 - \sum_i \alpha_i\Big) \cong f(\varkappa^2)\delta(a-1) + \bar{n}\bar{\alpha} f_a'(\varkappa^2)\delta(a-1). \tag{18}$$

Multiplication of the integral $a(\alpha, \beta, k_\perp)$ (5) by α leads to the loss of one logarithm, and therefore the mean value is $\overline{\alpha} \sim 1/\ln s$ (if (2) and (11) are taken into account, the resultant $\overline{\alpha}$ is even smaller). If we assume that the function f(s) varies in a power-law form with varying energy, then $f_a'(\kappa^2) \sim f(\kappa^2)$, and in order for the second term in (18) to be small compared with the first it is necessary to satisfy the condition

$$\bar{n}\frac{1}{\ln s} \ll 1 \quad \text{or} \quad \frac{\alpha_0}{2\pi} \ln s \ll 1. \tag{19}$$

this condition coincides with (3), i.e., no additional limitation arises. For the three-particle process (see Fig. 1), which does not depend on the momentum transfer, the discarding of the perpendicular components of the photon momenta likewise does not lead to additional limitations.

3. We have proved formula (4) using as an example the simplest three-particle process represented in Fig. 1. This formula, however, remains valid also for more complicated processes. Such processes contain a larger number of tensor invariants for the function $F_{\mu\nu}$ in (7), among which the invariant (9) is always present. Only this invariant contains poles $p_1k = 0$, $p_2k = 0$ and leads in (7) to an interval of the type (5). The remaining invariants do not lead to a d. l. contribution. This can be readily verified directly.

The Regge dependence of the asymptotic amplitude on the square of the momentum transfer $t = q^2$

$$f(s, t) = Bs^{A(t)\ln s} \approx Bs^{A(0)} e^{tA'(0)\ln s}, \tag{20}$$

leads to a new limitation on the perpendicular components of the photon momenta. Indeed, recognizing that at high energies and small momentum transfers the quantity q is a perpendicular vector[1,3], we replace $\kappa_\perp$ in (17) by $q_i - q_f$, where q_i and q_f are the momentum transfer at the start and at the end of the reaction.

As follows from the δ-functions in (17)

$$q_i = q_f + \sum_j k_{\perp j}.$$

At small q_i, if one of the $|k_{j\perp}^2| \gg q_i^2$, then $|q_j^2| \sim |k_{i\perp}^2| \gg q_i^2$. The behavior of the amplitude (20) will apparently be determined by the largest momentum transfer. Therefore, when $k_{j\perp}^2 \gg q_i^2$ the amplitudes (20) becomes negligibly small. Thus, if the asymptotic amplitude has the Regge behavior (20), then formula (4) is valid under the conditions

$$k_\perp^2 \ll 1 / A'(0) \ln s \qquad (21)$$

for each photon, and the contribution from larger $k_{j\perp}$ drops out[3)].

We call attention to the fact that in the case of a four-particle hadron process, the diagrams of Figs. 3a, b (3a, c) turn out to be small as the result of the decrease of the hadron amplitude $F_{\mu\nu}$ in (7) with increasing mass $s\alpha(s\beta)$, i.e., the contributions of the pole and of the cut with respect to $s\alpha(s\beta)$ cancel out. The main contribution, on the other hand, comes from the diagram of Fig. 3d, which does not contain any poles in $s\alpha(s\beta)$. The result obtained above signifies that after all the diagrams are added the contributions of the cuts cancel each other and only the contribution of the poles remains.

In the presence of several pairs of charged particles, which form large invariants $(p_i + p_j)^2 = s_{ij} \gg \mu^2$, formula (4) is valid in the d.l. approximation with respect to all these invariants, provided the photon momenta perpendicular to all the momenta of the charged particles are small:

$$-k_{ij\perp}^2 = \frac{4(p_i k)(p_j k)}{(p_i + p_j)^2} \ll \mu^2. \qquad (2')$$

It is then necessary to replace the integral a (5) in (4) by

$$A = \sum_{i>j} a_{ij} z_i z_j \theta_i \theta_j, \qquad (5')$$

where a_{ij} is defined by the same expression (5), with p_1, p_2, and s replaced by p_i, p_j, and s_{ij}; z_i and z_j are the particle charges, and θ_i equals +1 for the initial (incoming) particles and −1 for the final (outgoing) particles. The quantity A_1, corresponding to the contribution of the real photons, given by the same integral (5′), in which the limits of integration are determined by the experimental conditions (see the Appendix).

The author is grateful to A. A. Ansel'm, V. N. Gribov, and L. P. Lipatov for discussions and criticism.

APPENDIX

In (5) we introduced the experimental resolution ϵ, which cuts off the infrared-diverging integral. Such a cutoff means that an infinite number of photons is emitted, the energy of which in the rest systems of the two charge particles does not exceed ϵ. These photons are usually called infrared. The integral (5) in the region $p_1k < \mu\epsilon$, $p_2k < \mu\epsilon$ contains neither infrared divergences nor d.l. terms. The sum of the contributions from the virtual and real photons is completely canceled in this region. Figure 6 shows the d.l. contributions of different regions of the integral (5) in units of $(\alpha_0/2\pi)\ln^2 s$.

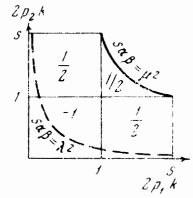

FIG. 6.

We note that in calculating the infrared-divergent part of the integral (5), using an arbitrarily small photon mass λ for regularization, there are no d.l. terms at all in the complete integral limited by the condition (2). Thus, in this case the d.l. terms from the regions p_1k, $p_2k < \epsilon\mu$ and p_1k, $p_2k > \epsilon\mu$ are equal and opposite in sign. Such a regularization signifies that an infinite number of infrared photons with $k_\perp^2 < \lambda^2$ is emitted; the integral (5) then becomes equal to

$$a = \frac{\alpha_0}{\pi} \ln\frac{s}{\mu^2} \ln\frac{\lambda^2}{\mu^2}.$$

[3)]The author is grateful to V.N. Gribov for pointing out this circumstance.

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