EXACT EXPRESSION FOR THE AMPLITUDE DESCRIBING THE SCATTERING OF A PHOTON BY A HYDROGEN ATOM AND THE LAMB SHIFT IN THE NONRELATIVISTIC APPROXIMATION

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Closed expressions are obtained for the amplitude describing the scattering (elastic and Raman) of a photon by a hydrogen atom; these formulae are exact in the sense of taking the external field into account. With their aid the Lamb shift of the energy levels of a hydrogen-like atom is calculated in the nonrelativistic approximation.

INTRODUCTION

As is well known, an expression for the amplitude describing the scattering of a photon by an arbitrary system was obtained by Kramers and Heisenberg in 1925, just before the development of quantum mechanics.[1] It has the form

$$A_{if} = \mathbf{e}_1\mathbf{e}_2\,\delta_{if} - \frac{1}{m}\sum_n\left[\frac{(\mathbf{pe}_1)_{fn}(\mathbf{pe}_2)_{ni}}{E_n - E_i + \omega_2} + \frac{(\mathbf{pe}_2)_{fn}(\mathbf{pe}_1)_{ni}}{E_n - E_i - \omega_1 - i\varepsilon}\right]. \quad (1)$$

Here $|n\rangle$ labels the complete set of states of the scatterer, E_n denotes its energy levels, $\mathbf{e}_{1,2}$ are the polarization vectors of the incident and scattered photons, and $\omega_{1,2}$ are their energies. The amplitudes are normalized such that $d\sigma/d\Omega = r_0^2|A|^2$, where $r_0 = e^2/mc^2$.

In general it is impossible to calculate the amplitude with the aid of a sum over states; in this connection, up to the present time the only way of using this formula has been an analysis of its individual terms. From here, in particular, originated the idea of resonance scattering, which is described by one of the terms in the second sum for $\omega_1 \approx E_n - E_i$. Here the remaining terms play the role of a nonresonant background. The derivation of the analytic properties of the scattering amplitude was another important application of formula (1).

With the aid of the Green's function for the scattering system,

$$G(\Omega) = \sum_n \frac{|n\rangle\langle n|}{E_n - \Omega} \quad (2)$$

one can represent the scattering amplitude in the following form:

$$A_{if} = e_{1a}e_{2b}\left\langle f\right|\delta_{ab} - \frac{1}{m}p_aG(E_i - \omega_2)p_b - \frac{1}{m}p_bG(E_i + \omega_1 + i\varepsilon)p_a\left|\,i\right\rangle. \quad (3)$$

The problem consists in whether it is possible to evaluate this matrix element. It turns out that if the scattering system is a hydrogen atom, then the problem is completely solvable. It is only necessary to use the coordinate representation for the wave functions and for the Coulomb Green's function[2]

$$G(\mathbf{r}_1, \mathbf{r}_2|\Omega) = \sum_{lm} Y_{lm}(\mathbf{n}_1)Y_{lm}^*(\mathbf{n}_2)G_l(r_1, r_2|\Omega), \quad (4)$$

$$G_l(r_1, r_2|\Omega) = \frac{2mi}{\sqrt{r_1r_2}}(-1)^{l+1}\int_1^\infty \frac{d\zeta}{\sqrt{\zeta^2-1}}\left(\frac{\zeta+1}{\zeta-1}\right)^{i\nu} \times e^{ik\zeta(r_1+r_2)}J_{2l+1}(2k\sqrt{r_1r_2}\sqrt{\zeta^2-1}) \quad (4a)$$

(here $k = \sqrt{2m\Omega}$, $\nu = Z\alpha m/k$, and the J_n are Bessel functions).

We shall calculate the amplitudes for scattering by a hydrogen atom in the 1s and 2s states, and also the amplitude for the 1S → 2S transition (Raman scattering) is calculated. In turn, knowing the scattering amplitude one can determine the radiative shift of a level, the so-called Lamb shift.

GROUND STATE

First we note that in the absence of external fields formula (3) may be written in the form

$$A_{if} = \mathbf{e}_1\mathbf{e}_2\left\langle f\right|1 - \frac{1}{3m}\mathbf{p}G(E_i - \omega_2)\mathbf{p} - \frac{1}{3m}\mathbf{p}G(E_i + \omega_1 + i\varepsilon)\mathbf{p}\left|\,i\right\rangle \quad (5)$$

and we shall evaluate the quantity

$$X(\Omega) = \frac{1}{3m}\langle f|\mathbf{p}G(\Omega)\mathbf{p}|i\rangle = \frac{1}{3m}\int \nabla\psi_f^*(\mathbf{r}_1)\cdot\nabla\psi_i(\mathbf{r}_2)G(\mathbf{r}_1, \mathbf{r}_2|\Omega)\,d\mathbf{r}_1\,d\mathbf{r}_2. \quad (6)$$

For the 1S state

$$\psi_i(\mathbf{r}) = \sqrt{\lambda^3/\pi}\,e^{-\lambda r}, \quad \lambda = Z\alpha m, \quad (7)$$

so that

$$\nabla\psi_f^*(\mathbf{r}_1)\cdot\nabla\psi_i(\mathbf{r}_2) = \pi^{-1}\lambda^5\cos\theta\, e^{-\lambda(r_1+r_2)}. \quad (7a)$$

Having substituted (7a) and (4) into (6), we obtain

$$X_{1S}(\Omega) = \frac{4\lambda^5}{3m}\int_0^\infty dr_1\int_0^\infty dr_2(r_1r_2)^2e^{-\lambda(r_1+r_2)}G_1(r_1, r_2|\Omega). \quad (8)$$

If we use expression (4a) for G_1, then the radial integral which is evaluated in Appendix A appears; from Appendix A we take the result

$$6k^3(\zeta^2-1)^{3/2}[(\lambda - ik\zeta)^2 + k^2(\zeta^2-1)]^{-4}. \quad (9)$$

Thus

$$X_{1S}(\Omega)=16i\lambda^5k^3\int_1^\infty d\zeta(\zeta+1)^{1+i\nu}(\zeta-1)^{1-i\nu}(\lambda^2-k^2-2ik\lambda\zeta)^{-4}$$
$$=128i\lambda^5k^3(\lambda-ik)^{-8}(2-i\nu)^{-1}F(4,2-i\nu;3-i\nu;\xi), \qquad (10)$$

where

$$\xi=\left(\frac{\lambda+ik}{\lambda-ik}\right)^2. \qquad (10a)$$

We have used the following integral representation of the hypergeometric function:

$$\int_1^\infty d\zeta(1+\zeta u)^a(\zeta+1)^b(\zeta-1)^c$$
$$=(1+u)^aF\left(-a,c+1;-a-b;\frac{1-u}{1+u}\right)$$
$$\times\frac{2^{b+c+1}\Gamma(c+1)\Gamma(-a-b-c-1)}{\Gamma(-a-b)}. \qquad (11)$$

Expression (10) was previously obtained by Gavrila, who used the momentum representation.[3] His method is overly complicated and does not enable one to calculate the scattering by excited states of hydrogen atom. In the following section we shall see that our method can be applied to any states without any complications.

With regard to formula (10), it is necessary to say that the quantity $X(\Omega)$ determined by it represents, to within a factor, the polarizability of a hydrogen atom so that formulas (5) and (10) contain the theory of dispersion for this atom. In particular, the poles of the hypergeometric function, which appear when $3-i\nu=0,-1,-2,\ldots$, describe resonances in the photon scattering.

THE 2S-STATE

In this case

$$\psi(r)=\sqrt{\mu^3/\pi}\,e^{-\mu r}(1-\mu r), \qquad \mu=Z\alpha m/2, \qquad (12)$$
$$\nabla\psi_f^*\cdot\nabla\psi_i=\pi^{-1}\mu^5\cos\theta e^{-\mu(r_1+r_2)}(2-\mu r_1)(2-\mu r_2), \qquad (13)$$
$$X_{2S}(\Omega)=\frac{4\mu^5}{3m}\int_0^\infty dr_1\int_0^\infty dr_2(r_1r_2)^2e^{-\mu(r_1+r_2)}(2-\mu r_1)(2-\mu r_2)G_1(r_1,r_2|\Omega). \qquad (14)$$

Having evaluated the radial integral (see Eq. (A.4)), we obtain

$$24k^3(\zeta^2-1)^{3/2}(\mu^2-k^2-2ik\mu\zeta)^{-6}[(\mu^2+k^2)^2-\mu^2k^2(\zeta^2-1)], \qquad (15)$$

and having used the representation (11) we find

$$X_{2S}(\Omega)=\frac{512i\mu^5k^3}{(\mu-ik)^{12}}\left\{2(\mu^2+k^2)^2\frac{\Gamma(2-i\nu)}{\Gamma(5-i\nu)}F(6,2-i\nu;5-i\nu;\xi)\right.$$
$$\left.-\frac{4\mu^2k^2}{3-i\nu}F(6,3-i\nu;4-i\nu;\xi)\right\}. \qquad (16)$$

In similar fashion one can evaluate even more complicated amplitudes, which describe elastic scattering by other excited levels of a hydrogen atom.

RAMAN SCATTERING 1S → 2S

In view of the orthogonality of the 1S and 2S states, the δ-function drops out of formula (3). In addition, the product of the gradients

$$\nabla\psi_f^*\cdot\nabla\psi_i=\pi^{-1}\sqrt{\lambda^5\mu^5}\cos\theta e^{-\lambda r_1-\mu r_2}(2-\mu r_2) \qquad (17)$$

loses its symmetry with respect to r_1 and r_2. Let us write down only the result of the radial integration:

$$-12k^3(\zeta^2-1)^{3/2}[\lambda\mu-k^2-ik\zeta(\lambda+\mu)]^{-5}[\lambda\mu+k^2+ik\zeta(\lambda-\mu)] \qquad (18)$$

and the final expression

$$\langle 2s|X|1s\rangle=-\frac{256ik^3\sqrt{\lambda^5\mu^5}}{(\lambda-ik)^5(\mu-ik)^5}\left\{\frac{(\lambda-ik)(\mu+ik)}{2-i\nu}F(5,2-i\nu;3-i\nu;\eta)\right.$$
$$\left.-\frac{(\lambda+ik)(\mu-ik)}{3-i\nu}F(5,3-i\nu;4-i\nu;\eta)\right\}, \qquad (19)$$

in which

$$\eta=\frac{\lambda+ik}{\lambda-ik}\frac{\mu+ik}{\mu-ik}. \qquad (20)$$

This amplitude may be used not only to calculate the Raman scattering, but it may also be used to calculate the probability for two-photon decay of the 2S-state. For this purpose, in formula (5) it is necessary to change the sign in front of ω_1, since both photons are emitted. In this connection, the arguments $\Omega_1=E_i-\omega_1$ and $\Omega_2=E_i-\omega_2$ both become negative, the momenta k_1 and k_2 are pure imaginary, but the amplitude is real.

RADIATIVE SHIFT OF THE 1S-STATE

The major part of the Lamb shift is made up of the well known Bethe term[4]

$$\Delta E_B=-\frac{2\alpha}{\pi m}\int_0^K\omega\,d\omega\cdot\frac{1}{3m}\sum_n\frac{|p_{0n}|^2}{E_n-E_0+\omega} \qquad (21)$$

(K is the cutoff energy), in which it is not difficult to see the already calculated above "crossing" part of the scattering amplitude $X(E_0-\omega)$. Taking into consideration the fact that $E_0-\omega<0$ and $k=\sqrt{2m(E_0-\omega)}=i\sqrt{2m(Ry+\omega)}$, it is convenient to write formula (10) in the form

$$X_{1S}(E_0-\omega)=128x^5(1+x)^{-8}(2-x)^{-1}F(4,2-x;3-x;\xi), \qquad (22)$$

where

$$x=\frac{\lambda}{|k|}=\sqrt{\frac{Ry}{Ry+\omega}}, \qquad \xi=\left(\frac{1-x}{1+x}\right)^2. \qquad (22a)$$

Changing from ω to x as the variable of integration in formula (21), we obtain the following expression for the shift of the 1s-level:

$$\Delta E_B(1S)$$
$$=\frac{4\alpha m(Z\alpha)^4}{3\pi}(-96)\int_{x_0}^1dx\frac{1-x}{(1+x)^7(2-x)}F(4,2-x;3-x;\xi), \qquad (23)$$

$$x_0=\sqrt{\frac{Ry}{Ry+K}}. \qquad (23a)$$

As $K\to\infty$ the integral diverges at the lower limit, and it is necessary to use regularization. In order to do this, we extract from the hypergeometric function the terms which are singular as $x\to 0$. In order to do this it is convenient to use the formula

$$\frac{6}{b}F(4,b;b+1;\xi)=\frac{2}{(1-\xi)^3}+\frac{3-b}{(1-\xi)^2}+\frac{(3-b)(2-b)}{1-\xi}$$
$$+(3-b)(2-b)(1-b)\sum_{n=0}^\infty\frac{\xi^n}{n+b} \qquad (24)$$

from Appendix B. The first three terms produce a contribution to $\Delta E(1S)$ which is given by

$$\frac{4\alpha m(Z\alpha)^4}{3\pi}\left\{-\frac{K}{4\,\mathrm{Ry}}+\ln\frac{K}{\mathrm{Ry}}-2\ln 2-\frac{11}{6}\right\}, \tag{25}$$

in which all the divergences are contained.

The linearly divergent term represents a correction to the mass of a free electron and, in accordance with the concept of mass renormalization, it should be discarded. The logarithmically divergent term arises as a consequence of the inapplicability of the dipole approximation in the region of high frequencies—upon matching the Bethe part of the level shift with the relativistic result, m/2 appears instead of K.

Thus, after mass renormalization the following formula is obtained for the level shift:

$$\Delta E_{\mathrm{B}}(1S)=\frac{4\alpha m(Z\alpha)^4}{3\pi}\left\{\ln(Z\alpha)^{-2}+\ln\frac{\mathrm{Ry}}{\langle E\rangle_{1S}}\right\}, \tag{26}$$

containing the Bethe logarithm

$$\ln\frac{\mathrm{Ry}}{\langle E\rangle_{1S}}=-2\ln 2-\frac{11}{6}+16\int_0^1 dx\,\frac{x(1-x)^2}{(1+x)^6(2-x)}F(1,2-x;\,3-x;\,\xi). \tag{27}$$

This expression replaces the definition given by Bethe[4]

$$\ln\frac{\mathrm{Ry}}{\langle E\rangle}=\frac{3n_0^3}{16}\sum_n f_{0n}\nu_{0n}^2\ln|\nu_{0n}|, \tag{28}$$

which contains a summation over the complete set of states of a hydrogen atom and is usually found numerically with the aid of tables of oscillator strengths. Formula (27) permits us to avoid this laborious work, and at the same time it enables us to attain a higher degree of accuracy.

We note that the terms already found, $-2\ln 2 - (^{11}/_6) = -3.2196$, constitute a good approximation to the value -2.9842 found by numerical methods.[5] It is not difficult to improve this result by separating the following series in Eq. (27):

$$16\int_0^1 dx\,\frac{x(1-x)^2}{(1+x)^6}\sum_{n=0}^{\infty}\frac{\xi^n}{n+2}=4\sum_{n=2}^{\infty}\frac{1}{n(4n^2-1)}=8(\ln 2-{}^2/_3)=0.2118. \tag{29}$$

This gives

$$\ln\frac{\mathrm{Ry}}{\langle E\rangle_{1S}}=-3.0078+16\int_0^1 dx\,\frac{x^2(1-x)^2}{(1+x)^6}\sum_{n=0}^{\infty}\frac{\xi^n}{(n+2)(n+2-x)}. \tag{30}$$

Repeating this step, we obtain the following correction:

$$2\sum_{n=2}^{\infty}\frac{1}{n^3(4n^2-1)}=2\left(8\ln 2-\frac{13}{3}-\zeta(3)\right)=0.0196. \tag{31}$$

so that

$$\ln\frac{\mathrm{Ry}}{\langle E\rangle_{1S}}=-2.9882+16\int_0 dx\,\frac{x^3(1-x)^2}{(1+x)^6}\sum_{n=0}\frac{\xi^n}{(n+2)^2(n+2-x)} \tag{32}$$

As is evident from Eqs. (29) and (31), each step gives a correction of the order of a small quantity, and the whole process converges rapidly.

SHIFT OF THE 2S-LEVEL

In order to calculate the displacement of this level it is necessary to use the scattering amplitude (16), in which it is again convenient to introduce the variable $x = \mu/|\mathbf{k}|$. Then the following expression is obtained:

$$\Delta E_{\mathrm{B}}(2S)=\frac{\alpha m(Z\alpha)^4}{6\pi}(-192)\int_{x_1}^1 dx\,\frac{(1-x)}{(1+x)^{11}}\left\{\frac{4x^2}{3-2x}F(6,3-2x;4-2x;\xi)+2\frac{\Gamma(2-2x)}{\Gamma(5-2x)}(1-x^2)^2F(6,2-2x;5-2x;\xi)\right\}, \tag{33}$$

$$x_1=\sqrt{\frac{\mathrm{Ry}}{\mathrm{Ry}+4K}}. \tag{33a}$$

Extracting the singular terms by the same method used previously, we arrive at the result (some of the details are given in Appendix B)

$$\frac{\alpha m(Z\alpha)^4}{6\pi}\left\{-\frac{K}{2\,\mathrm{Ry}}+\ln\frac{K}{\mathrm{Ry}}-\frac{22}{7}\right\}, \tag{34}$$

so that after renormalization we obtain

$$\Delta E_{\mathrm{B}}(2S)=\frac{\alpha m(Z\alpha)^4}{6\pi}\left\{\ln(Z\alpha)^{-2}+\ln\frac{\mathrm{Ry}}{\langle E\rangle_{2S}}\right\}, \tag{35}$$

$$\ln\frac{\mathrm{Ry}}{\langle E\rangle_{2S}}=-\frac{22}{7}+\frac{32}{5}\int_0^1 dx\,\frac{x(1-x)^2(1-4x^2)}{(1+x)^{10}}\sum_{n=0}^{\infty}\xi^n\left[4\frac{1-4x^2+x^4}{n+3-2x}+\frac{(3+2x)(1+x)(1-x^2)}{n+2-2x}+\frac{(3-2x)(1-x)(1-x^2)}{n+4-2x}\right]. \tag{36}$$

Term-by-term integration in this formula leads to a slowly converging series which was previously obtained in article[6] by a clever but more complicated method. The convergence can be improved if one uses the above-described method for extraction of the integrable parts. We note that since the term $-22/7 = -3.1428$ for the 1S-state is very close to -2.8177, hence the integral term only amounts to a ten percent correction.

CONCLUSION

It was shown above how it is possible to obtain a closed expression for the scattering amplitude, taking the external Coulomb field in which the electron is bound into account exactly. It is easy to see that two factors make this result possible:

a) Use of the coordinate representation instead of the momentum representation which is usually applied in scattering problems, but which is inadequate for the physical situation realized in a hydrogen atom:

b) Application of a non-closed expression as the Coulomb Green's function, and expansion in partial waves, as a consequence of which the majority of integrations are trivial to perform. It should be especially emphasized that perhaps both of these circumstances also remain in force in the relativistic theory.

On the other hand, the presence of an expression for the amplitude, which contains in it all Coulomb effects without approximations of the Born type, enables us in a new fashion to raise and solve the question of evaluation of the radiative corrections to the binding energy. As a result formulae are developed for the principal part of the Lamb shift, and these formulae give the Bethe expression.

APPENDIX A

A radial integral of the form

$$I=\int_0^{\infty}dr_1\int_0^{\infty}dr_2\,e^{-\alpha r_1-\beta r_2}r_1^p r_2^q J_n(2\gamma\sqrt{r_1r_2}) \tag{A.1}$$

is easier to evaluate by making a series expansion of the Bessel function

$$I=\int_0^\infty dr_1\int_0^\infty dr_2\, e^{-\alpha r_1-\beta r_2} r_1^p r_2^q \sum_{k=0}^\infty \frac{(-1)^k}{k!(n+k)!}(\gamma\sqrt{r_1r_2})^{2k+n}$$

$$=\sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{\gamma^{2k+n}}{(n+k)!}\frac{\Gamma(p+k+1+n/2)}{\alpha^{p+k+1+n/2}}\frac{\Gamma(q+k+1+n/2)}{\beta^{q+k+1+n/2}}$$

$$=\frac{\gamma^n\Gamma(p+1+n/2)\Gamma(q+1+n/2)}{\alpha^{p+1+n/2}\beta^{q+1+n/2}\Gamma(n+1)}$$

$$\times F(p+1+n/2,\, q+1+n/2;\, n+1;\, -\gamma^2/\alpha\beta). \qquad \text{(A.2)}$$

Actually, the resulting hypergeometric function turns out to be elementary. Thus, in the integral appearing in Eq. (8) we have

$$\alpha=\beta=\lambda-ik\zeta,\quad p=q={}^3/_2,\quad n=3,\quad \gamma=k\sqrt{\zeta^2-1},$$

$$I(8)=\frac{\gamma^3}{\alpha^8}\Gamma(4)F\left(4,4;4;-\frac{\gamma^2}{\alpha^2}\right)$$

$$=6\gamma^3(\alpha^2+\gamma^2)^{-4}=\frac{6k^3(\zeta^2-1)^{3/2}}{[(\lambda-ik\zeta)^2+k^2(\zeta^2-1)]^4}. \qquad \text{(A.3)}$$

Similarly, in formula (14) the integral becomes

$$I(14)=\frac{4\gamma^3}{\alpha^8}\Gamma(4)F\left(4,4;4;-\frac{\gamma^2}{\alpha^2}\right)-\frac{4\mu\gamma^3}{\alpha^9}\Gamma(5)F\left(4,5;4;-\frac{\gamma^2}{\alpha^2}\right)$$

$$+\frac{\mu^2\gamma^3}{\alpha^{10}}\frac{\Gamma(5)\Gamma(5)}{\Gamma(4)}F\left(5,5;4;-\frac{\gamma^2}{\alpha^2}\right)$$

$$=\frac{\gamma^3\Gamma(5)}{(\alpha^2+\gamma^2)^6}[(\alpha^2+\gamma^2-2\mu\alpha)^2-\mu^2\gamma^2]. \qquad \text{(A.4)}$$

APPENDIX B

In order to extract the terms which are singular at $\xi = 1$ from the hypergeometric function, we utilize the following series expansion:

$$F(4,b;b+1;\xi)=\frac{b}{6}\sum_{n=0}^\infty \xi^n\frac{(n+1)(n+2)(n+3)}{n+b}, \qquad \text{(B.1)}$$

in which we successively divide each factor in the numerator by the denominator:

$$\frac{6}{b}F(4,b;b+1;\xi)=\sum_{n=0}^\infty \xi^n(n+1)(n+2)+(3-b)\sum_{n=0}^\infty \xi^n\frac{(n+1)(n+2)}{n+b}$$

$$=\frac{2}{(1-\xi)^3}+(3-b)\sum_{n=0}^\infty \xi^n(n+1)+(3-b)(2-b)\sum_{n=0}^\infty \xi^n\frac{n+1}{n+b}$$

$$=\frac{2}{(1-\xi)^3}+\frac{3-b}{(1-\xi)^2}+\frac{(3-b)(2-b)}{1-\xi}$$

$$+(3-b)(2-b)(1-b)\sum_{n=0}^\infty\frac{\xi^n}{n+b} \qquad \text{(B.2)}$$

which then leads to formula (24) given in the text.

In formula (33) as a preliminary let us transform the expression appearing inside curly brackets

$$\Gamma(6)\left\{\frac{4x^2}{3-2x}F(6,3-2x;4-2x;\xi)\right.$$

$$\left.+2\frac{\Gamma(2-2x)}{\Gamma(5-2x)}(1-x^2)^2F(6,2-2x;5-2x;\xi)\right\}$$

$$=\sum_{n=0}^\infty \xi^n\frac{\Gamma(n+6)}{n!}\left\{\frac{4x^2}{n+3-2x}+2(1-x^2)^2\frac{\Gamma(n+2-2x)}{\Gamma(n+5-2x)}\right\}$$

$$=\sum_{n=0}\xi^n(n+1)(n+2)\ldots(n+5)\left(\frac{A}{n+a}+\frac{B}{n+a-1}+\frac{C}{n+a+1}\right) \qquad \text{(B.3)}$$

where

$$a=3-2x,\quad A=4x^2-2(1-x^2)^2,\quad B=(1-x^2)^2=C. \qquad \text{(B.4)}$$

Again dividing in succession by the denominators, we obtain

$$K_5\frac{4!}{(1-\xi)^5}+K_4\frac{3!}{(1-\xi)^4}+\ldots+\frac{K_1}{1-\xi}$$

$$+\sum_{n=0}^\infty \xi^n\left(\frac{\alpha}{n+a}+\frac{\beta}{n+a-1}+\frac{\gamma}{n+a+1}\right), \qquad \text{(B.5)}$$

where

$$K_5=A+B+C=4x^2,\quad K_4=A(5-a)+B(6-a)+C(4-a)$$
$$=8x^2(1+x),$$
$$K_3=A(5-a)(4-a)+B(6-a)(5-a)+C(4-a)(3-a)$$
$$=8x^2(1+x)(1+2x)+2(1-x^2)^2,$$
$$K_2=2(1+2x)[3(1-x^2)^2+8x^3(1+x)],$$
$$K_1=4x(1+2x)[4x^2(1+x)(2x-1)+6(1-x^2)^2],$$
$$\alpha=A(5-a)(4-a)\ldots(1-a),$$
$$\beta=B(6-a)(5-a)\ldots(2-a),$$
$$\gamma=C(4-a)(3-a)\ldots(-a). \qquad \text{(B.6)}$$

These same expressions may also be obtained from the integral Mellin-Barnes representations.

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