SOVIET PHYSICS JETP VOLUME 29, NUMBER 2 AUGUST, 1969

STIMULATED RAMAN EMISSION IN AN OPTICAL RESONATOR

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Submitted September 11, 1968

Zh. Eksp. Teor. Fiz. 56, 683–693 (February, 1969)

Equations are derived which describe the dynamics of stimulated emission of Raman radiation in an optical resonator excited by an external monochromatic ray. Solutions are obtained which correspond to stationary field oscillations, and their stability is investigated. It is found that in these conditions the phenomenon under consideration possesses some characteristic singularities.

INTRODUCTION

THE phenomenon of stimulated Raman emission (SRE) depends in an essential way on the conditions under which it occurs.[1] In particular, one of the types of this kind of emission is stimulated Raman scattering (SRS). This scattering arises when a sufficiently intense light ray is incident on the substance under investigation situated outside some optical resonator. The peculiarity of SRS is that it has radiative point sources (independent molecules—dipoles of the first Stokes frequency) that steadily emit non-interfering spherical waves. Stimulated effects appear in relation to the radiated spherical waves and lead, in particular, to their spatial amplification in the medium. In essence, the Stokes radiation obtained in this way has the properties of light scattered by individual molecules without interference.

Another picture of the phenomenon will be observed if the substance is placed inside an optical resonator with highly reflecting mirrors. Because of the amplifying properties of the medium (at the Stokes frequency) self-excitation of the system can take place, resulting in laser action at the Stokes frequency. The stationary regime of such generation will obviously be determined by the nonlinear interaction of the fields of the exciting light and of the generated Stokes component. Hence, it is clear to begin with that SRE in an optical resonator may depend on whether its mirrors are transparent to the incident external ray or whether they form at this frequency an optical resonator in which the external ray excites natural oscillations. A theoretical investigation of the first of these cases for plane-parallel mirrors was offered by Alekseev and Sobel'man.[2] In this paper we treat the case when plane or spherical mirrors form an open optical resonator at the frequency of the exciting ray and at the Stokes frequency simultaneously, and the exciting ray is incident along the resonator axis onto one of the slightly transparent mirrors (e.g., interference filters).

It is shown below that SRE under these conditions differs essentially from that considered in[2]. It has characteristic features which may be of practical interest. We first derive a system of equations describing the dynamics of the SRE process in the optical resonator and then analyze this system in the framework of oscillation theory.

1. DERIVATION OF EQUATIONS

Let an optical resonator containing a Raman-active substance be excited by an external monochromatic beam of a given intensity, so that this beam is incident along the axis of the resonator. We shall assume that a selection of axial modes has been established in the resonator (by a diaphragm or some other means), so that a sufficiently high Q is possessed only by those natural oscillations with the smallest transverse numbers and differing only by the values of the axial indices. Accordingly, we shall consider below only these (axial) types of oscillation. If the reflection coefficients of the mirrors R_1 and R_2 are close to unity, the interval between adjacent eigenfrequencies of the axial modes will greatly exceed their frequency width associated with damping due to incomplete reflection from the mirrors and radiative losses. In this case the longitudinal monochromatic beam with a frequency near one of these eigenfrequencies of the resonator, obviously, will excite only the axial mode corresponding to this eigenfrequency. In other words, in the expansion of the electromagnetic field in the resonator over the modes, we may keep only the one whose natural frequency is close to the frequency of the exciting beam.

The light waves at the frequency of the exciting beam may turn out to be sufficiently intense to evoke noticeable negative absorption at the first Stokes frequency. More accurately, negative absorption will arise not at a discrete frequency, but over the entire interval of frequencies determined by the width of the line of spontaneous Raman scattering. Thus, in the resonator all axial modes whose natural frequencies lie within the limits of the width of this line can arise. Depending on the specific substance and the distance between the mirrors the number of such modes can run from one to several hundred. Therefore we shall consider the most general case, i.e., when an arbitrary number n of such eigenfrequencies of the resonator lies within the line width of spontaneous Raman scattering; accordingly, we shall take into account all the corresponding modes in the field expansion.

We hasten to point out that if the oscillations at the first Stokes frequency are sufficiently intense, then obviously they can play the role of the exciting radiation with respect to the field of the second Stokes frequency, i.e., they can produce generation of the second

Stokes component. However, we shall assume that the reflection coefficient of the mirrors at this frequency is sufficiently small (easily accomplished practically, by using interference mirrors), or that an absorber at this frequency is introduced into the resonator. Then electromagnetic oscillations of the second Stokes component will not be excited.

Thus we arrive at the following expression for the electromagnetic field in the resonator:

$$\mathbf{E} = \mathscr{E}_0(t)\mathbf{E}_0(\mathbf{r}) + \sum_{s=1}^{n} \mathscr{E}_{-1s}(t)\mathbf{E}_{-1s}(\mathbf{r}), \qquad (1)$$

where $\mathbf{E}_0(\mathbf{r})$ is an axial mode whose frequency is close to that of the exciting beam, $\mathbf{E}_{-1s}(\mathbf{r})$ are axial modes with natural frequencies lying close to the first Stokes frequency $\omega_{-1} = p - \omega_r$ (p is the frequency of the external beam, ω_r is the frequency of a vibrational transition in the substance). The spectrum of the function $\mathscr{E}_0(t)$ is centered near the frequency p; the spectra of all the functions $\mathscr{E}_{-1s}(t)$ are centered in the neighborhood of frequency ω_{-1}.

The first task is to find a closed system of equations that describe the variation of the quantities $\mathscr{E}_0$ and $\mathscr{E}_{-1s}$. This system must be determined from the equations of the field in the resonator and from the material equation. The field equations may be written down in the form of the following system of ordinary differential equations for the coefficients $\mathscr{E}_\tau(t)$ from the expansion of this field over the modes $\mathbf{E}_\tau(\mathbf{r})$, $\mathbf{H}_\tau(\mathbf{r})$ (see[3,4]):

$$\ddot{\mathscr{E}}_\tau + \frac{\omega_\tau}{Q_\tau}\dot{\mathscr{E}}_\tau + \omega_\tau^2\mathscr{E}_\tau = \frac{\omega_\tau^2}{N_\tau}\int (\mathbf{P}\mathbf{E}_\tau - \mathbf{M}\mathbf{H}_\tau)\,dV, \qquad (2)$$

where

$$N_\tau = \frac{1}{4\pi}\int \varepsilon \mathbf{E}_\tau^2\,dV = -\frac{1}{4\pi}\int \mu \mathbf{H}_\tau^2\,dV;$$

$\mathbf{P} = \mathbf{P}(\mathbf{r}, t)$, $\mathbf{M} = \mathbf{M}(\mathbf{r}, t)$ are the additional parts of the polarization and magnetization of the medium with respect to those basic parts which are taken into account in the specification of the modes; ω_τ is the natural frequency belonging to the particular mode, Q_τ is its quality factor; the subscript τ in this case takes on values from 0 to −1s. In this case $\mathbf{P} = \mathbf{P}_{nl} + \mathbf{P}_{ex}$ and $\mathbf{M} = \mathbf{M}_{ex}$, where $\mathbf{P}_{nl}$ is the nonlinear part of the polarization of the medium filling the resonator, $\mathbf{P}_{ex} = \mathrm{Re}(\mathbf{P}_0 e^{-pt})$ and $\mathbf{M}_{ex} = \mathrm{Re}(\mathbf{M}_0 e^{-pt})$ are the given external polarization and magnetization, determined by the exciting beam.

The material equation for the medium, which is Raman-active, i.e., the equation for the nonlinear part of its polarization, has the form (see, for example,[1,5]):

$$\mathbf{P}_{nl} = Nx\frac{d\alpha}{dx}\mathbf{E}, \qquad \ddot{x} + 2h\dot{x} + \omega_r^2 x = \frac{1}{m}\frac{d\alpha}{dx}E^2. \qquad (3)$$

The coordinate x describes the molecular vibration, α is the polarizability of this molecule for a given (in general, nonequilibrium) arrangement of the nuclei, m is the reduced mass, N is the density of molecules of the substance, 2h is the total width of the line of spontaneous Raman scattering ($2h \ll \omega_r$); the derivative $d\alpha/dx$ is taken at the point $x = 0$ and is a constant coefficient. Below we shall assume that the medium is homogeneous, i.e., the quantity N does not depend on $\mathbf{r}$.

Substitute (1) into (3). The quantity E^2 appearing in the right hand side of second equation in (3) is replaced by

$$2\sum_{s=1}^{n}\mathscr{E}_0\mathscr{E}_{-1s}(\mathbf{E}_0\mathbf{E}_{-1s}),$$

i.e., we leave out the nonresonant (with respect to the oscillator in the left member) terms

$$\mathscr{E}_0^2\mathbf{E}_0^2, \quad \Big(\sum_{s=1}^{n}\mathscr{E}_{-1s}\mathbf{E}_{-1s}\Big)^2.$$

Then it is easily seen that the second of the equations in (3) is satisfied by

$$x = \sum_{s=1}^{n} x_s(\mathbf{E}_0\mathbf{E}_{-1s}), \qquad (4)$$

where $x_s(t)$ are functions satisfying the equations

$$\ddot{x}_s + 2h\dot{x}_s + \omega_r^2 x_s = \frac{1}{m}\frac{d\alpha}{dx}\mathscr{E}_0\mathscr{E}_{-1s}, \quad s = 1, 2, \ldots, n. \qquad (5)$$

Hence the second equation of system (3) may be replaced by (4) and (5), in which the variables x_s are functions of time only. Substituting now Eqs. (1) and (4) into the first of Eqs. (3), we find an expression for the polarization $\mathbf{P}_{nl}$:

$$\mathbf{P}_{nl} = N\frac{d\alpha}{dx}\Big\{\sum_{s=1}^{n} x_s\mathscr{E}_0(\mathbf{E}_0\mathbf{E}_{-1s})\mathbf{E}_0 + \sum_{s,s'=1}^{n} x_s\mathscr{E}_{-1s'}(\mathbf{E}_0\mathbf{E}_{-1s})\mathbf{E}_{-1s'}\Big\}, \qquad (6)$$

which together with Eqs. (2) gives the following system of relations for $\mathscr{E}_\tau$ and x_s:

$$\ddot{\mathscr{E}}_\tau + \frac{\omega_\tau}{Q_\tau}\dot{\mathscr{E}}_\tau + \omega_\tau^2\mathscr{E}_\tau$$

$$= N\frac{d\alpha}{dx}\frac{\omega_\tau^2}{N_\tau}\Big(\sum_{s=1}^{n} a_{\tau,s}x_s\mathscr{E}_0 + \sum_{s,s'=1}^{n} b_{\tau,ss'}x_s\mathscr{E}_{-1s'}\Big) + \frac{\omega_\tau^2}{N_\tau}\mathrm{Re}(f_\tau e^{-ipt}), \qquad (7)$$

where

$$a_{\tau,s} = \int (\mathbf{E}_\tau\mathbf{E}_0)(\mathbf{E}_0\mathbf{E}_{-1s})\,dV, \quad b_{\tau,ss'} = \int (\mathbf{E}_\tau\mathbf{E}_{-1s'})(\mathbf{E}_{-1s}\mathbf{E}_0)\,dV,$$

$$f_\tau = \int (\mathbf{P}_0\mathbf{E}_\tau - \mathbf{M}_0\mathbf{H}_\tau)\,dV. \qquad (8)$$

We note that the integral in (8) for f_τ will in fact be a surface integral. However, since specific values of f_τ are not necessary for what follows, we shall not write out the corresponding expressions in detail.

Equation (7) can be simplified if we take into account the spatial structure of the modes under consideration. We take a system of coordinates the origin of which lies on the surface of one of the mirrors and the z axis coincides with the resonator axis. Then for plane-parallel mirrors the following equalities hold with sufficient accuracy (see[4]):

$$\mathbf{E}_\tau = \mathbf{g}_\tau(x, y)\sin k_\tau z, \qquad k_\tau = \pi n_\tau/L, \qquad (9)$$

where L is the spacing between the mirrors, n is a large integer (in this case $n_\tau \neq n_{\tau'}$ for $\tau \neq \tau'$). Using (9) it is easy to show that as a result of integration over z in the expressions in (8) for $a_{\tau,s}$ and $b_{\tau,ss}$, part of the latter goes to zero. Only $a_{-1s,s}$ and $b_{0,ss}$ ($s = 1, 2, \ldots, n$) are different from zero, and $a_{-1s,s} = b_{0,ss}$. It is also easy to show that for closely spaced spherical mirrors (confocal type) the situation is analogous: practically the only non-zero coefficients are $a_{-1s,s} \equiv b_{0,ss}$.

We now consider the term with coefficient f_τ in the right hand side of Eq. (7). This term is harmonic with frequency p close to frequency ω_0 for any value of τ. Thereby it is nonresonant with respect to the oscillators in the left hand side of Eq. (7) that are obtained for $\tau = -1s$, and consequently, we keep such a term only for $\tau = 0$.

As a result, Eqs. (7) and (5) give the following system of equations determining the time variation of $\mathscr{E}_0$, $\mathscr{E}_{-1s}$, x_s:

$$\ddot{\mathscr{E}}_0 + \frac{\omega_0}{Q_0}\dot{\mathscr{E}}_0 + \omega_0^2\mathscr{E}_0 = N\frac{d\alpha}{dx}\frac{\omega_0^2}{N_0}\sum_{s'=1}^{n} a_{s'}x_{s'}\mathscr{E}_{-1s'} + \frac{\omega_0^2}{N_0}\operatorname{Re}(f_0e^{-ipt}),$$

$$\ddot{\mathscr{E}}_{-1s} + \frac{\omega_{-1s}}{Q_{-1s}}\dot{\mathscr{E}}_{-1s} + \omega_{-1s}^2\mathscr{E}_{-1s} = N\frac{d\alpha}{dx}\frac{\omega^2_{-1s}}{N_{-1s}}a_sx_s\mathscr{E}_0,$$

$$\ddot{x}_s + 2h\dot{x}_s + \omega_r^2x_s = \frac{1}{m}\frac{d\alpha}{dx}\mathscr{E}_0\mathscr{E}_{-1s}. \tag{10}$$

Here $a_s = a_{-1s,s} \equiv b_{0,ss}$; $s = 1, 2, \ldots, n$. This nonlinear system consists of $2n + 1$ ordinary differential equations and describes the dynamics of the SRE process of interest to us under the given conditions. Our next task is to investigate the solutions of the system (10) that pertain to stationary field oscillations and their stability.

2. STATIONARY FIELD OSCILLATIONS

In Eq. (10) we make the replacement of the independent variable $t = t_1/p$, and for simplicity we write the dimensionless time t_1 again with the symbol t. Then these equations take the form

$$\ddot{\mathscr{E}}_0 + \mathscr{E}_0 = -(\Omega_0^2 - 1)\mathscr{E}_0 - 2\mu_0\dot{\mathscr{E}}_0 + 4\Omega_0\gamma_0\left(\sum_{s'=1}^{n} a_{s'}x_{s'}\mathscr{E}_{-1s'}\right) + 2\operatorname{Re}(Fe^{-it}),$$

$$\ddot{\mathscr{E}}_{-1s} + \Omega_{-1s}^2\mathscr{E}_{-1s} = -2\mu_{-1s}\dot{\mathscr{E}}_{-1s} + 4\Omega_{-1s}\gamma_{-1s}a_sx_s\mathscr{E}_0, \tag{11}$$
$$\ddot{x}_s + 2\mu\dot{x}_s + \Omega^2x_s = 4\eta\Omega\mathscr{E}_0\mathscr{E}_{-1s},$$

where

$$\Omega_\tau = \frac{\omega_\tau}{p},\ \Omega = \frac{\omega_r}{p},\ \mu_\tau = \frac{\omega_\tau}{2pQ_\tau},\ \mu = \frac{h}{p},$$

$$\gamma_\tau = \frac{N}{4}\frac{d\alpha}{dx}\frac{\Omega_\tau}{N_\tau},\ \eta = \frac{1}{4mp^2\Omega}\frac{d\alpha}{dx},\quad F = \frac{\omega_0^2}{2p^2N_0}f_0. \tag{12}$$

In all cases of practical interest, the system (11) is close to the conservative one having the general solution

$$\mathscr{E}_0 = {}^1\!/_2 Z\exp\{-it\} + \text{c.c.}$$
$$\mathscr{E}_{-1s} = {}^1\!/_2 Y_s\exp\{-i\Omega_{-1s}t\} + \text{c.c.} \tag{13}$$
$$x_s = {}^1\!/_2 X_s\exp\{-i(1-\Omega_{-1s})t\} + \text{c.c.}$$

($1 - \Omega_{-1s} \approx \Omega$, since $|1 - \Omega_{-1s} - \Omega| \lesssim \mu \ll \Omega$). Hence for the solution of the problem we use the Van der Pol method, treating the constant quantities Z, Y_s, X_s in (13) as "slowly changing amplitudes." Substituting (13) into (11) and averaging over the fast oscillations, we find equations for Z, Y_s, X_s:

$$\dot{Z} = -(\mu_0 + i\Delta_0)Z + i\gamma_0\sum_{s'=1}^{n} a_{s'}X_{s'}Y_{s'} + iF,$$
$$\dot{Y}_s = -\mu_{-1s}Y_s + i\gamma_{-1s}a_sZX_s^*,$$
$$\dot{X}_s = -(\mu + i\Delta_{-1s})X_s + i\eta ZY_s^*, \tag{14}$$

where

$$\Delta_0 = \Omega_0 - 1,\quad \Delta_{-1s} = \Omega_{-1s} + \Omega - 1. \tag{15}$$

The principal advantage of this system of equations over the original one is that, because of (13), its equilibrium positions correspond to the stationary field oscillations we are interested in. Thus, the problem reduces to investigating the equilibrium states of (14) and their stabilities. First we transform (14) somewhat.

In the following we shall assume fulfillment of the following conditions, which are of practical interest,

$$\mu_0,\ \mu_{-1s} \ll \mu, \tag{16}$$

which say that the width of the resonance curve of each of the considered modes of the resonator is considerably less than the line width of spontaneous Raman scattering. Since (in accordance with what was said above) we also suppose that $|\Delta_0| \lesssim \mu_0$, it is easily seen that under these conditions (and not too large values of $|F|$), Eq. (14) represents a problem of the type

$$\dot{u} = G(u, v),\quad \beta\dot{v} = H(u, v), \tag{17}$$

where $u = \{Z, Y_s\}$, $v = \{X_s\}$, $G \sim H$, and β is a small parameter. It is known that the phase space in this case divides into regions of "fast" and "slow" motions.[6] The fast motions are determined by the last n equations in (14) (in this it is necessary to formally set u = const). These equations represent a linear system with characteristic indices $p_s = -\mu - i\Delta_{-1s}$, i.e., $\operatorname{Re} p_s = -\mu < 0$. Consequently, whatever the initial conditions, the considered system will by fast motion arrive in the region of slow motions and will be found there for all subsequent time. Its equations of motion in this region will be

$$\dot{u} = G(u, v),\qquad 0 = H(u, v). \tag{18}$$

Expressing v in terms of u and substituting this expression into G(u, v), we find

$$\dot{Z} = -(\mu_0 + i\Delta_0)Z - \eta\gamma_0\left(\sum_{s'=1}^{n}\frac{a_{s'}}{\mu + i\Delta_{-1s'}}|Y_{s'}|^2\right)Z + iF,$$

$$\dot{Y}_s = -\mu_{-1s}Y_s + \frac{\eta\gamma_{-1s}a_s}{\mu - i\Delta_{-1s}}|Z|^2Y_s. \tag{19}$$

From the point of view of the initial physical system, these transformations express the fact that the possible rate of change of the amplitudes of the oscillations of field in the resonator is sufficiently small that at each moment a quasistationary response of polarization (3) can be established in the medium. Correspondingly, the equations in (9) describe only the changes of these amplitudes.

Symbolizing $w_s = |Y_s|^2$, separating out the real and imaginary parts of the complex amplitude $Z = Z_1 + iZ_2$, and using (19), we arrive at the following system of equations for the quantities Z_1, Z_2, and w_s:

$$\dot{Z}_1 = A_1(Z_1, Z_2, w_{s'}),\quad \dot{Z}_2 = A_2(Z_1, Z_2, w_{s'}),$$
$$\dot{w}_s = B_s(Z_1, Z_2, w_{s'}), \tag{20}$$

where

$$A_1 = -\mu_0(Z_1 - Z_0^{(1)}) + \Delta_0(Z_2 - Z_0^{(2)}) - \sum_{s'=1}^{n}(\rho_{s'}'Z_1 - \rho_{s'}''Z_2)w_{s'},$$

$$A_2 = -\Delta_0(Z_1 - Z_0^{(1)}) - \mu_0(Z_2 - Z_0^{(2)}) - \sum_{s'=1}^{n}(\rho_{s'}''Z_1 + \rho_{s'}'Z_2)w_{s'},$$

$$B_s = -2\mu_{-1s}w_s + 2\Omega_{-1s}\rho_s'(Z_1^2 + Z_2^2)w_s. \tag{21}$$

The quantities $Z_0^{(1)}$, $Z_0^{(2)}$ and ρ_s', ρ_s'' appearing in these equations are respectively the real and imaginary parts of the numbers $Z_0 = Z_0^{(1)} + iZ_0^{(2)}$, $\rho_s = \rho_s' + \rho_s''$, and

$$Z_0 = \frac{iF}{\mu_0 + i\Delta_0},\quad \rho_s = \frac{\eta\gamma_0a_s}{\mu + i\Delta_{-1s}}\quad (\eta\gamma_0a_s > 0) \tag{22}$$

the norm N_T for all modes is taken to be the same: $N_{-1s} = N_0$, $s = 1, 2, \ldots, n$). As is easily seen, the quantity Z_0 is the complex amplitude of those field oscillations which would be excited in the resonator by the external beam if oscillation at the Stokes frequency were absent. Since $|Z_0|^2$ is proportional to the intensity of the external beam, we shall give the intensity of this beam by specifying values of $|Z_0|^2$.

We now consider the equilibrium positions of the system (20). They are determined from the conditions

$$A_1 = 0, \quad A_2 = 0, \quad B_s = 0. \tag{23}$$

All solutions of this system of algebraic equations are easily found. There are n + 1 of them. Below we shall indicate by a tilde the values of Z_1, Z_2, and w_s that correspond to them. The first of the equilibrium states is specified by the equalities

$$\tilde{Z} = Z_0, \quad \tilde{w}_s = 0 \quad (s = 1, 2, \ldots, n). \tag{24}$$

SRE is obviously absent in the corresponding oscillation regime. The remaining n equilibrium positions, distinguished by values of the index $\zeta = 1, 2, \ldots, n$ have the following form:

$$\tilde{Z}_1^2 + \tilde{Z}_2^2 = |\tilde{Z}_\zeta|^2 \equiv \frac{\mu_{-1\zeta}}{\Omega_{-1\zeta}\rho_\zeta'} \qquad \tilde{w}_s = 0 \text{ for all } s \neq \zeta,$$

$$\tilde{w}_\zeta = \frac{1}{|\rho_\zeta|^2}\left\{-\rho_\zeta'\mu_0 - \rho_\zeta''\Delta_0 \pm \left[(\mu_0^2+\Delta_0^2)\left(\frac{|Z_0|^2}{|\tilde{Z}_\zeta|^2} - 1\right)|\rho_\zeta|^2 + (\rho_\zeta'\mu_0 + \rho_\zeta''\Delta_0)^2\right]^{1/2}\right\} \tag{25}$$

($\tilde{w}_\zeta \geq 0$). These states determine the possible stationary regimes of SRE. It is interesting that in each of these states the value $|\tilde{Z}|$ of the amplitude of field oscillation in the resonator with the frequency of the exciting beam is independent of $|Z_0|^2$, i.e., on the intensity of the exciting beam. The second peculiarity of these states is that in the Stokes component of the field only one mode of the resonator is excited, and the amplitudes of the remaining modes are equal to zero.

3. STABILITY OF THE STATIONARY FIELD OSCILLATIONS

Practically realizable stationary field oscillations in the initial physical system correspond only to stable equilibrium positions of the system (20). Hence we shall investigate below the stability of all the solutions obtained.

As is well known, the stability of an equilibrium position is determined by the characteristic polynomial of the system of equations of the linear approximation for deviations from this position. The characteristic polynomial has the form

$$D(\lambda) = |A - \lambda E|, \tag{26}$$

where

$$A = \begin{pmatrix} \frac{\partial A_1}{\partial Z_1} & \frac{\partial A_1}{\partial Z_2} & \frac{\partial A_1}{\partial w_{s'}} \\ \frac{\partial A_2}{\partial Z_1} & \frac{\partial A_2}{\partial Z_2} & \frac{\partial A_2}{\partial w_{s'}} \\ \frac{\partial B_s}{\partial Z_1} & \frac{\partial B_s}{\partial Z_2} & \frac{\partial B_s}{\partial w_{s'}} \end{pmatrix} \tag{27}$$

($s, s' = 1, 2, \ldots, n$). The partial derivatives in the matrix A must be taken at a point $\tilde{Z}_1$, $\tilde{Z}_2$, $\tilde{W}_s$ corresponding to the equilibrium state being investigated. Since A is a matrix of (n + 2)-nd order, the characteristic polynomial has n + 2 roots λ_k which determine the stability.

Consider first the equilibrium position specified by Eq. (24). Substituting (24) into the matrix A and expanding the determinant in (26), we find the following expression for the polynomial $D(\lambda)$:

$$D(\lambda) = [(\mu_0 + \lambda)^2 + \Delta_0^2]\prod_{s=1}^{n}(\Lambda_s - \lambda), \tag{28}$$

where

$$\Lambda_s = 2(\Omega_{-1s}\rho_s'|Z_0|^2 - \mu_{-1s}). \tag{29}$$

Below we shall attach the index ζ_0 to that one of the modes $\mathbf{E}_{-1s}$ for which the ratio $\mu_{-1s}/\Omega_{-1s}\rho_s'$ is a minimum (the value of ρ_s' is always positive). It follows from (28) and (29) that the investigated equilibrium state is stable for $|Z_0|^2 < |Z_0|^2_{thr}$ and unstable for $|Z_0|^2 > |Z_0|^2_{thr}$, where

$$|Z_0|^2_{thr} = \mu_{-1\zeta_0}/\Omega_{-1\zeta_0}\rho_{\zeta_0}'. \tag{30}$$

Therefore, SRE should arise in the resonator when $|Z_0|^2 > |Z_0|^2_{thr}$.

We now consider the stability of each of the stationary regimes of oscillation specified by (25). It can be shown that the characteristic polynomial for the corresponding equilibrium states has the form

$$D(\lambda) = -D_\zeta(\lambda)\prod_{s=1}^{n}{}'(\Lambda_s^{(\zeta)} - \lambda), \tag{31}$$

where

$$\Lambda_s^{(\zeta)} = 2(\Omega_{-1s}\rho_s'|\tilde{Z}_\zeta|^2 - \mu_{-1s})$$

$$D_\zeta(\lambda) = \lambda^3 + 2q_1\lambda^2 + [q_1^2 + q_2^2 + 4\Omega_{-1\zeta}(\rho_\zeta')^2\tilde{w}_\zeta|\tilde{Z}_\zeta|^2]\lambda + 4\Omega_{-1\zeta}\rho_\zeta'(q_1\rho_\zeta' + q_2\rho_\zeta'')\tilde{w}_\zeta|\tilde{Z}_\zeta|^2,$$

$$q_1 = \mu_0 + \rho_\zeta'\tilde{w}_\zeta, \quad q_2 = \Delta_0 + \rho_\zeta''\tilde{w}_\zeta. \tag{32}$$

The prime in Eq. (31) means that the index s runs through all values except $s = \zeta$.

It is easy to see that all the considered equilibrium states obtained for $\zeta \neq \zeta_0$ are unstable. The stability of the equilibrium states obtained for $\zeta = \zeta_0$, on the other hand, is determined by the roots $\lambda_{1,2,3}$ of the polynomial $D_{\zeta_0}(\lambda)$. To investigate the roots of this polynomial one can use the Routh-Hurwitz criterion, which leads to the following conditions for stability:

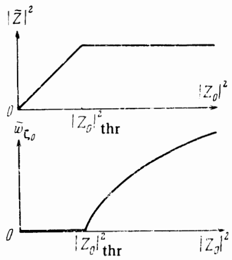
$$q_1\rho_{\zeta_0}' + q_2\rho_{\zeta_0}'' > 0,$$

$$q_1(q_1^2 + q_2^2) + 2\Omega_{-1\zeta_0}\rho_{\zeta_0}'\tilde{w}_{\zeta_0}|\tilde{Z}_{\zeta_0}|^2(q_1\rho_{\zeta_0}' - q_2\rho_{\zeta_0}'') > 0. \tag{33}$$

With the aid of (33) it is easy to show that if the inequalities

$$|\Delta_0| < \mu_0, \quad |\Delta_{-1\zeta_0}| < \mu, \tag{34}$$

are fulfilled then the given equilibrium state is always stable. Note that the inequalities in (34) mean: the natural frequency of the mode with index ζ_0 does not stand farther away from the center of the line of spontaneous Raman scattering than the halfwidth of this line; the frequency of the external exciting beam does not stand farther away from the position of the maximum of the resonance curve of the nearest axial mode than its half-width. Obviously, the conditions (34) are of the most practical interest. We also note that under these conditions the stationary regime of SRE itself (specified by (25) for $\zeta = \zeta_0$) exists only for values of $|Z_0|^2$ that exceed $|Z_0|^2_{thr}$. Then, as $|Z_0|^2$ increases, the value of $\widetilde{w}_{\zeta_0}$ increases monotonically (from zero), whereas the value of $|\widetilde{Z}|^2$ remains constant (see figure).



Thus, the analysis carried out above permits an elucidation of the following features of SRE under the conditions considered. With respect to the exciting radiation, the possibility of an ideal limitation to the intensity of its forward portion is clarified; with respect to the Stokes component—the existence of a stable generation regime in one mode of the resonator. It is also possible to show that with respect to the back-reflected portion of the exciting beam the considered system represents a "nonlinear mirror," the reflection coefficient of which increases with increasing incident intensity.

We have assumed above that the intensity of the exciting beam, as well as the phase of the complex amplitude determined by it, is constant in time. It is clear that for a slow (with respect to the characteristic times for establishment of a stationary regime) change of these quantities the solution will be quasistationary, i.e., it will be determined by Eqs. (24) and (25) (with the results of the stability investigation taken into account). The characteristic times for the establishment of a stationary regime are determined by the roots of the polynomial (31). Depending on the parameters of the resonator and the substance filling it, these times can vary over wide limits. In particular, they cannot exceed 10^{-10} sec. In this case a quasistationary regime will be observed for pulses of the exciting radiation of length 10^{-9} sec and greater. In general, we see that the establishment times determined by the roots of the polynomial $D_{\zeta_0}(\lambda)$ are of order $(\mu_0 p)^{-1}$, $(\mu_{-1\zeta_0} p)^{-1}$, and the establishment time for the function Z is determined solely by these roots. The characteristic times determined by the remaining roots $\Lambda_S^{(\zeta_0)}$ can also be of this same order. However, from (32) for $\Lambda_S^{(\zeta_0)}$ it is easily seen that this will be only in case not more than two or three eigenfrequencies of the resonator lie with the line width of spontaneous Raman scattering. For a larger number of such eigenfrequencies the characteristic times determined by the roots of $\Lambda_S^{(\zeta_0)}$ markedly increase, and then it is just these that superpose the limitation to the allowable (for quasistationary regime) pulse length of the exciting radiation. We also note that if the arrangement of eigenfrequencies of the resonator relative to the center of the line of spontaneous Raman scattering is symmetrical, the Q's of the modes corresponding to them are the same, and there is no eigenfrequency at the center of the line, then the establishment time of the stationary regime is infinite irrespective of the number of eigenfrequencies that occur within the line width of spontaneous Raman scattering.

In conclusion we remark that if the inequalities in (34) are not fulfilled, then the stationary regime specified by (25) (for $\zeta = \zeta_0$) can be unstable in certain cases. For example, under the condition

$$\rho_{\zeta_0}'\mu_0 + \rho_{\zeta_0}''\Delta_0 < 0, \tag{35}$$

when $\widetilde{w}_{\zeta_0}$ can be double-valued, the equilibrium state corresponding to the minus sign before the square root in Eq. (25) is always unstable, whereas the plus sign makes the state stable. Then with respect to the Stokes component we obtain the typical pattern of stable self-excitation.

There is also a relation connecting the parameters of this system for which Eq. (25) for $\widetilde{w}_{\zeta_0}$ is single-valued and at the same time in a certain finite interval of values of $|Z_0|^2$ belonging to the region $|Z_0|^2 > |Z_0|^2_{thr}$, the corresponding equilibrium state is unstable. In this interval not one of the stationary regimes of oscillation specified by Eqs. (24) and (25) is stable. Evidently, pulsations in the intensity of the Stokes component will arise in this case.

The author expresses his sincere thanks to A. M. Prokhorov for his interest in this work and for discussions which played an essential role in the development of the ideas set forth.

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Translated by L. M. Matarrese
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