# HEAT TRANSFER AND SECOND SOUND IN DIELECTRICS AT LARGE DRIFT VELOCITIES OF THE PHONON GAS

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Nonlinear hydrodynamic equations are derived, which describe the motion of a phonon gas in a dielectric at low temperatures. The equations are used for analyzing the thermal conductivity of a dielectric under conditions when the drift velocity of the phonon gas V is comparable with that of sound c (it is shown in the paper that such conditions can be obtained experimentally). For an arbitrarily large temperature difference between the ends of a rod, the phonon drift velocity cannot exceed a critical value smaller than c. For an isotropic phonon dispersion law, the critical velocity is equal to the velocity of second sound. Because of the existence of a maximum drift velocity, which is smaller than c, the thermal flux which can be obtained by maintaining a temperature difference between the ends of the rod also does not exceed a certain maximum value. As another application of the linear hydrodynamic equations, the propagation of second sound waves in a phonon gas in the presence of a strong stationary thermal flux is considered. It is shown that the velocity of second sound in the drifting phonon gas can be obtained from the drift velocity and the velocity of second sound in a quiescent phonon gas by adding the drift velocities in accord with a law which is similar to the relativistic law of addition of velocities.

## 1. INTRODUCTION

IN pure dielectrics at low temperatures, collisions of phonons, in which the phonon quasimomentum is conserved (N-processes), take place more frequently than collisions with nonconservation of the phonon quasimomentum (U-processes). Intense N-processes lead to the result that the phonon distribution function has the form of a "Planck function with drift:"

$$N_{\mathbf{q}j}^{0} = N^{0}\left(\frac{\hbar\omega_{\mathbf{q}j} - \hbar\mathbf{q}\mathbf{V}}{T}\right) = \left[\exp\left(\frac{\hbar\omega_{\mathbf{q}j} - \hbar\mathbf{q}\mathbf{V}}{T}\right) - 1\right]^{-1}. \quad (1)$$

Here $\omega_{qj}$ is the frequency of a phonon with quasimomentum $\hbar q$, belonging to branch j; T is the temperature, **V** the drift velocity of the phonon gas.

The state of the phonon gas with $V \neq 0$ is a nonequilibrium state. Without external influence, the drift velocity relaxes to zero through U-processes. However, if a temperature difference $\Delta T$ is maintained between the ends of the sample, then a stationary drift velocity and a heat flux **Q** arise. Ordinarily, when problems of thermal conduction are solved, i.e., when we are interested in the dependence of **Q** on $\Delta T$, it is assumed that $\Delta T$ is small and therefore the drift velocity of the phonons is much less than the sound velocity c. In this case, we can expand the distribution (1) in a series in V and limit ourselves to the linear term in the expansion. Then, from the linearized kinetic equation, which is written down in the approximation of the relaxation time for U-processes

$$\frac{\partial\omega_{\mathbf{q}j}}{\partial q_x}\frac{d(N_{\mathbf{q}j}^{0})_{V=0}}{dx} = -\frac{(\partial N_{\mathbf{q}j}^{0}/\partial\mathbf{V})_{V=0}\mathbf{V}}{\tau_U}, \quad (2)$$

we can find V:

$$V = -\frac{cl_U}{T}\frac{dT}{dx} \approx -c\frac{l_U}{L}\frac{\Delta T}{T}. \quad (3)$$

Here L is the length of the sample in the x direction, along which the temperature gradient is applied, $l_U = c\tau_U$ is the path length of the phonon relative to the U-processes. The linear relation between Q and $\Delta T$ follows naturally from (3).

Using the result of the linear theory (3), we shall show that at low temperatures the condition for the applicability of this theory $V \ll c$ is easily violated. As is known, in a pure crystal, $l_U \sim e^{bT_D/T}$ ($T_D$ is the Debye temperature, b a constant of order unity), so that $l_U$ can be larger than L at sufficiently low temperatures. Then, for a large temperature drop $\Delta T \lesssim T$, it follows from (3) that $V \gtrsim c$. This means that the linear theory is inapplicable in the case under consideration and a nonlinear theory of thermal conduction is necessary, which can describe the case of drift velocities comparable with the sound velocity.

We emphasize that the inequality $l_U > L$ assumed above does not always mean that the phonons lose their quasimomentum on the boundaries of the sample more frequently than in the volume. If the path length of the phonons relative to the N-processes $l_N$ is smaller than the characteristic transverse dimension of the sample d, then the distance traversed by the phonon before collision with the boundaries is equal to $d^2/l_N$ and can appreciably exceed $l_U$. Here the quasimomentum of the phonons is lost in the volume of the crystal, and the flux of the phonon gas is homogeneous over the cross section of the sample. Only such a case, in which the boundaries are unimportant, will be considered in this work.

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The nonlinear theory of thermal conductivity for the case in which the loss of the quasimomentum of the phonons takes place on the boundaries, and the motion of the phonon gas resembles Poiseuille flow of a liquid, which was discussed previously in[1,2].

Thus we shall be interested in the situation in which the conditions

$$d^2 / l_N \gg l_U \gg l_N, \quad L, \ d \gg l_N.$$

are satisfied. These conditions were realized for solid $He^4$ in the experiment of Mezhov-Deglin.[3] They exist for temperatures somewhat higher than the temperature for which the maximum of the thermal conductivity was observed in[3]. For such a situation, we use the nonlinear hydrodynamic equations which describe the motions of the phonon gas. We then investigate the character of the dependence of the heat flow **Q** on $\nabla T$ for arbitrary $\nabla T$. The only additional condition superimposed on the quantity $\nabla T$ will be the condition that the characteristic length of change of temperature $T|\nabla T|^{-1}$ be much greater than $l_N$. This is necessary in order that the concepts of temperature and drift velocity have meaning. In addition to the theory of thermal conductivity, we consider the propagation of second sound in the presence of a strong stationary heat flow of the phonon gas. It will be shown that the velocity of second sound in the drifting phonon gas can be obtained from the drift velocity V and the velocity of second sound in the quiescent phonon gas by addition of these velocities according to a law similar to Einstein's relativistic law for addition of velocities.

## 2. THERMAL CONDUCTION

We shall start from the kinetic equation

$$\frac{\partial N_{\mathbf{q}j}}{\partial t} + \frac{\partial \omega_{\mathbf{q}j}}{\partial \mathbf{q}} \frac{\partial N_{\mathbf{q}j}}{\partial \mathbf{r}} = -\hat{\tau}_N^{-1} N_{\mathbf{q}j} - \hat{\tau}_U^{-1} N_{\mathbf{q}j}, \tag{4}$$

where $-\hat{\tau}_N^{-1}$ and $-\hat{\tau}_U^{-1}$ are operators for N- and U-processes, respectively. We multiply (4) first by $\hbar q_i$ and then by $\hbar \omega_{qj}$, and integrate both equations over all q. Taking into account the properties of the collision operators (conservation of energy for N- and U-processes and conservation of quasimomentum for N-processes), we obtain the equations of hydrodynamics of a phonon gas:

$$\frac{\partial P_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = \left(\frac{\partial P_i}{\partial t}\right)_{\text{coll}}, \quad \frac{\partial E}{\partial T} + \operatorname{div} \mathbf{Q} = 0. \tag{5}$$

Here **P** and E are the quasimomentum and the energy per unit volume; $\Pi_{ik}$ is the quasimomentum flux density tensor; $(\partial \mathbf{P}/\partial t)_{\text{coll}}$ is the dissipation of the quasimomentum as a result of the U-processes. The exact distribution function $N_{qj}$ enters into these quantities. However, since we assume that $\tau_N$ is less than all the remaining characteristic times, $N_{qj}$ can be replaced, with sufficient accuracy, by the distribution function (1). We thus obtain

$$\mathbf{P}(V, T) = \sum_j \int (dq) \hbar \mathbf{q} N_{\mathbf{q}j}^0, \quad E(V, T) = \sum_j \int (dq) \hbar \omega_{\mathbf{q}j} N_{\mathbf{q}j}^0,$$

$$\Pi_{ik}(V, T) = \sum_j \int (dq) \hbar q_i \frac{\partial \omega_{\mathbf{q}j}}{\partial q_k} N_{\mathbf{q}j}^0,$$

$$\mathbf{Q}(V, T) = \sum_j \int (dq) \hbar \omega_{\mathbf{q}j} \frac{\partial \omega_{\mathbf{q}j}}{\partial \mathbf{q}} N_{\mathbf{q}j}^0,$$

$$\left(\frac{\partial P_i}{\partial t}\right)_{\text{coll}} = -\sum_j \int (dq) \hbar q_i \hat{\tau}_U^{-1} N_{\mathbf{q}j}^0, \quad (dq) = \frac{d^3 q}{(2\pi)^3}. \tag{6}$$

Inasmuch as we are dealing with low temperatures, $T \ll T_D$, we shall consider only the acoustic branch of the phonon spectrum with the dispersion laws $\omega_{qj} = qc_j(\theta, \varphi)$, where j = 1, 2, 3 and $\theta$ and $\varphi$ are the angles of the spherical coordinates with polar axis x. It is then seen that the absolute value of the drift velocity never exceeds the value

$$c_0 = \min \left\| \frac{\omega_{\mathbf{q}j}}{q \cos \theta} \right\| = \min \left\| \frac{c_j(\theta, \varphi)}{\cos \theta} \right\|.$$

Actually, if $V \geq c_0$, then, for certain angles $\theta$ and $\varphi$, the difference $\hbar\omega_{qj} - \hbar \mathbf{q} \mathbf{V}$ entering into the argument $N_{qj}^0$ vanishes. Here the function $N_{qj}^0$ (1) is seen to be unnormalized and the values of (6) diverge upon integration over the angles, which is impossible. In the case of a spectrum consisting of a single branch with an isotropic dispersion $\omega_q = cq$, it is evident that $c_0 = c$. For an isotropic body whose spectrum consists of two transverse and one longitudinal branches with velocities $c_t$ and $c_l$, $c_0 = c_t$.

We apply Eq. (5) to the solution of the problem of the stationary heat conduction of a cylindrical rod between the ends of which is applied a temperature difference $\nabla T$, while the lateral surface is thermally isolated. For simplicity, we assume that the generatrix of the cylinder is parallel to the axis of symmetry of the crystal (the x axis). Then only the quantities $V_x$, $P_x$, $Q_x$, $\Pi_{xx}$ differ from zero (in what follows, we shall denote these simply by V, P, Q, Π), and these quantities, as also E and T, depend only on the coordinate x. Omitting the time derivatives in (5), we get

$$\frac{\partial \Pi}{\partial T} \frac{dT}{dx} + \frac{\partial \Pi}{\partial V} \frac{dV}{dx} = \left(\frac{\partial P}{\partial t}\right)_{\text{coll}}, \tag{7a}$$

$$\frac{\partial Q}{\partial T} \frac{dT}{dx} + \frac{\partial Q}{\partial V} \frac{dV}{dx} = 0. \tag{7b}$$

To find Q as a function of $\Delta T$, it would have been necessary to find such a solution of the set (7), T(x), V(x) which satisfies the boundary conditions $T_{x=0} = T_0 + \Delta T$, $T_{x=L} = T_0$, and then to compute Q. However, for this purpose, it is necessary to specify the form of the operator $\hat{\tau}_U^{-1}$. Further, the solution of the set of equations (7) presents great difficulties. Therefore, we shall limit ourselves below to the qualitative study of the given problem.

With the help of the definition (6) of $(\partial P/\partial t)_{\text{coll}}$, it is easy to show that under the condition $\tau_N \ll \tau_U$ the following equation is valid

$$\left(\frac{\partial P}{\partial t}\right)_{\text{coll}} = -\frac{T}{V}\left(\frac{\partial S}{\partial t}\right)_{\text{coll}}, \tag{8}$$

where

$$S = \sum_j \int (dq) \{(N_{\mathbf{q}j} + 1) \ln (N_{\mathbf{q}j} + 1) - N_{\mathbf{q}j} \ln N_{\mathbf{q}j}\}$$

is the entropy per unit volume of the phonon gas and $(\partial S/\partial t)_{\text{coll}}$ is the change in entropy as the result of collisions. Eliminating dV/dx from (7) and using (8), we obtain

$$\alpha(V, T) \frac{dT}{dx} = -\frac{T}{V}\left(\frac{\partial S}{\partial t}\right)_{\text{coll}}, \tag{9a}$$

where

$$\alpha(V, T) \equiv \frac{\partial \Pi}{\partial T} - \frac{\partial \Pi}{\partial V} \frac{\partial Q/\partial T}{\partial Q/\partial V}. \tag{9b}$$

Further qualitative investigation is based on Eq. (9a) and on the properties of the function $\alpha(V, T)$, which we immediately formulate. First of all, using the definitions (9b) and (6), it is easy to see that $\alpha(V, T) = T^3\beta(V)$ (this is a consequence of the acoustical dispersion law). As $V \to 0$, the function $\alpha(V, T)$ becomes identical with S. Actually, taking it into account that $\partial\Pi/\partial V|_{V=0} = 0$ because of the odd nature of the integrand in $q_x$, we have

$$\alpha(V, T)|_{V=0} = \frac{\partial\Pi(V, T)}{\partial T}\Big|_{V=0} = \sum_j \int (dq)\hbar q_x \frac{\partial\omega_{\mathbf{q}j}}{\partial q_x} N^0\left(\frac{\hbar\omega_{\mathbf{q}j}}{T}\right) \cdot\left[N^0\left(\frac{\hbar\omega_{\mathbf{q}j}}{T}\right)+1\right]\frac{\hbar\omega_{\mathbf{q}j}}{T^2} = S(V=0, T) > 0. \qquad (10)$$

Finally, the last general property of $\alpha(V, T)$ that we need is the behavior of this function as $V \to c_0 - 0$. It is possible to study it in the general form without making specific the spectrum, thanks to the fact that as $V \to c_0 - 0$, of the quantities $\partial\Pi/\partial T$, $\partial\Pi/\partial V$, $\partial Q/\partial T$, $\partial Q/\partial V$ in the angular integrals, only the narrow range of angles close to the angles $\theta_0$, $\varphi_0$ are important for which $|c_j(\theta, \varphi)/\cos\theta|$ takes on its minimum value $c_0$. If we introduce a new spherical set of coordinates $\theta'$, $\varphi'$ with polar axis directed along $\theta_0$, $\varphi_0$, then, close to $\theta' = 0$, we get by means of a series expansion

$$\hbar\omega_{\mathbf{q}j} - \hbar\mathbf{q}\mathbf{V} = \hbar q\cos\theta\left(\frac{c_j(\theta, \varphi)}{\cos\theta} - V\right) = \hbar q\cos\theta\,[c_0 - V + (1-\cos\theta')f(\sin 2\varphi')]. \qquad (11)$$

Substituting (11) in (9b), integrating over q and estimating the angular integrals, we can show that $\alpha(V, T) \to -\infty$ as $V \to c_0 - 0$ (we omit the details of the calculation because of their cumbersome nature).

It follows from the considered properties of $\alpha(V, T)$ that for some critical value of the velocity $V = V_c$ not dependent on T, the value of $\alpha(V, T)$ changes sign from positive to negative. Below, $\alpha(V, T)$ will be calculated for a particular form of the spectrum. The graph $\alpha(V)$ for this spectrum, constructed in Fig. 1, allows us to represent the character of $\alpha(V)$ in the general case.

We now return to (9a). We shall assume that the drift of the phonon gas takes place in the positive direction of the x axis ($V > 0$). Then, assuming that $(\partial S/\partial t)_{coll}$ is always positive, we obtain from (9a)

$$\frac{dT}{dx} < 0 \text{ for } V < V_c,$$
$$\frac{dT}{dx} > 0 \text{ for } V > V_c. \qquad (12)$$

Thus, although the velocity does not exceed the critical value, the drift of the phonon gas and the thermal flux are directed from the "hot" end to the "cold" end. At velocities above critical, the phonon gas on the other hand flows in the direction of increasing temperature. It is not difficult also to investigate how the drift velocity changes along the flow. Since $\partial Q/\partial T$, $\partial Q/\partial V > 0$, then (7b) gives

$$\frac{dV}{dx} > 0 \text{ for } V < V_c,$$
$$\frac{dV}{dx} < 0 \text{ for } V > V_c. \qquad (13)$$

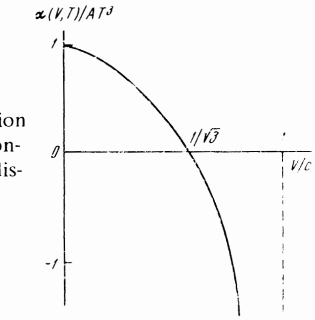


FIG. 1. Graph of the function $\alpha(V)$ for a phonon spectrum consisting of a single branch with dispersion law $\omega_q = cq$.

i.e., for subcritical motion, the velocity of flow increases with the flow while for supercritical motion, it falls off.

As is known ([4], Sec. 91), regularities similar to (12) and (13) take place for viscous motion of a compressed gas along the tube. The role of $V_c$ in this case is played by the sound velocity in the gas. For subsonic motion, the gas pressure falls downward along the flow, and increases in the supersonic case. The velocity of motion, on the other hand, increases along the tube for subsonic motion and falls for supersonic motion. The analogy pointed out allows us to clarify the problem of the character of the dependence of Q on $\Delta T$ by the method set forth in[4].

We introduce into consideration the entropy flux density $j_S$. For the stationary case,

$$\frac{dj_S}{dx} = \left(\frac{\partial S}{\partial t}\right)_{coll} > 0. \qquad (14)$$

Along the tube, $j_S$ is a function of T, which depends on Q as on a parameter. Indeed, with the help of (14) and (9a) we get

$$\frac{dj_S}{dT} = -\frac{V}{T}\alpha(V, T), \qquad (15)$$

where V is connected with T by the equation $Q(V, T) = \text{const}$. For $V = V_c$, $\alpha = 0$, and the function $j_S(T)$ has an extremum. By computing the second derivative of this function, we establish the fact that it is negative for $V = V_c$:

$$\frac{d^2 j_S}{dT^2}\Big|_{V=V_c} = \frac{V}{T}\frac{\partial Q/\partial T}{\partial Q/\partial V}\frac{\partial\alpha}{\partial V}\Big|_{V=V_c} < 0. \qquad (16)$$

Thus $j_S(T)$ has a maximum for $V = V_c$. The curves of the $j_S(T)$ dependence are shown in Fig. 2. To the right of the maxima there is a region $V < V_c$, to the left, $V > V_c$, and, as follows from (7b), V falls off with increase in T.

We now follow the motion of the phonon gas along the sample. Let $V < V_c$ at the input to the sample. For motion along the sample, the entropy flux increases and the

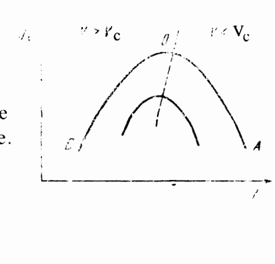


FIG. 2. Typical form of the dependence of the entropy flux density on the temperature as it varies along the sample.

temperature decreases, which corresponds to motion along the right branch of the curve from A to O. Such can be continued so long as the entropy flux does not reach a maximum, and the velocity the value $V_c$. Further motion along the curve behind the point O is impossible, since this would correspond to a decrease in the entropy flux for motion along the sample. Thus the velocity $V_c$ is the maximum drift velocity achievable under such conditions. If the temperatures $T_0 + \Delta T$ and $T_0$ are maintained at the ends of the sample, then $V_c$ is quite generally not reached or is reached at the output of the sample (when $\Delta T$ is sufficiently large). It is clear here that for such suitably large $\Delta T$, the heat flux cannot exceed the value $Q_{max} = Q(V_c, T_0)$. The dependence of Q on $\Delta T$, which should be observed experimentally, is shown schematically in Fig. 3. We emphasize that the conclusion as to the boundedness of Q, which is the principal result of the qualitative investigation, is essentially connected with the fact that $V_c < c_0$. If the drift velocity were to approach $c_0$ in the experiment on thermal conductivity, then the heat flux would increase without limit.

The case considered of subcritical velocity corresponds to the ordinary experimental setup for the measurement of the thermal conductivity, when the phonon gas "gets going" from the externally applied temperature gradient. The other case is theoretically possible in which the phonons are "injected" at the input to the sample with velocity $V > V_c$. Here a temperature gradient arises directed counter to the flow of phonons and retarding them. Such a motion of the phonon gas along the sample can be described by the transfer along the curve of Fig. 2 from the point B to the point O. In order to make clear the possibility of the practical realization of such a regime, it is necessary to analyze processes which take place at the input contact of the sample. However, this question lies beyond the scope of the present paper.

We now write down the results of the calculation of the thermodynamic quantities and the function $\alpha(V, T)$ for a spectrum consisting of one acoustic branch with an isotropic dispersion law $\omega_q = cq$. These results illustrate the general conclusions drawn above and allow us to determine the value of $V_c$ for the spectrum considered. Moreover, they will be needed below in the study of second sound. Calculating the integrals (6), we have

$$P = \frac{AT^4}{c}\frac{\gamma}{(1-\gamma^2)^3}, \quad E = AT^4\frac{3/4 + 1/4\gamma^2}{(1-\gamma^2)^3},$$

$$\Pi = AT^4\frac{3/4\gamma^2 + 1/4}{(1-\gamma^2)^3}, \quad Q = c^2P; \tag{17a}$$

$$\alpha(V, T) = \frac{AT^3}{(1-\gamma^2)^2}\frac{1-3\gamma^2}{1+5\gamma^2}. \tag{17b}$$

Here

$$A = \frac{2\pi^2}{45}\frac{1}{(\hbar c)^3}, \quad \gamma = \frac{V}{c},$$

and $|\gamma| < 1$.

The graph of the dependence of $\alpha(V)$ is constructed in Fig. 1. It is seen from (17b) that $\alpha = 0$ for $\gamma^2 = V^2c^2 = 1/3$. Thus we come to the conclusion that for the case considered, the critical velocity $V_c$ is exactly the same as the velocity of second sound $U_{II} = c/\sqrt{3}$. For maximum heat flow, we have here

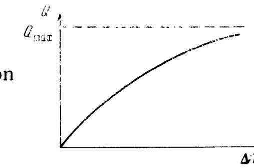


FIG. 3. Schematic diagram of the predicted dependence of the heat flux on the temperature difference between the ends of the sample.

$$Q_{max} = 9/8\sqrt{3}cAT_0^4.$$

In the case of a more complicated spectrum, for example, the spectrum of an isotropic body, strictly speaking, $V_c \neq U_{II}$. However, because of the fact that $c_l > \sqrt{2}c_t$ (see[4], p. 744), the velocity of second sound in an isotropic body is

$$U_{II} = \frac{1}{\sqrt{3}}\left(\frac{c_l^{-3} + 2c_t^{-3}}{c_l^{-5} + 2c_t^{-5}}\right)^{1/2}$$

and the values of (17) are determined in practice only by the contribution of the transverse branches; $V_c \approx c_t/\sqrt{3} \approx U_{II}$. We think it likely that $V_c$ is close to the velocity of second sound for crystals also.

## 3. SECOND SOUND

It is known that if a continuous medium moves with velocity V, then the velocity of sound that is propagated in this medium relative to a fixed observer is determined by the formula $c_{1,2} = V \pm c$, where c is the sound velocity in the medium at rest (the one-dimensional case is considered). Such a law of addition of velocities is connected with Galilean invariance. In the case of the drift of a phonon gas, as is easily established, "Galilean" invariance is absent, i.e., the thermodynamic functions of a phonon gas drifting with velocity V, computed in a set of coordinates moving with the same velocity, are different from the thermodynamic functions of the gas at rest. Therefore, for the velocity of second sound, the simple law of velocity addition

$$(U_{II})_{1,2} = V \pm U_{II}^0, \tag{18}$$

is invalid. Here $U_{II}^0$ is the velocity of the second sound in the phonon gas at rest. The invalidity of Eq. (18) is also clear from the fact that it does not satisfy the following obvious condition: the velocity of second sound can never exceed the maximum velocity of the phonons. Thus, for example, for a spectrum consisting of a single branch with the dispersion law $\omega_q = cq$ and for $V = c/2$, we would have from (18)

$$(U_{II})_1 = \frac{c}{2} + \frac{c}{\sqrt{3}} > c, \tag{19}$$

which is impossible.

In this section, we obtain the dispersion equation for second sound in the presence of a stationary heat flux and find the law for the addition of the velocities of second sound and the drift for the simplest form of the spectrum. Rewriting Eq. (5) for the one-dimensional case, we have

$$\frac{\partial P}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial P}{\partial V}\frac{\partial V}{\partial t} + \frac{\partial \Pi}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial \Pi}{\partial V}\frac{\partial V}{\partial x} = \left(\frac{\partial P}{\partial t}\right)_{\text{coll}},$$

$$\frac{\partial E}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial E}{\partial V}\frac{\partial V}{\partial t} + \frac{\partial Q}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial Q}{\partial V}\frac{\partial V}{\partial x} = 0. \tag{20}$$

Let the stationary quantities T(x), V(x) have small oscillatory contributions $T_1(x)e^{-i\omega t+ikx}$, $V_1(x)e^{-i\omega t+ikx}$, while the condition $\tau_U^{-1} \ll \omega \ll \tau_N^{-1}$ is satisfied, which means that at the frequency $\omega$ the hydrodynamic description is still applicable and the damping of second sound both from N-processes and from U-processes is small. We also assume that the characteristic length of change of the stationary quantities T(x), V(x), and consequently the amplitudes $T_1(x)$, $V_1(x)$ are much larger than the wavelength of second sound $2\pi/k$. This second condition does not contradict the first, since we assume that $l_N \ll L$.

Linearizing (20) in $T_1$, $V_1$ and omitting the term $(\partial P/\partial t)_{coll}$, which leads to small damping, we obtain

$$\left[-i\omega\frac{\partial P}{\partial T}+ik\frac{\partial\Pi}{\partial T}\right]T_1+\left[-i\omega\frac{\partial P}{\partial V}+ik\frac{\partial\Pi}{\partial V}\right]V_1=0,$$

$$\left[-i\omega\frac{\partial E}{\partial T}+ik\frac{\partial Q}{\partial T}\right]T_1+\left[-i\omega\frac{\partial E}{\partial V}+ik\frac{\partial Q}{\partial V}\right]V_1=0. \quad (21)$$

Equating the determinant of (21) to zero, we obtain an equation for the determination of the velocities of the two branches of second sound:

$$\left(\frac{\omega}{k}\right)^2\left[\frac{\partial P}{\partial T}\frac{\partial E}{\partial V}-\frac{\partial P}{\partial V}\frac{\partial E}{\partial T}\right]+\frac{\omega}{k}\left[\frac{\partial P}{\partial V}\frac{\partial Q}{\partial T}+\frac{\partial E}{\partial T}\frac{\partial\Pi}{\partial V}-\frac{\partial Q}{\partial V}\frac{\partial P}{\partial T}-\frac{\partial\Pi}{\partial T}\frac{\partial E}{\partial V}\right]+\left[\frac{\partial\Pi}{\partial T}\frac{\partial Q}{\partial V}-\frac{\partial\Pi}{\partial V}\frac{\partial Q}{\partial T}\right]=0. \quad (22)$$

Taking into account the definition of $\alpha(V, T)$ (9b), we see that for $V = V_c$, Eq. (22) has a root $\omega/k = 0$, i.e., the velocity of one branch of second zero is zero.

We solve Eq. (22) for the case of a spectrum consisting of one isotropic branch. The calculation of the coefficients of Eq. (22), with the help of (17a) gives

$$(\omega/k)^2(3-\gamma^2)-4\gamma(\omega/k)+3\gamma^2-1=0. \quad (23)$$

The roots of Eq. (23) are equal to

$$(U_{II})_{1,2}=\left(V\pm\frac{c}{\sqrt{3}}\right)\Big/\left(1\pm\frac{c}{\sqrt{3}}V/c^2\right)=(V\pm U_{II}^0)\Big/\left(1\pm\frac{U_{II}^0V}{c^2}\right). \quad (24)$$

Thus we come to the conclusion that the law of addition of the velocity of the drift of a phonon gas and the velocity of second sound has the same form as the relativistic formula of velocity addition of Einstein, with this difference, however, that in place of the velocity of light in (24), we have the maximum attainable velocity in the phonon system—the velocity of sound. Equation (24) remains approximately valid even for an isotropic body in which, as has already been said, the contribution of the longitudinal branch to the dispersion equation for second sound can be neglected. In this case, naturally, we understand by c in (24) the value $c_t$.

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Translated by R. T. Beyer
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## ERRATA

Article "Premature" Disappearance of Antiferromagnetic Resonance in Hematite, by V. I. Ozhogin and V. G. Shapiro, Sov. Phys.-JETP 28, No. 5, 915 (1969)

In lines 19 and 20 of the abstract, p. 915, read: "...effective field strengths: $H_E$ = 8960 kOe; $H_D$ = 22.7 kOe; $2H_{A1}$ (77°K) = 01382 kOe; $2H_{A2}$ (77°K) = 0.222 kOe."

Pages 917 and 918 should be interchanged.

In line 7 from the bottom, right-hand column of p. 920, read: "...the parameters $\frac{1}{2}B$ = 8960 kOe..."

In the next-to-last and last lines of the right-hand column of p. 920, read: "...pure-spin value $M_0$ = 175 G-cm$^2$/g."

The ordinates of Fig. 7 on p. 921 should be reduced by one-half, and the corresponding numbers on the ordinate scale should be 0, 0.1, 0.2, and 0.3 kOe.