CONCERNING THE THEORY OF PROXIMITY EFFECTS IN SUPERCONDUCTORS

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The Josephson current is calculated for an arbitrary barrier transparency near T_c. The critical current is calculated for a system of two identical superconductors separated by a layer of normal metal.

THE physical nature of the proximity effects,[1] and in particular that of the Josephson effect,[2] can be most fully investigated near the critical temperature T_c, because in this case the equations of the theory of superconductivity[3] can be solved by expanding them in the order parameter Δ. The beginnings of such investigations were made by De Gennes.[1]

The purpose of this article is a further development of this approach which will make it possible to obtain a formula for the Josephson current for an arbitrary barrier transparency, and an investigation of the behavior of the order parameter near the barrier. Using the same method it is possible to calculate for a superconducting junction with a layer of normal metal the dependence of the critical current on the thickness of the normal layer.

Making use of the usual procedure of expanding in powers of Δ,[3] we arrive at the following equation for the order parameter:

$$\Delta(\mathbf{r}) = gT\sum_{\omega}\int d^3\mathbf{r}'\,\overline{G_0(\mathbf{r},\mathbf{r}',\omega)G_0(\mathbf{r},\mathbf{r}'-\omega)}\,\Delta(\mathbf{r}')+\ldots \qquad (1)$$

Here the dots denote the terms nonlinear in Δ whose explicit form is, as will become clear below, unimportant for us; G_0 is the temperature Green's functions of the system in the normal state; the bar denotes averaging over the atomic distances. The essential nature of such an averaging is connected with the fact that only changes of the order parameter at distances of the order of the coherence length have physical significance.[4]

1. THE JOSEPHSON EFFECT

Expanding G_0 in wave functions of the one-particle problem in the presence of a barrier and making use of the circumstance that at distances large compared with the atomic distances only the asymptotes of the wave functions are important, after discarding rapidly oscillating terms, we rewrite (1) in the form

$$\Delta(z) - \pi N(0)gT\frac{1}{v_0}\sum_{\omega}\int_0^1\frac{dx}{x}\int_{-\infty}^{\infty}\exp\left\{-\frac{2|\omega|}{v_0x}\zeta\right\}\Delta(z+\zeta)\,d\zeta$$
$$= \operatorname{sign} z\,\pi N(0)gT\frac{1}{v_0}\sum_{\omega}\int_0^1\frac{dx}{x}R(x)\exp\left\{-\frac{2|\omega|}{v_0x}|z|\right\}$$
$$\cdot\int_{-\infty}^{\infty}\operatorname{sign}\zeta\exp\left\{-\frac{2|\omega|}{v_0x}|\zeta|\right\}\Delta(\zeta)\,d\zeta+\ldots \qquad (2)$$

We have directed the z axis perpendicular to the plane of the junction and taken into account the symmetry of the problem. In Eq. (2) $R(x)$ is the reflection coefficient which will for the sake of simplicity be considered independent of x; $N(0)$ is the density of states on the Fermi surface; v_0 is the Fermi velocity.

We note that since the appreciable $\omega \sim T_c$, the main contribution to the integral over ζ is due to $\zeta \sim \xi_0 \sim v_0/T_c$. Let us first consider $|z| \gg \xi_0$. Then the term set out in the right-hand side of (2) is exponentially small and can be discarded, and $\Delta(z+\zeta)$ can be expanded under the integral in a series in ζ. Retaining in the expansion terms no higher than quadratic in ζ, we obtain

$$\Delta(z)\left(1-\pi N(0)gT\sum_{\omega}\frac{1}{|\omega|}\right)-\frac{7}{8}\zeta(3)\frac{1}{6}\left(\frac{v_0}{\pi T_c}\right)^2\frac{d^2\Delta}{dz^2}+\ldots=0, \qquad (3)$$

where $\zeta(3)$ is a particular value of the Riemann ζ function. By virtue of the definition of the critical temperature

$$\pi N(0)gT\sum_{\omega}\frac{1}{|\omega|}=1,$$

therefore the main term in the expansion in $\tau = 1 - T/T_c$ (linear in Δ) cancels and the first term in (3) reduces to $\tau\Delta(z)$, i.e., it is of the order of Δ^3. Clearly the nonlinear terms denoted by the dots must also be taken into account, being obviously taken in the local form. The function $\Delta(z)$ varies over distances $\xi(\tau) \sim \xi_0/\sqrt{\tau}$ in the indicated range of z. We come, thus, to the conclusion that the usual Ginzburg–Landau equation is valid for distances $|z| \gg \xi_0$.

In the region $|z| \sim \xi$ the behavior of $\Delta(z)$ is described by a linear integral equation, since in this region terms of the order of $\sqrt{\tau}$ do not cancel and the nonlinear terms can be discarded as representing only small corrections. In the linear integral equations one can assume $T = T_c$.

Dividing $\Delta(z)$ into even and odd parts $\Delta = \Delta_s + \Delta_a$ we reduce (2) to the following system of equations:

$$\Delta_s(z) = \int_{-\infty}^{\infty}K(z-z')\Delta_s(z')\,dz', \qquad (4)$$

$$\Delta_a(z) = \int_0^{\infty}L(z,z')\Delta_a(z')\,dz', \quad z>0. \qquad (5)$$

Here

$$K(z) = \pi N(0)gT_c\frac{1}{v_0}\sum_{\omega}\int_0^1\frac{dx}{x}\exp\left(-\frac{2|\omega|}{v_0x}|z|\right), \qquad (6)$$

$$L(z,z') = K(z-z') + (1-2D)K(z+z'). \qquad (7)$$

Equations (4) and (5) determine the behavior of the order parameter near the barrier at distances of the order of $\xi_0 \sim v_0/T_c$. They have the following readily establishable properties:

(a) the asymptotic behavior of the solution $\Delta(z)$ for $z \to \infty$ is of the form

$$\Delta(z) = \Delta_- + z\Delta_-', \quad z \to -\infty; \quad \Delta(z) = \Delta_+ + z\Delta_+', \quad z \to +\infty;$$

(b) the only solution of the equation for the symmetric part is a constant; consequently, $\Delta_+' = \Delta_-'$;

(c) $\Delta_a(z) = \Delta_+' z + \frac{1}{2}(\Delta_+ - \Delta_-)$, $z \to \infty$; if the coefficient of z is specified, the solution is uniquely determined. There exists, therefore, a relation (linear because of the linearity of the equation) relating Δ_+' with $\Delta_+ - \Delta_-$.

Thus we obtain the following matrix relation:

$$\begin{pmatrix} \Delta_+ \\ \Delta_+' \end{pmatrix} = \hat{M} \begin{pmatrix} \Delta_- \\ \Delta_-' \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}. \tag{8}$$

The coefficient α can be readily found explicitly in the two limiting cases of high and low transparency. Assuming

$$\Delta_a(z) = \Delta_+' z + \frac{1}{2}(\Delta_+ - \Delta_-) + \varphi(z),$$

we obtain after simplification

$$\varphi(z) - \int_0^\infty L(z, z')\varphi(z')\,dz'$$
$$= 2R\Delta_+' \int_0^\infty z' K(z+z')\,dz' - D(\Delta_+ - \Delta_-)\int_0^\infty K(z+z')\,dz'. \tag{9}$$

Using for the solution of Eq. (9) perturbation theory in D, we obtain from the solvability condition of the first-approximation equation

$$D\frac{\Delta_+ - \Delta_-}{2}\int_0^\infty dz \int_0^\infty dz' K(z+z') = \Delta_+' \int_0^\infty dz \int_0^\infty dz' K(z+z')z'. \tag{10}$$

We emphasize that the explicit form of the kernel K(z) has thus far not been used; therefore, Eq. (10) retains its validity also for the more general case (for example, in the presence of impurities). Using (6), we find

$$\Delta_+' = \frac{3\pi^3 T_c}{14\zeta(3) v_0} D(\Delta_+ - \Delta_-), \quad D \ll 1. \tag{11}$$

Analogously for high transparency we have

$$\Delta_+ - \Delta_- = \frac{\pi^3 v_0}{112\zeta(3) T_c} R\Delta_+'. \tag{12}$$

In the intermediate case one can readily obtain a better approximation for α by the variational method.

Let us now note that since the characteristic distance for the Ginzburg–Landau equation is $\xi(\tau) \gg \xi_0$, and the solutions of the integral equations (4) and (5) take on the asymptotic form for $\xi(\tau) \gg |z| \gg \xi_0$, the linear asymptote of the integral equations should be identified with the linear part of the expansion of the solution of the Ginzburg–Landau equation into a Taylor series near $z = \pm 0$. Thus relation (8) is a boundary condition for the Ginzburg–Landau equation. It must not be thought that the boundary condition (8) is only an effective boundary condition. Indeed, as is clear from the above considerations, there exists a region in which both the linear integral equation and the Ginzburg–Landau equation reduce to the equation $d^2\Delta/dz^2 = 0$. As such a z one can take $|z| \sim \xi_0 \tau^{-1/4}$. The above statement also remains valid when one takes into account the intrinsic magnetic field, since all the characteristic lengths appearing in this case are much larger than those essential for the above treatment.

We note that because of the crudeness of the approximation made by De Gennes[1] in the estimate of α he obtained the boundary conditions very inaccurately even for low transparency. However, he indicated the general form of the boundary conditions correctly.

In order to solve the Ginzburg–Landau equation we make the substitution

$$\Delta(z) = \exp(2im\chi \pm i\varphi/2)\Delta_\infty f(z), \tag{13}$$

where χ is the continuous component of the phase, and $d\chi/dz = v_s$. The boundary conditions are consistent by virtue of the relation $\sin\varphi = 2\alpha m v_s(0)$ which expresses the law of conservation of current and reduce to the equalities

$$f_+ = f_- \equiv f_0, \quad f_+' = -f_-', \quad \alpha f_+' = f_0(1 - \cos\varphi), \tag{14}$$

The current is given by the expression

$$j = \frac{4eC}{\alpha}\Delta_\infty^2 f_0^2 \sin\varphi, \quad C = \frac{7\zeta(3)}{48\pi^2}\frac{N(0)v_0^2}{T_c^2}. \tag{15}$$

It is convenient to find f_0^2 from the first integral of the Ginzburg–Landau equation. Making use of it, we find for the currents that are much smaller than the thermodynamic critical current the expression

$$j = n\frac{e}{m}\frac{\tau}{\alpha}\sin\varphi\left\{1 + \frac{4}{\alpha^2}\xi^2(\tau)\sin^4\frac{\varphi}{2} + \frac{2\sqrt{2}}{\alpha}\xi(\tau)\sin^2\frac{\varphi}{2}\sqrt{1 + \frac{2}{\alpha^2}\xi^2(\tau)\sin^4\frac{\varphi}{2}}\right\}, \tag{16}$$

where n is the total number of electrons. The obtained formula is general. For low transparency α is large, and we obtain Josephson's result. For high transparency α is small, but φ is also small, so that one must retain all terms in (16). Of course, $\sin\varphi$ can be replaced by the phase.

In conclusion we note that the formal investigation of the linear equations (4)–(5) was carried out by Zaĭtsev[5] who obtained boundary conditions for arbitrary transparency D which coincide for $D \ll 1$ with (11) and for $D \sim 1$ differ from (12) by about 6%. The latter is related to a different choice of the trial function.

2. THE PROXIMITY EFFECT

Let us consider the problem of calculating the superconducting current in a system consisting of two superconductors separated by a layer of normal metal of thickness d. We shall consider the case of the same superconductors. We take the usual model in which we ignore the difference between the effective masses, the Debye temperatures, and the density of levels N(0), of the superconducting and normal metal, assuming that they differ only in their constant of effective attraction which changes abruptly. Assuming that there is no reflection on passing through the boundary and that in the region of space occupied by the normal metal $g = 0$, we obtain a linear integral equation for the order parameter near the boundary

$$\Delta(z) = \int_{-\infty}^{-d/2} K(z - z')\Delta(z')\,dz' + \int_{d/2}^{\infty} K(z - z')\Delta(z')\,dz'. \tag{17}$$

By virtue of the invariance with respect to reflection of z, the even and odd parts of the function $\Delta(z)$ separately satisfy Eq. (17) which can therefore be considered only on the $z > 0$ semiaxis. The asymptotic behavior of the solutions of (17) is of the form (8), and in accordance with this we have for $\Delta_s(z)$ and $\Delta_a(z)$

$$\Delta_s(z) = C_1(z + \xi_0 q_{1\infty}), \quad \Delta_a(z) = C_2(z + \xi_0 q_{2\infty}), \quad z \to \infty.$$

Since, as usual, $q_{1\infty}$ and $q_{2\infty}$ are uniquely determined, we obtain the boundary conditions in the form

$$(\Delta_+' - \Delta_-')\xi_0 q_{1\infty} = \Delta_+ + \Delta_-, \quad (\Delta_+' - \Delta_-')\xi_0 q_{2\infty} = \Delta_+ - \Delta_-. \quad (18)$$

Taking into account the fact that $d \gg \xi_0$, which makes it possible to simplify the kernel of the equation, introducing the dimensionless variable $\zeta = z/\xi_0$, $\xi_0 = v_0/2\pi T_c$, and assuming that $\Delta_{s,a}(\zeta) = C_{1,2}[\zeta + \xi_0 q_{1,2}(\zeta)]$, we arrive at the equations

$$q_1(\zeta) = \frac{1}{2}E_1(\zeta) + \frac{1}{2}\int_0^\infty d\zeta' L_1(\zeta, \zeta') q_1(\zeta'),$$

$$E_1(\zeta) = E(\zeta) + \rho\frac{\xi_0}{d}\exp\left\{-\frac{d}{\xi_0} - \zeta\right\},$$

$$E(\zeta) = \sum_n \frac{\rho}{(2n+1)^2}\int_1^\infty \frac{dy}{y^3}\exp\{-|2n+1|y\zeta\},$$

$$L_1(\zeta, \zeta') = L(\zeta - \zeta') + \rho\frac{\xi_0}{d}\exp\left\{-\frac{d}{\xi_0} - \zeta - \zeta'\right\}, \quad \rho = gN(0). \quad (19)$$

For $q_{1\infty}$ we have the relation

$$q_{1\infty}\sum_n \frac{1}{|2n+1|^3} = \frac{3}{2}\left\{\frac{1}{4}\sum_n \frac{1}{(2n+1)^4} + \frac{\xi_0}{d}e^{-d/\xi_0} + (2\rho D_{1\,min})^{-1}\right\}, \quad (20)$$

where D_1 is a functional, the minimization of which leads to (19). We also have an analogous equation for $q_{2\infty}$. Making use of the variational method,[6] we obtain

$$q_{1,2\infty} = q_\infty \pm k\frac{\xi_0}{d}e^{-d/\xi_0}, \quad k \cong 0.98, \quad q_\infty \cong 0.64.$$

The current turns out to be

$$j = 4eC\frac{k}{q_\infty^2}\frac{1}{d}e^{-d/\xi_0}|\Delta_+|^2\sin\varphi. \quad (21)$$

Since $d \gg \xi_0$, one can substitute for Δ_+ the known solution of the problem of the semi-infinite superconductor.[7] We thus obtain for the critical current

$$j_c = A\frac{ne}{md}e^{-d/\xi_0}\tau^2. \quad (22)$$

The numerical constant $A = 6k/7\zeta(3)$ is of the order of unity. The critical current decreases exponentially with the thickness of the normal layer. The temperature dependence is determined by the factor τ^2 and not by τ as in the case of the Josephson current. The latter fact has been noted by De Gennes.[1]

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