TRANSPORT PHENOMENA IN TOROIDAL MAGNETIC SYSTEMS

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Expressions are given for the particle flux and the energy flux across the magnetic field in various toroidal magnetic systems (tokomak, levitron, stellarator, and bumpy torus). Both fully ionized and weakly ionized plasmas are considered. The expressions that are obtained indicate that taking account of the toroidal geometry leads to a substantial increase in the diffusion coefficient and the thermal conductivity at low collision frequencies as compared with straight systems (corresponding to $\delta = r/R = 0$). In this connection it is found that in systems that are not axially symmetric (stellarator, bumpy torus) there exists a parameter region (low collision frequencies or large electric fields) for which the transport coefficients do not, in general, depend on the magnitude of the magnetic field and are substantially higher (several orders of magnitude) than the values that obtain when the toroidal geometry is not taken into account.

1. INTRODUCTION

MANY experiments carried out in recent years for the purpose of investigating plasma confinement in closed (toroidal) systems have indicated that these systems are characterized by a very short lifetime; this finding implies the rapid loss of plasma across the magnetic field. Attempts to explain this rapid loss within the framework of the theory of binary collisions and stable plasmas have not been successful and have led many authors to the conclusion that the enhanced plasma loss is due to various kinds of instabilities that develop in a plasma, that is to say, that the diffusion and thermal conductivity derive from turbulence phenomena. We are very much of the opinion that it is important and necessary to develop a turbulence theory for transport phenomena although, unfortunately, a quantitative analysis of this problem has still not been initiated. It should be noted that the potentialities of a theory that takes account only of binary collisions or, as it is sometimes called, a classical theory of transport phenomena, have still not been fully investigated and it would be premature to overlook such a theory in attempts to explain experimental results that have been obtained.

There are basically two arguments that support the notion that the diffusion and thermal conductivity are associated with turbulence phenomena. 1) The extremely small value of the theoretically computed diffusion coefficient as compared with that which is measured experimentally, and 2) the fundamental differences in the dependence of the theoretical and experimental plasma lifetime (and consequently the diffusion coefficient) on the magnetic field and a number of other plasma parameters. However, it should be noted that the comparisons between theory and experiment are frequently carried out making use of transport coefficients which are computed for straight systems rather than toroidal systems.[1)] As is well known, the diffusion coefficient and the thermal conductivity for a straight system are directly proportional to the collision frequency (i.e., density) and inversely proportional to the square of the magnetic field and the square-root of the temperature. But any such comparison implies that the toroidal geometry does not have an important effect (in any case, for a sufficiently small ratio of the minor radius of the torus r to the major radius R) on the magnitude and structure of the particle and energy fluxes. However, this assumption is found to be quite improper even when the toroidal parameter is small ($\delta = r/R \ll 1$). In very many cases there are fundamental changes in the magnitude and structure of the fluxes and these lead to results that are fundamentally different from those obtained for straight systems ($\delta = 0$) both as regards the absolute values of the transport coefficients as well as the nature of the dependence of these transport coefficients on the parameters that characterize the magnetic field and the plasma.

Actually, as far back as 1951 Tamm[1] and Budker[2] called attention to the fact that in toroidal systems the particle flux and heat flux across a strong magnetic field could be considerably larger than for analogous straight systems. This enhancement is due to an effect called "mixing"[2] which arises because of the toroidal drift and is manifest as differences between trajectories corresponding to different values of the particle velocities. In turn, the latter effect can lead to a situation in which, as a result of a collision, a particle is displaced over a distance greater than its Larmor radius. The net effect is to increase the effective particle drift rate across the magnetic field.

Subsequently, more exact calculations carried out by Pfirsch and Schluter[3] (see also [4,5]) and Shafranov[6] verified these qualitative predictions. However, the results of these investigations, which were obtained within the framework of a hydrodynamic approximation, are valid only for a dense plasma, in which the mean free path is much smaller than the characteristic dimensions of the system. For this reason it is of interest to carry out analogous calculations using kinetic theory in order to obtain expressions for the particle flux and the heat flux that hold for a low density plasma. Galeev and

[1)]There are investigations in which the toroidal geometry has been taken in to account; however, there are hydrodynamic calculations which have been improperly extrapolated into the region of low collision frequencies.

Sagdeev[7] have carried out a mathematical analysis of this problem for the case of an axially symmetric system and a fully ionized plasma of low density. However, an improper linearization of the collision integral leads these authors to results that differ from those obtained below (cf. Sec. 2B). For certain particular cases, estimates of the diffusion coefficient and the thermal conductivity for systems that are not axially symmetric were obtained in a paper by Galeev, Sagdeev and Furth,[8] which was devoted to a discussion of the diffusion of a fully ionized plasma in an $l = 3$ stellarator.

In the present work we discuss transport across a strong magnetic field in various kinds of toroidal systems. The second part of the paper is devoted to axially symmetric systems, in which we include such systems as tokomak, levitron, and zeta, while the third section is devoted to systems that are not axially symmetric such as the $l = n$ stellarator and the bumpy torus.[9]

It would be beyond the scope of the present work to go into any great detail concerning the actual mathematical calculations, which are extremely complicated.[2)] Here, we shall limit ourselves to a description of the general scheme of the calculation and present the final expressions for the particle and energy fluxes and also the limits of applicability of the results, together with a brief discussion.

The general features of the calculation are as follows. We start with the stationary kinetic equation for particles of species j (the subscript j = e, i refers to electrons and ions respectively):

$$\mathbf{v}\frac{\partial f_j}{\partial \mathbf{r}} + e_j\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}]\right\}\frac{\partial f_j}{\partial \mathbf{v}} = \mathrm{St}_j, \qquad (1)^*$$

where f_j is the distribution function for particles of species j, **E** and **H** are the electric and magnetic fields and St_j is the kinetic collision integral, which takes account of collisions between particles of species j and all other particles (electrons, ions, and neutrals).

Without dwelling on the details, we indicate here only that collisions with neutrals are taken into account by means of a collision integral written in the form given by Bhatnagar, Gross and Krook[10] while collisions between charged particles are taken account of by the Landau collision integral,[11] these then being linearized in accordance with the procedure used below by expansion of the solution in powers of a small parameter $\sim 1/\omega_j$. It is important to emphasize that the first moment of the exact distribution function (i.e., the mean directed velocity) although small, is not equal to zero and for axially symmetric systems (Sec. 2) is of the same order of smallness (in terms of the parameter $1/\omega_j$) as the corrections of interest to the distribution function in the zeroth approximation.[3)] This means that in the derivation of the linearized collision integral, as in the initial integral, it is necessary to make sure that one does not violate momentum conservation. In particular, the linearized collision integral must satisfy the two following physically obvious requirements: 1) collisions between particles of the same j-th species must not change the total momentum of the j-th component of the plasma, and 2) collisions between particles of two different species must not change the total momentum of both plasma components.

However, these requirements were not always kept in mind:[7] although the initial collision integral was written properly, in the subsequent use of the linearized form as well as in the "τ-approximation" the requirements given above were violated. This is the basis, in particular for the difference between the results obtained in Sec. 2B of the present work, according to which the diffusion of the plasma is always (that is to say, aside from the dependence on the magnitude of the electric field) ambipolar in nature, and the results of [7], from which a contradictory result is obtained.

In Eq. (1) we can make use of the so-called drift variables,[12] in which the equations of motion of the particles do not contain the fast phase α, that is to say, these are averaged over the fast Larmor gyration) and we can write $f_j = \bar{f}_j + \tilde{f}_j$ where $\bar{f}_j$ is the average part of the distribution function while $\tilde{f}_j$ is the variable part (with respect to α) of the function. We then obtain the following equations for $\bar{f}_j$ and $\tilde{f}_j$:[4)]

$$-\omega_j\frac{\partial \tilde{f}_j}{\partial \alpha} = \widetilde{\mathrm{St}}_j, \quad \omega_j = \frac{e_jH}{m_jc}, \qquad (2)$$

$$\hat{L}_j\bar{f}_j = \overline{\mathrm{St}}_j, \qquad (3)$$

where $\overline{\mathrm{St}}_j$ and $\widetilde{\mathrm{St}}_j$ are the mean and variable (with respect to α) parts of the collision integral while the operator $\hat{L}_j$, which after conversion to the new variable becomes integrals of the drift equations (i.e., the energy $\mathscr{E}$ and the transverse adiabatic invariant μ) is given by

$$\hat{L}_j = \mathbf{v}_{dj}\frac{\partial}{\partial \mathbf{r}}, \qquad (4)$$

where $\mathbf{v}_{dj}$ is the familiar expression for the drift velocity of the guiding center in weakly inhomogeneous electric and magnetic fields.[12]

The particle flux

$$\mathbf{S}_j = \int \mathbf{v}f_j\,d\mathbf{v}$$

and the energy flux

$$\mathbf{\Pi}_j = \int \frac{m_jv^2}{2}\mathbf{v}f_j\,d\mathbf{v}$$

can also be written in the form of a sum of two terms:

$$\mathbf{S}_j = \bar{\mathbf{S}}_j + \tilde{\mathbf{S}}_j, \quad \mathbf{\Pi}_j = \bar{\mathbf{\Pi}}_j + \tilde{\mathbf{\Pi}}_j, \qquad (5)$$

one of which ($\bar{\mathbf{S}}_j$, $\bar{\mathbf{\Pi}}_j$) is associated with the slow drift motion while the second ($\tilde{\mathbf{S}}_j$, $\tilde{\mathbf{\Pi}}_j$) is associated with the fast gyration. Assuming that $\widetilde{\mathrm{St}}_j \sim 1/\omega_j$ and that in the zeroth approximation the distribution function is Maxwellian, we can integrate Eq. (2) for $\tilde{f}_j$ and derive ex-

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[2)] The detailed calculations applying to the material described here will be published separately.

[3)] In the usual way, we take this to be a local Maxwellian distribution with density and temperature depending on coordinates, and mean velocity equal to zero.

*$[\mathbf{v}\,\mathbf{H}] \equiv \mathbf{v} \times \mathbf{H}$.

[4)] In the usual way we assume that the magnetic field is large so that the ratio of the particle Larmor radius to the characteristic dimensions of the system is much smaller than unity.

pressions for the fluxes $\tilde{S}_j$ and $\tilde{\Pi}_j$:[5)]

$$\tilde{S}_j = \int \mathbf{v}\tilde{f}_j d\mathbf{v} = -\frac{\nu_{jn}}{m_j\omega_j^2}(\nabla_\perp p_j - e_j N_j \mathbf{E}_\perp) - \frac{\nu_{ei}\nabla_\perp p}{m_e\omega_e^2}, \qquad (6)$$

$$\tilde{\Pi}_j = \int \frac{m_j v^2}{2}\mathbf{v}\tilde{f}_j\, d\mathbf{v} = \frac{5}{2}T_j\tilde{S}_j - \frac{5}{2}\nu_j N_j T_j \frac{\nabla_\perp T_j}{m_j\omega_j^2}, \qquad (7)$$

where $j = e, i$ and the subscript $\perp$ means that we take the component of a vector perpendicular to the magnetic field; $p = p_e + p_i$; $p_j = N_j T_j$, N_j and T_j are the density and temperature for particles of species j, $\nu_j = \nu_{jn} + \nu_{je} + \nu_{ji}$ is the effective collision frequency and

$$\nu_{ei} = \left(\frac{e_i}{e_e}\right)^2 \nu_{ee}, \quad \nu_{ie} = \frac{m_e}{m_i}\nu_{ei}, \quad \nu_{jj} = \frac{4\sqrt{2\pi}e_j^4 N_j \mathscr{L}}{3m_j^{1/2}T_j^{3/2}}. \qquad (8)$$

The expressions (6) and (7) for the fluxes $\tilde{S}_j$ and $\tilde{\Pi}_j$ are essentially independent of the "fine structure" of the magnetic field and give the well-known expressions for the diffusion coefficient and the thermal conductivity for transport phenomena in systems in which the mixing effect is not important. However, the contribution to the transport phenomena due to mixing is given completely by the fluxes $\overline{S}_j$ and $\overline{\Pi}_j$ and this contribution depends on the degree to which the drift trajectories depart from the magnetic surfaces. The calculation of these fluxes and the determination of the conditions for which these are appreciably greater than the fluxes $\tilde{S}_j$ and $\tilde{\Pi}_j$ will then be the fundamental problem with which we shall be concerned.

We make use of a quasi-toroidal coordinate system r, φ, and ζ (cf. Fig. 1) and assume that the magnetic field can be written as a superposition position of a uniform toroidal field $\mathbf{H}_\zeta = \mathbf{e}_\zeta H_0/(1 + \delta\cos\varphi)$ where $\delta = r/R \ll 1$, and a small stabilizing field $\mathbf{H}_1$, where $H_1 \ll H_0$. Now, Eq. (3) for $\overline{f}_j$ can be solved by successive approximations by expansion in the small parameters δ, H_1/H_0 and $1/\omega_j$. Applying this procedure we can find $\overline{f}_j$ and then the fluxes $\overline{S}_j$ and $\overline{\Pi}_j$, which, in addition to depending on the subscript of the magnetic surface, for example, its mean (with respect to ζ) radius r_0 will, in general, depend on the variables φ and ζ. However, since the rate of plasma loss (or energy loss) is obviously determined by the mean component of the flux $\overline{S}_j$ ($\overline{\Pi}_j$) averaged over a magnetic surface, this flux being perpendicular to the magnetic surface, the expressions for $\overline{S}_j$ and $\overline{\Pi}_j$ that are obtained can be simplified appreciably by taking averages over φ and ζ [for fixed $r_0 = r_0(r, \varphi, \zeta)$][6)] and then projecting in the direction of the normal to the magnetic surface $\mathbf{n}_0 = \nabla r_0/|\nabla r_0|$. The expressions for these averaged "radial" fluxes, which we will denote by $S^j_{r_0}$ and $\Pi^j_{r_0}$ will be given below.

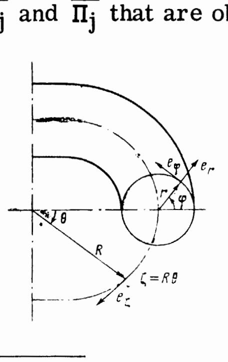

FIG. 1. Coordinate System

2. AXIALLY SYMMETRIC SYSTEMS

Let us assume that the magnetic field is axially symmetric, that is to say, that it is independent of the variable ζ:

$$\mathbf{H} = \mathbf{e}_r 0 + \mathbf{e}_\varphi H_1 + \mathbf{e}_\zeta \frac{H_0}{1 + \delta\cos\varphi},$$

where $\delta = r/R \approx r_0/R \ll 1$. In this relatively simple case we can obtain general expressions for the fluxes $\Pi^j_{r_0}$ and $S^j_{r_0}$ which, in practice, are valid for a plasma with arbitrary density and ionization. However, in view of the fact that these expressions are extremely complicated, we shall make use of them only for a number of particular cases in which they can be simplified appreciably. Having in view the fact that real devices (tokomak, zeta, levitron) are characterized by a rotational transform $i = 2\pi R H_1/r_0 H_0$ which is rather large, we assume in this section that the radial electric field satisfies the condition[7)]

$$c\frac{E_{r_0}}{H_0} \ll v_j\frac{H_1}{H_0}, \qquad v_j = \sqrt{\frac{T_j}{m_j}}. \qquad (9)$$

A. Weakly Ionized Plasma

If the following condition is satisfied:

$$\nu_{jn} \gg \nu_{ei}\delta^{-1}\frac{m_e}{m_j}\frac{T_e + T_i}{T_j}, \qquad j = e, i, \qquad (10)$$

the expressions for the "radial" fluxes of particles and energy are of the following form:

$$S_{r_0}{}^j = -\nu_{jn}N_j\frac{2\pi^2}{i^2}\rho_j^2\Big\{(3 + q_2{}^j + q_1{}^j c_j^2)\Big(\frac{\partial \ln N_j}{\partial r_0} - \frac{e_j E_{r_0}}{T_j}\Big) + \Big(4 + q_3{}^j - \frac{3}{2}q_2{}^j\Big)\frac{\partial \ln T_j}{\partial r_0}\Big\} \qquad (11)$$

$$\Pi_{r_0}{}^j = -\nu_{jn}N_j T_j\frac{2\pi^2}{i^2}\rho_j^2\Big\{\Big(\frac{17}{2} + q_3{}^j + \frac{3}{2}q_1{}^j c_j^2\Big)\Big(\frac{\partial \ln N_j}{\partial r_0} - \frac{e_j E_{r_0}}{T_j}\Big) + \Big(20 + q_4{}^j - \frac{3}{2}q_3{}^j\Big)\frac{\partial \ln T_j}{\partial r_0}\Big\}, \qquad (12)$$

where

$$\rho_j^2 = \frac{v_j^2}{\omega_j^2}, \qquad c_j = \nu_j\frac{r_0 H_0}{v_j H_1} = \nu_j\frac{2\pi R}{i v_j}; \qquad (13)$$

$$q_s{}^j = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}x^s\,dx\int_0^\infty \frac{e^{-t^2/2}\,dt}{c_j^2 + t^2 + \delta x/4} \approx \begin{cases} s!/c_j^2 & \text{for} \quad c_j \gg 1 \\ \sqrt{\frac{\pi}{2}}\,s!/c_j & \text{for} \quad \delta^{1/2}/2 \ll c_j \ll 1, \\ \sqrt{2\pi}\,(s - 1/2)!/\delta^{1/2} & \text{for} \quad c_j \ll \delta^{1/2}/2 \end{cases}$$

where the magnitude of the electric field is determined,

[5)]We note that after conversion to the drift variable and separation of the collision integral into mean and oscillating parts, the quantity $\tilde{S}t_j$: contains terms that are proportional to the electric field and the gradients of the density, temperature, and magnetic field.

[6)]The equality $r_0(r, \varphi, \zeta)$ = const obviously represents the equation for a magnetic surface.

[7)]In the opposite limit the fluxes are reduced appreciably (especially in the region $c_j \ll 1$).

in the usual way, from the condition for ambipolar diffusion $S_{r_0}^e = S_{r_0}^i$.

It follows from Eqs. (11)–(13) that when $c_j \gg 1$ the diffusion coefficient and the thermal conductivity coincide with those obtained in the hydrodynamic model; however, when the collision frequency ν_{jn} is diminished, the diffusion coefficient and the thermal conductivity become much larger than the latter (cf. Fig. 2a). The greatest difference appears in the region $c \ll \delta^{1/2}/2$ where the expressions in (11) and (12) lead to a diffusion coefficient D_j and a thermal conductivity κ_j that are approximately $\delta^{-1/2}$ times larger than those obtained in the hydrodynamic theory when extrapolated to the region of small ν_{jn}.

B. Highly Ionized Plasma

We now consider the case of a highly ionized plasma, in which the following conditions are satisfied:

$$\nu_{jn} \ll \frac{m_e}{m_j} \nu_{ei} \frac{T_e + T_i}{T_j}, \quad j = e, i. \tag{14}$$

For arbitrary collision frequencies the expressions for the particle flux and the energy flux are extremely complicated. However, in view of the fact that these expressions are already known[3–5] for the region $c_j \gg 1$, in which the conditions for the hydrodynamic approximation are satisfied, here we shall limit ourselves to low frequencies, in which kinetic theory effects become important.

Thus, assuming that $c_j = \nu_j r_0 H_0 / v_j H_1 \ll 1$ and neglecting quantities of order unity as compared with $\hat{q}_s(c_j) \gg 1$, we find

$$S_{r_0}{}^e = S_{r_0}{}^i = -\nu_{ei} \frac{2\pi^2}{\iota^2 m_e \omega_e^2} \left\{ \frac{\nu_e \nu_i \hat{q}_2(c_e) \hat{q}_2(c_i)}{\nu_e \nu_{ie} \hat{q}_2(c_e) + \nu_i \nu_{ei} \hat{q}_2(c_i)} \right\}$$
$$\times \sum_j \left\{ T_j \frac{\partial N_j}{\partial r_0} + \left[\frac{\hat{q}_3(c_j)}{\hat{q}_2(c_j)} - \frac{3}{2} \right] N_j \frac{\partial T_j}{\partial r_0} \right\}, \tag{15}$$

$$\Pi_{r_0}{}^j = \frac{\hat{q}_3(c_j)}{\hat{q}_2(c_j)} T_j S_{r_0}{}^j - \nu_j N_j T_j \rho_j^2 \frac{2\pi^2}{\iota^2} \hat{q}_3(c_j) \frac{\partial \ln T_j}{\partial r_0} \tag{16}$$

where the function

$$\hat{q}_s(c_j) = \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{e^{-x} x^s \, dx}{(c_j^2 + \delta^3 x/4)^{1/2}} \approx \begin{cases} \sqrt{\frac{\pi}{2}} \frac{s!}{c_j} & \text{for } \frac{\delta^{3/2}}{2} \ll c_j \ll 1 \\ \frac{\sqrt{2\pi}(s - 1/2)!}{\delta^{3/2}} & \text{for } c_j \ll \frac{\delta^{3/2}}{2} \end{cases} \tag{17}$$

It follows from Eq. (15) that the electron flux and the ion flux do not depend explicitly on the strength of the electric field E_{r_0} and that these fluxes are equal, that is to say, that to first order in ω_j^{-2} the diffusion is always ambipolar (cf. [7]). The general nature of the dependence of the diffusion coefficient D_j and the thermal conductivity κ_j on collision frequency is similar to the case of a weakly ionized plasma (cf. Fig. 2b).[8)] However,

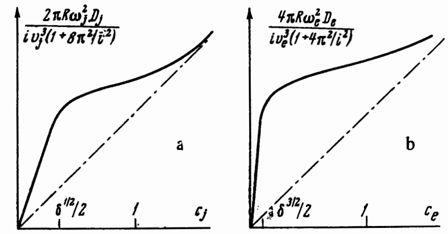

FIG. 2. Qualitative dependence of the diffusion coefficient D_j on the collision frequency $c_j = 2\pi R\nu_j/\iota v_j$ for the fixed values of T_j, r_0, R, ι, and H for axially symmetric systems: a) weakly ionized plasma and b) highly ionized plasma.

the difference between the true and hydrodynamic transport coefficients reaches a maximum in the region of still lower collision frequencies $\nu_j \ll \delta^{3/2} v_j H_1 / 2r_0 H_0$ and is then considerably larger:[9)] for a fully ionized plasma the ratio of the diffusion coefficients and the thermal conductivities is approximately $\delta^{-3/2}$.

Thus, in axially symmetric systems, although the toroidal geometry leads to an increase in the transport coefficients, the dependence of these coefficients on the strength of the magnetic field, with i = const, remains unchanged (D_j, $\kappa_j \sim H_0^{-2}$).

3. AXIALLY ASYMMETRIC SYSTEMS

We now wish to consider transport phenomena in axial asymmetric systems, of which we shall consider the following to be the most characteristic examples: a) the $l = n$ stellarator, and b) a toroidal system with a bumpy field. The specific difference of the first of these systems from an axially symmetric system lies in the fact that the former always exhibits a group of "trapped" particles which oscillate in a region bounded in ζ. In general, these particles do not "see" the rotational transform. As is well known,[13] the trajectories of these particles depart noticeably from the magnetic surfaces so that in the case of low collision frequencies and weak electric fields [cf. Eq. (20)] it is specifically these particles that determine the effectiveness of "mixing" and make the basic contribution to the flux. In bumpy-field systems there is generally no rotational transform, the stabilization of the toroidal drift in such systems being provided by the inhomogeneity in the longitudinal field (that is to say, by the "bump") while the effectiveness of the mixing for strong electric fields [cf. below Eq. (38)] it is determined by all the particles in equal degree.

The results given below apply for a plasma with an arbitrary degree of ionization.[10)]

A. Particle and Energy Fluxes in an l = n Stellarator

As is well known, the magnetic field in an l = n stellarator is given by

$$\mathbf{H} = \frac{\mathbf{e}_\zeta H_0}{1 + \delta \cos \varphi} + \mathbf{H}_1, \quad \mathbf{H}_1 = \nabla \Phi, \tag{18}$$
$$\Phi = \varepsilon \alpha^{-1} H_0 I_n(n\alpha r) \sin [n(\varphi - \alpha\zeta)],$$

[8)] We note that Eqs. (15)-(17) imply that when $\delta^{3/2}/2 \ll c_j \ll 1$ the fluxes are independent of collision frequency. The point here, as in the case of a weakly ionized plasma, is that there is a monotonic increase of these quantities with increasing ν_j in this region as well (cf. Fig. 2b). This can easily be shown if account is taken of small terms $\sim c_j$, which we have neglected in the expression for the fluxes.

[9)] Similar results have been obtained earlier in [7].

[10)] The only exception is the case of very low collision frequencies $\nu_j r_0 \lesssim \gamma V_{\lambda j}$ [(cf. below Eqs. (26) and (27)] in which the differential nature of the Coulomb collision integral becomes important.

where $I_n(x)$ is the modified Bessel function of order n while the parameters ϵ and α characterize the intensity and spatial periodicity of the helical field. If the toroidal geometry is neglected the equation for the magnetic surfaces is given by

$$r_0^2 = r^2 - \frac{2\varepsilon r}{\alpha^2 n}\frac{\partial}{\partial r} I_n(n\alpha r)\cos n(\varphi - \alpha\zeta) = \text{const.} \qquad (19)$$

We assume that the following inequalities are satisfied:

$$\delta \ll 2\varepsilon n I_n, \quad \delta_i \ll 2\pi\alpha R n^2 \varepsilon I_n, \quad \nu_j^2 r_0^2 + V_E^2 \ll \left(\frac{i\delta\nu_j}{2\pi}\right)^2 a_j n\varepsilon I_n, \qquad (20)$$

where $V_E = cE_{r_0}/H_0$; $\nu_j^2 = T_j/m$; i, as before, is the rotational transform, that is to say the angle through which a line of force rotates around the magnetic axis in one circuit around the principal axis of the torus, while

$$a_j = \begin{cases} \nu_j/\nu_{jn} & \text{for } \nu_{jn} \gtrsim \dfrac{\nu_{ei}}{\varepsilon n I_n}\dfrac{m_e}{m_j}\dfrac{T_e+T_i}{T_j} \\ \varepsilon n I_n \dfrac{m_j T_j \nu_j}{m\nu_e \nu_{ei}(T_e+T_i)} & \text{for } \nu_{jn} \lesssim \nu_{ei}\dfrac{m_e}{m_j}\dfrac{1}{\varepsilon n I_n}\dfrac{T_e+T_i}{T_j} \end{cases} \qquad (20')$$

The physical significance of these inequalities is very simple. The first inequality implies that the deviation of the trajectories of the trapped particles from the magnetic surfaces is small. The second implies that the basic role is played by particles which are trapped between maxima of the helical field rather than the longitudinal field. If these inequalities are violated the number of such particles becomes relatively small and the dominant role is played by "toroidal" trapped particles, that is to say, particles which reflect the maxima in the basic longitudinal field H_ζ. Because of the averaging effect in motion along the lines of force, for these particles the local inhomogeneities of the helical field are unimportant. In this case the specific features of the stallarator such as the lack of axial symmetry disappear and the expressions for the fluxes assume forms similar to those obtained above for axially symmetric systems. Finally, the third inequality is a statement of the condition that the basic contribution to the transport processes comes from trapped particles rather than transiting particles.

In addition to assuming the conditions in (20), we shall also assume that the equipotentials are close to the magnetic surfaces, or, more precisely, that the tangential component $E_t \ll T_j|e_j|R$. This is a reasonable assumption for a rarefield plasma because of the very high conductivity along the magnetic lines of force.

If the conditions given above are satisfied the expressions for the particle flux and the energy flux become

$$S_{r_0}{}^j = -\nu_j N_j\left(\frac{r_0}{R}\right)^2\sqrt{\frac{2\varepsilon n I_n}{\pi}}\rho_j^2 v_j^2 Q_2{}^j\left\{\frac{\partial \ln N_j}{\partial r_0} + \left[\frac{Q_3{}^j}{Q_2{}^j} - \frac{3}{2}\right]\frac{\partial \ln T_j}{\partial r_0} - \frac{e_j E_{r_0}}{T_j}\right\}, \qquad (21)$$

$$\Pi_{r_0}{}^j = -\nu_j N_j T_j\left(\frac{r_0}{R}\right)^2\sqrt{\frac{2\varepsilon n I_n}{\pi}}\rho_j^2 v_j^2 Q_3{}^j\left\{\frac{\partial \ln N_j}{\partial r_0} + \left[\frac{Q_4{}^j}{Q_3{}^j} - \frac{3}{2}\right]\frac{\partial \ln T_j}{\partial r_0} - \frac{e_j E_{r_0}}{T_j}\right\}. \qquad (22)$$

The explicit form of the functions Q_s^j that appear in these expressions depends on the magnitude of the self-consistent "radial" electric field E_{r_0} which, in the stationary diffusion regime, must be determined, in the usual way, from the requirement that the electron and ion fluxes be equal.[11] In the general case the expression for Q_s^j is extremely complicated and we shall consider it here only for two particular cases. If the electric field is sufficiently large so that

$$\frac{|e_j r_0 E_{r_0}|}{T_j} \gg \lambda_n = \varepsilon n r_0 \frac{\partial}{\partial r_0} I_n(n\alpha r_0), \qquad (23)$$

the expression for Q_s^j assumes the simple form

$$Q_s{}^j = \int_0^\infty e^{-x} x^{s+1/2}\frac{dx}{V_E^2 + r_0^2 \nu_j^2} = \frac{(s+1/2)!}{V_E^2 + r_0^2\nu_j^2}, \quad \frac{Q_{s+1}^j}{Q_s{}^j} = s + \frac{3}{2}. \qquad (24)$$

In the opposite limit (weak fields), in which $|e_j r_0 E_{r_0}| \ll \lambda_n T_j$ or, what is the same thing,

$$V_E^2 \ll V_{\lambda j}^2 = \lambda_n^2 v_j^2 \frac{\rho_j^2}{r_0^2}, \qquad (25)$$

the functions Q_s^j are determined as follows:[12]

$$Q_s{}^j = \frac{1}{\pi}\int_0^\infty e^{-x} x^{s+1/2}\,dx \int_0^1 \frac{K(t)\,dt^2}{[\gamma^2 + (t^2 - t_0^2)^2]\,x^2 V_{\lambda j}^2 + r_0^2 \nu_j^2}$$
$$\approx \begin{cases} \dfrac{(s+1/2)!}{r_0^2\nu_j^2} & \text{for } \nu_j r_0 \gg V_{\lambda j} \\ \dfrac{K(t_0)(s-1/2)!}{r_0 \nu_j V_{\lambda j}} & \text{for } \gamma V_{\lambda j} \ll \nu_j r_0 \ll V_{\lambda j}, \\ \dfrac{K(t_0)(s-3/2)!}{\gamma V_{\lambda j}^2} & \text{for } \nu_j r_0 \ll \gamma V_{\lambda j} \end{cases} \qquad (26)$$

where $t_0 = 0.91$ is the root of the equation $2E(t) - K(t) = 0$, K and E are the complete elliptic integrals of the first and second kind [$K(t_0) = 2.3$], and the parameter $\gamma^2 \ll 1$ depends on the degree of ionization of the plasma and is of the following order of magnitude:

$$\gamma^2 \approx \begin{cases} \dfrac{\delta}{4n\varepsilon I_n} & \text{for } \nu_{jn} > \dfrac{4\varepsilon n I_n}{\delta}\nu_{jj} \\ \left(\dfrac{\delta}{4n\varepsilon I_n}\right)^3 & \text{for } \nu_{jn} < \nu_{jj}. \end{cases} \qquad (27)$$

The difference between Eqs. (24) and (26) is connected physically with the different nature of the motion of the trapped particles in the cases described by (23) and (25). In a strong field in which the condition in (23) is satisfied the electric drift rate exceeds the magnetic drift rate and the drift trajectories of all particles are approximately circles. In the inverse case there is a group of so-called resonance particles[13] whose trajectories are similar to those of trapped particles in axially symmetric systems, being crescent-like. These do not encircle the axis of the system. These resonance particles, like the trapped particles in axially symmetric systems, exhibit the most marked deviations from magnetic surfaces and (in the case of low collision frequencies) make the basic contribution to the particle flux and the energy flux. This is the origin of the similarity between Eq. (26) for Q_s^j and Eqs. (13) and (17)

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[11] In this case the diffusion rate of the plasma as a whole is obviously determined by the diffusion of the component that has the slower diffusion rate.

[12] We note that in the particular case n = 3, $\nu_{jn} = 0$, T_j = const and $\gamma V_{\lambda j} \ll \nu_j r_0 \ll V_{\lambda j}$ Eq. (21) leads to the results that have been obtained earlier in [8]

for q_s^j and $\hat{q}_s$ [cf. Eq. (27) and Eqs. (10) and (14)]. We note that if the quantity γ is not very small, then for quantitative estimates Eqs. (24) and (26) can be approximated by a single expression which is valid for arbitrary values of V_E. Under these conditions we find[13)]

$$Q_s^j = \frac{(s+{}^1/_2)!}{V_E^2 + r_0^2\nu_j^2 + V_{\lambda j}}, \quad \frac{Q_{s+1}^j}{Q_s^j} = s + \frac{3}{2}. \qquad (27')$$

In discussing Eqs. (21) and (22) we may note the following most characteristic features of the transport processes in a toroidal stellarator.

1. In contrast with axially symmetric systems, the particle flux in the stellarator is proportional to the total collision frequency $\nu_j = \nu_{jn} + \nu_{je} + \nu_{ji}$ which means that even in the presence of only one particle species ($\nu_j = \nu_{jj}$) and a uniform temperature ($\partial T_j/\partial r_0 = 0$) there is no equilibrium state in a toroidal stellarator. In accordance with this result, even in the absence of a neutral gas ($\nu_{jn} = 0$) and with $E_{r_0} = 0$ the electron and ion fluxes are generally not equal; this leads to the production of a radial electric field E_{r_0} which, in the stationary diffusion regime, is determined by the relation $S^e_{r_0} = S^i_{r_0}$.

2. As the collision frequency is increased from zero to

$$\nu_j^* \approx \frac{(V_E^2 + V_{\lambda j}^2)^{1/2}}{r_0} \qquad (28)$$

the diffusion coefficient first increases essentially linearly with collision frequency ν_j and then, when $\nu_j \approx \nu_j^*$, reaches a maximum value of order

$$D_j^m = \frac{15}{8}\sqrt{\frac{\varepsilon n I_n}{2}}\left(\frac{r_0}{R}\right)^2 \rho_j^2 \frac{v_j^2}{r_0\sqrt{V_E^2 + V_{\lambda j}^2}}, \qquad (29)$$

Finally, it starts to diminish to a value for which the last inequality in (20) is violated, after which, as follows from the hydrodynamic analysis, it again increases (cf. Fig. 3). We note that for a sufficiently strong magnetic field D_j^m can be many orders of magnitude greater than the diffusion coefficient $D_{ny} \approx \nu_{ei}\rho_e^2(1 + 4\pi^2/i^2)$ as computed in the hydrodynamic approximation.

3. The third characteristic feature of transport processes in the systems being considered is the existence of a region in which the particle flux and the energy flux are essentially independent of the strength of the magnetic field H. This is the region of low collision frequencies or weak magnetic fields, in which the particle flux and the energy flux increase linearly with frequency. This region is defined by the inequality (c is the velocity of light)

$$H < H_j^* = \frac{c}{r_0^2\nu_j|e_j|}\begin{cases} |r_0 e_j E_{r_0}| & \text{for } |r_0 e_j E_{r_0}| \gg \lambda_n T_j \\ \gamma\lambda_n T_j & \text{for } |r_0 e_j E_{r_0}| \ll \lambda_n T_j \end{cases} \qquad (30)$$

In this case the diffusion coefficient $D_j \approx D_j^*$ where

$$D_j^* = \nu_j r_0^2\sqrt{\frac{\varepsilon n I_n}{2}}\left(\frac{r_0}{R}\right)^2 \begin{cases} \frac{15}{4}\left(\frac{T_j}{r_0 e_j E_{r_0}}\right)^2 & \text{for } |r_0 e_j E_{r_0}| \gg \lambda_n T_j \\ \frac{K(t_0)}{\gamma\lambda_n^2} & \text{for } |r_0 e_j E_{r_0}| \ll \lambda_n T_j \end{cases} \qquad (31)$$

Thus, as the magnetic field is increased the particle flux and the heat flux will exhibit the following behavior.[14)] When $H < H_j^*$ these fluxes are generally independent of the magnetic field and when $H \gtrsim H_j^*$ they diminish slowly; it is only when $H \gg H_j^*$ that they become inversely proportional to the square of the magnetic field (cf. Fig. 4).

4. The fourth distinguishing feature of the transport coefficients is the rather complicated dependence these coefficients exhibit on temperature which, for various values of the parameters, can vary over rather broad limits. Thus, for example, for a fully ionized plasma in which $\nu_j \sim T_j^{-3/2}$, in the low-frequency region $\nu_j \ll \nu_j^*$ the diffusion coefficient $D_j \sim T_j^{-3/2}$; however, when $\nu_j > \nu_j^*$ we find $D_j \sim T_j^{7/2}$.

5. As we have already indicated when $E_{r_0} = 0$ the electron flux and the ion flux are generally not equal. Consequently, in the stationary regime, in which the diffusion is ambipolar in nature and the electron and ion fluxes are equal, the plasma must become charged. Under these conditions it is found that depending on a number of parameters the charge can be either positive or negative. Analysis of Eq. (21) shows that when[15)]

$$\nu_e > \nu_i, \quad e_e^2 T_i^2 \nu_e < e_i^2 T_e^2 \nu_i \qquad (32)$$

the plasma is always charged positively and for the opposite limit

$$\nu_e < \nu_i, \quad e_e^2 T_i^2 \nu_e > e_i^2 T_e^2 \nu_i \qquad (33)$$

the plasma is always charged negatively. However, in the region of intermediate parameter values, depending on the magnetic field and the density, the plasma charge can be either greater or less than zero. The value of the magnetic field H_p (or density) for which the plasma charge vanishes can be found from the equations

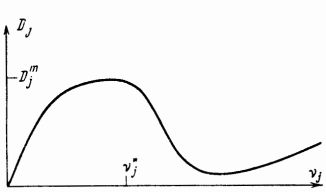

FIG. 3. Qualitative dependence of the diffusion coefficient D_j on the collision frequency for fixed values of γ_j, E_{r_0}, and H for an axially asymmetric system.

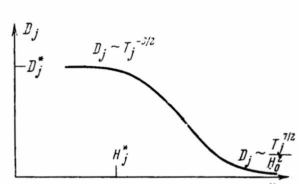

FIG. 4. Qualitative dependence of the diffusion coefficient on the magnetic field for fixed values of γ_j, v_j, and E_{r_0}.

13) This approximation has been used in a work of the author. [14]

14) Obviously this description holds only when the other plasma parameters (density, temperature, electric field, etc.) are independent of the strength of the magnetic field.

15) In a fully ionized hydrogen plasma the condition in (32) is equivalent to the inequality $T_i < 80^{-2/7} T_e$.

$$S_{r_0}{}^e = S_{r_0}{}^i, \quad E_{r_0} = 0. \tag{34}$$

For reasons of simplicity we limit ourselves to the approximate expression for Q_S^j (27′) and find that

$$H_p = \frac{\lambda_n c}{r_0^2 \sqrt{\nu_e \nu_i}} \frac{T_e}{|e_e|} \left\{ \frac{\nu_e - \nu_i}{\nu_e - \nu_i (e_i T_e / e_e T_i)^2} \right\}^{1/2}. \tag{35}$$

Thus, when $\nu_e > \nu_i$ and $\nu_e > \nu_i(e_i T_e/e_e T_i)^2$ in the region of strong magnetic fields $H > H_p$ the plasma is charged negatively (assuming that $E_{r_0} = 0$ in the initial state) while for weak fields $H < H_p$ the plasma is charged positively.

In Fig. 5 we show the qualitative dependence of the dimensionless electric field $x = e_i r_0 E_{r_0}/T_i$ in the plasma as a function of the parameter $\zeta = \nu_e \nu_i \omega_i^2 r_0^4/\nu_i^4$ for various relations between the quantities $A = \nu_e/\nu_i$ and $\beta^2 = e_e^2 T_i^2/e_i^2 T_e^2$. The dashed curve shows the unstable part of the function $\zeta(x)$: in the region where one value of ζ corresponds to two values of x the magnitude of the electric field is determined by the initial conditions.

In concluding this section we wish to present a numerical example for an $l = 2$ stellarator. Substituting the following values in Eq. (21): n = 2, $\epsilon = 0.5$, $r_0 = 4$ cm, $\alpha = 0.12\ \text{cm}^{-1}$, R = 60 cm, $N_e = N_i = 10^{11}\ \text{cm}^{-3}$, $T_e = 16$ eV, $T_i = 25$ eV, $H_0 = 5 \times 10^3$ G and assuming

$$\frac{\partial \ln N_e T_e}{\partial r_0} = \frac{\partial \ln N_i T_i}{\partial r_0} = 3\ \text{см}^{-1},$$

we obtain the following values for the plasma lifetime and the magnitude of the electric field

$$\tau = \frac{\pi r_0^2 N_j}{2\pi r_0 S_{r_0}{}^j} \approx 3 \cdot 10^{-3}\ \text{sec} \quad E_{r_0} \approx -3\ \text{V/cm}$$

We note that the hydrodynamic theory would predict a lifetime that is three orders of magnitude larger; however, the value of the lifetime observed in experiment is very close to the one derived here.

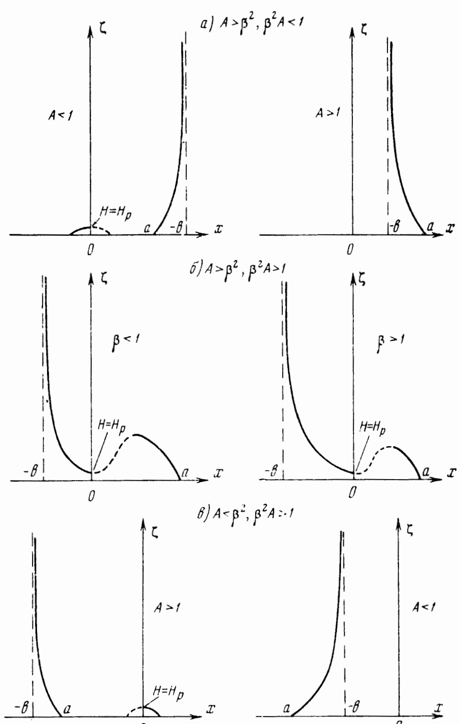

FIG. 5. Qualitative dependence of the electric field on density and magnetic field:

B. Particle Flux and Energy Flux in a Bumpy Field System

A system of this kind is characterized by a magnetic field

$$\mathbf{H} = \frac{\mathbf{e}_\zeta H_0}{1 + \delta \cos \varphi} + \mathbf{H}_1, \quad \mathbf{H}_1 = \nabla \Phi, \qquad \Phi = -\varepsilon \alpha^{-1} H_0 I_0(\alpha r) \sin \alpha \zeta, \tag{36}$$

while a "magnetic surface" can be understood to be a surface defined by the equation

$$r_0^2 = r^2 - \frac{2\varepsilon r}{\alpha^2} \frac{\partial}{\partial r} I_0(\alpha r) \cos \alpha \zeta = \text{const}. \tag{37}$$

In computing the particle flux and the energy flux, in contrast with the earlier section we shall limit ourselves to the case of a strong electric field that satisfies the condition

$$\frac{|e_j r_0 E_{r_0}|}{T_j} \gg \lambda_0 = \varepsilon r_0 \frac{\partial}{\partial r_0} I_0(\alpha r_0). \tag{38}$$

However, we will not necessarily assume that the equipotentials coincide with magnetic surfaces (37) and allow the possibility of small (of order $T_j/|e_j|R$) azimuthal electric fields E_φ.

Thus, if (58) is satisfied and if the following conditions are satisfied:

$$\delta \ll 2\varepsilon I_0, \qquad r_0^2 \nu_j^2 + V_E^2 \ll \left(\frac{\delta r_0 \alpha v_j}{2\pi} \right)^2 \cdot \times \begin{cases} \dfrac{\nu_j}{\nu_{jn}} & \text{for } \nu_{jn} \gtrsim \nu_{ei} \dfrac{m_e}{m_j} \dfrac{T_e + T_i}{T_j} \\ \dfrac{m_j T_j \nu_j}{m_e \nu_{ei} (T_e + T_j)} & \text{for } \nu_{jn} \lesssim \nu_{ei} \dfrac{m_e}{m_j} \dfrac{T_e + T_i}{T_j} \end{cases} \tag{39}$$

the expressions for the particle flux and energy flux assume the form

$$S_{r_0}{}^j = -\frac{1}{6} \nu_j N_j \left(\frac{r_0}{R} \right)^2 \rho_j^2 \frac{v_j^2}{r_0^2 \nu_j^2 + V_E^2} \left\{ \frac{\partial \ln N_j T_j^2}{\partial r_0} - \frac{e_j E_{r_0}}{T_j} \right\}, \tag{40}$$

$$\Pi_{r_0}{}^j = -\frac{7}{12} \nu_j N_j T_j \left(\frac{r_0}{R} \right)^2 \rho_j^2 \frac{v_j^2}{r_0^2 \nu_j^2 + V_E^2} \left\{ \frac{\partial \ln N_j T_j^{101/7}}{\partial r_0} - \frac{e_j E_{r_0}}{T_j} \right\}. \tag{41}$$

A comparison of these expressions with Eqs. (21) and (22) show that the structure of the fluxes in the case of a bumpy field is very similar to the structure of the fluxes in a stellarator field, differing from the latter only in numerical factors. In accordance with this result all of the remarks made in the preceding section with regard to the diffusion coefficient and the thermal conductivity remain valid for this case.

In conclusion we note that in Eqs. (40) and (41) we have given the component of the electric field E_{r_0} that is normal to the "magnetic surface." This means that a weak lack of equivalence of the "magnetic surfaces" ($E_\varphi \neq 0$) will not have an effect on the magnitude of the particle and energy fluxes.

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