# *UNIFIED DESCRIPTION OF GRAVITATION AND ELECTROMAGNETISM*

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Submitted September 29, 1968

Zh. Eksp. Teor. Fiz. 56, 1046—1056 (March, 1969)

A new variant is proposed for a unified geometrical description of gravitational and electromagnetic fields. Both fields are described in an affinely-connected space with non-zero tensors of curvature and torsion. The expression for the coefficients of affine connection obtained depends on both fields. The extent of the effect of the electromagnetic field on the geometry is determined by a new constant with the dimensions of length, $l_0$, for which experiment sets the limitation $l_0 \lesssim 10^{-14}$ cm. Possible experiments which would permit a judgement on the validity of the theory are discussed. Finally, the author discusses the possible relation between geometrization of the electromagnetic field and violation of T-invariance.

## 1. INTRODUCTION

SINCE Einstein's development of the basic theory of general relativity there has been an almost endless number of attempts to generalize the original theory so as to give geometrical meaning to the electromagnetic field. Such schemes came to be known as unified field theories (cf. e.g., Einstein, Collected Works[1]). In our times, the use of this all-inclusive term for theories of gravitation and electromagnetism seems excessive since other elementary interactions (strong, weak) are now known to exist; yet, in our view such a theory, in the original sense and in itself, retains significant interest. The justification for studying a unified description of the electromagnetic and gravitational field, in as a first step towards a possible geometrical theory of all interactions[1)], is to be found in the comparatively simple interpretation of the electromagnetic field (as compared to lepton and hadron fields) and the similarity of certain of its properties with those of the gravitational field: both are describable by tensors of second order, correspond to zero rest-mass, satisfy gauge-invariance conditions, etc.

In this paper we examine a new method for describing gravitation and electromagnetism. Our procedure is based on the hypothesis that the gravitational and electromagnetic fields (the latter even in the absence of the gravitational one) constitute a measure of the geometry of space-time. Gravitation without the electromagnetic field is described by the general theory of relativity. The geometrical description of the electromagnetic field, in the absence of the gravitational field, has been developed in our previous article[3] (henceforth referred to as I). Here we formulate a unified geometrical theory which contains, as special cases, the general theory of relativity and the scheme developed in I.

Since it is neither our purpose nor feasible to recount here the content of our previous work[1)], we shall only present a brief account of the essential assumptions and results: We investigate a Minkowski space (metric defined by the line element $ds^2 = (dx^0)^2 - d\mathbf{x}^2$) with affine connection (defined by the operation of parallel-transfer $\delta A^i = -L^i_{jk} A^j \delta x^k$) and absolute parallelism (the curvature tensor $B^i_{jkl}$, defined by the coefficients of affine-connection $L^i_{jk}$, cf. Eq. (2.4), is zero). To avoid misunderstanding, we emphasize that our space is non-Riemannian (we do not impose the conditions $L^i_{jk} = L^i_{kj}$ and the $L^i_{jk}$ are not the Christoffel symbols, for which we reserve the notation $\Gamma^i_{jk}$).

Under the foregoing assumptions the most general form of $L^i_{jk}$

$$L^i_{jk} = \partial_k F_{j\cdot}^{\cdot s}\,(\delta^i_s + F^{i\cdot}_{\cdot s}), \tag{1.1}$$

where $\partial_k = \partial/\partial x^k$, $F_{j\cdot}^{\cdot s} = \delta^{si} F_{ji}$, $F^{i\cdot}_{\cdot s} = \delta^{im} F_{ms}$ ($\delta^{ij} = 0$, $i \neq j$, $\delta^{00} = 1$, $\delta^{ii} = -1$, $i = 1, 2, 3$) and the tensor $F_{mn}$ is such that the matrix $I + F$ is orthogonal. The orthogonality condition imposes 10 restraints on the tensor $F_{mn}$, leaving only 6 free components. We may select the components antisymmetric part of $F_{mn}$, viz., $f_{mn} = \frac{1}{2}(F_{mn} - F_{nm})$, as the free ones, and then relate the tensor $f_{mn}$ to the electromagnetic field $E_{mn}$; the simplest assumption, of course, is that $f_{mn}$ is simply proportional to $E_{mn}$ (cf. infra):

$$f_{mn} = \lambda E_{mn}, \quad \lambda = \pm l_0^2/e. \tag{1.2}$$

Here we introduce a new constant $l_0$, with the dimensions of length; the appearance of the electron charge e in the definition arises only from considerations of convenience.

A characteristic feature of the affinity (1.1) is that it leads to a non-vanishing torsion tensor of the space:

$$\Omega^i_{jk} = \frac{1}{2}(L^i_{jk} - L^i_{kj}) \neq 0. \tag{1.3}$$

In this way electromagnetism is described in a space with torsion but without curvature, in contrast to gravitation in which the converse situation obtains. The natural union of these two cases is a common space with both curvature and torsion.

## 2. PARAMETRIZATION OF THE AFFINITY

We shall study a 4-dimensional space with coordinates $x = \{x^0, x^1, x^2, x^3\}$[2)] and the standard metric form

[1)]Earlier attempts at geometrization were made, for example, for the case of weak interactions [2].

[2)]Information on spaces with affine-connection can be found in standard works (cf. [4]).

(and notation)

$$ds^2 = g_{ij}dx^i dx^j. \tag{2.1}$$

The operations of parallel-transfer and covariant differentiation are, as usual, determined by the coefficients $L^i_{jk}$, which are taken to transform according to:

$$L^i_{jk} = \frac{\partial x^i}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^j}\frac{\partial x'^\lambda}{\partial x^k}L'^\mu_{\nu\lambda} + \frac{\partial x^i}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^j\,\partial x^k}. \tag{2.2}$$

In order that the metric tensor $g_{ij}(x)$ obtain from the metric tensor $g'_{ij}(x')$ by the operation of parallel-transfer, we require

$$g_{ij;\,k} = 0; \tag{2.3}$$

where the notation ;k indicates covariant differentiation. These conditions establish the correspondence between the metric properties of the space and the properties of parallel-transfer. As is well-known, the coefficients $L^i_{jk}$ do not constitute a tensor but from them we can construct two tensors which define the geometry of the space, viz., the curvature tensor

$$B^i_{jkl} = -\partial_l L^i_{jk} + \partial_k L^i_{jl} - L^s_{jk}L^i_{sl} + L^s_{jl}L^i_{sk} \tag{2.4}$$

and the torsion tensor given by Eq. (1.3). If we impose the conditions $\Omega^i_{jk} = 0$, then we have a Riemannian space and from Eq. (2.3) the coefficients of the affine-connection are the usual Christoffel symbols

$$L^i_{jk} = \Gamma^i_{jk} \equiv {}^1/_2\, g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) \tag{2.5}$$

and the space is that of general relativity.

Here, we shall not assume that $\Omega = 0$, i.e., we allow a non-symmetrical affine-connection and, of course, Eq. (2.5) does not apply.

Since our goal is a unified description of gravitation and electromagnetism, the coefficients $L^i_{jk}$ must now be determined not only by the metric tensor $g_{ij}$ (Riemannian case) but also a certain quantity describing the electromagnetic field. Based on our prior work, as sketched in the introduction, we take for such a quantity a second-rank tensor $F_{mn}$ on which we impose the conditions:

$$(\delta^k_s + F^{k\cdot}_{\cdot s})(\delta^s_l + F^{\cdot s}_{l\cdot}) = \delta^k_l \tag{2.6}$$

which generalize the previous orthogonality condition; the raising and lowering of indices following the rules: $F^{k\cdot}_{\cdot s} = g^{km}F_{ms}$, $F^{\cdot s}_{l\cdot} = g^{sm}F_{lm}$.

Thus, our first task is to determine an affinity depending on $g_{ij}$, $F_{mn}$, and their derivatives, and satisfying the transformation law of Eq. (2.2) as well as the conditions given by Eq. (2.3). Furthermore, it is clear that when $F_{mn} = 0$ we must obtain Eq. (2.5) and when $g_{ij} = \delta_{ij}$ we must obtain Eq. (1.1). Various methods may be used to obtain the form of the affine-connection satisfying the conditions imposed. Here we describe a method, based on the Lagrangian variational procedure, for a Lagrangian suitably generalized to our problem from the original Einstein form, depending on the invariance of the Lagrangian with respect to variations of the $L^i_{jk}$, $g_{ij}$ and $F_{mn}$.

It is well-known[1] that in the general theory of relativity one may consider the $g_{ij}$ and $L^i_{jk}$ as independent at first and that the (Einstein) Lagrangian density

$$\mathcal{L}_0 = \sqrt{g}\, B^i_{jik}\, g^{jk} \tag{2.7}$$

(where the notation is conventional and $B^i_{jkl}$ is given by Eq. (2.4)) yields Eq. (2.5) under variation of the $L^i_{jk}$ and the gravitational field equations under variation of the $g^{ik}$.

Now, as indicated above, we propose to generalize the Lagrangian (2.7) to the form:

$$\mathcal{L} = \sqrt{g}\, B^i_{jlk}(\delta^l_i + F^{\cdot l}_{i\cdot})(g^{jk} + F^{jk}), \tag{2.8}$$

and we shall try to find the expression for $L^i_{jk}$ in terms of $g_{ij}$ and $F_{ij}$ as solutions of the equations

$$\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i L^m_{nk})} - \frac{\partial \mathcal{L}}{\partial L^m_{nk}} = 0. \tag{2.9}$$

After obtaining the solution we shall show that the necessary conditions, as formulated in the preceding section, are satisfied. With the use of Eq. (2.6) to simplify the result, Eq. (2.9) gives us

$$L^n_{ml} = {}^1/_2\,(g^{kn} + F^{nk})(\delta^s_m + F^{\cdot s}_{m\cdot})(\partial_l g_{ks} + \partial_s g_{kl} - \partial_k g_{sl}) + (\delta^n_k + F^{n\cdot}_{\cdot k})\,\partial_l F^{\cdot k}_{m\cdot} \tag{2.10}$$

which, using Eq. (2.5) for $\Gamma^i_{jk}$, can be reduced to the more convenient form

$$L^n_{ml} = (\delta^n_r + F^{n\cdot}_{\cdot r})(\delta^s_m + F^{\cdot s}_{m\cdot})\Gamma^r_{sl} + (\delta^n_k + F^{n\cdot}_{\cdot k})\,\partial_l F^{\cdot k}_{m\cdot}. \tag{2.11}$$

This is the sought expression for the $L^i_{jk}$ in terms of the $g_{ij}$ and $F_{ij}$. We note that Eq. (2.11) could be obtained using only the general properties of the affinity without reference to the Lagrangian formalism; however, our derivation reflects better the physical nature of the theory.

## 3. CHARACTERISTICS OF THE AFFINITY. EQUATIONS OF THE GRAVITATIONAL FIELD.

First of all, it is easy to verify that the affinity (2.11) reduces to well-known results in the appropriate special cases. When there is no electromagnetic field $F_{mn} = 0$ and $L^i_{jk} = \Gamma^i_{jk}$, as required for a purely gravitational field. When there is no gravitational field one can always choose a system of coordinates such that $\Gamma^i_{jk} = 0$ everywhere, and $L^i_{jk}$ reduces to (1.1) as required for a purely electromagnetic field.

We shall, further, check the validity of the transformation law (2.2), i.e., we shall verify that (2.11) is an affinity. Under the transformation $x \to x'$ the quantities appearing in Eq. (2.11) transform according to:

$$F^{\cdot s}_{m\cdot} = \frac{\partial x^s}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^m}F'^{\cdot\alpha}_{\beta\cdot};$$

$$\Gamma^r_{sl} = \frac{\partial x^r}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^s}\frac{\partial x'^\gamma}{\partial x^l}\Gamma'^\alpha_{\beta\gamma} + \frac{\partial x^r}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^s\,\partial x^l}. \tag{3.1}$$

Substituting these expressions in Eq. (2.11) we find that certain terms mutually cancel and the rest take the form

$$L^n_{mk} = \frac{\partial x^n}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^m}\frac{\partial x'^\gamma}{\partial x^k}L'^\alpha_{\beta\gamma} + \frac{\partial x^n}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^m\,\partial x^k}$$
$$+ \frac{\partial x^n}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^k\,\partial x^m}(F'^{\mu\cdot}_{\cdot\nu} + F'^{\cdot\mu}_{\nu\cdot} + F'^{\cdot\alpha}_{\nu\cdot}F'^{\mu\cdot}_{\cdot\alpha}). \tag{3.2}$$

The last term is identically zero by virtue of the conditions of Eq. (2.6), and this completes the proof.

In similar fashion, it is not difficult to verify that the second set of conditions, Eq. (2.3), are also satisfied, viz.,

$$g_{mn;\,k} = \partial_k g_{mn} - L^l_{mk} g_{nl} - L^l_{nk} g_{ml} = 0. \quad (3.3)$$

Here, again, the conditions of Eq. (2.6) are essential.

Thus, the affine-connection (2.11) derived, satisfies all the imposed conditions and we can formulate our first basic postulate, viz., the affinity of a space in the presence of both gravitational and electromagnetic fields[3] is given by (2.11).

As remarked before, the affinity determines both curvature and torsion tensor. The latter is given by

$$\Omega^m_{nk} = {}^1/_2 (\delta^m_l + F^{m}_{\cdot\cdot l}) [\partial_k F_{n\cdot}^{\cdot l} - \partial_n F_{k\cdot}^{\cdot l} + F_{n\cdot}^{\cdot s} \Gamma^l_{sk} - F_{k\cdot}^{\cdot s} \Gamma^l_{sn}]. \quad (3.4)$$

To obtain the curvature tensor one may substitute (2.11) in (2.4) and carry out the necessary calculations; however, we shall take a simpler approach. We write (2.11) in the form

$$\partial_k F_{m\cdot}^{\cdot l} = (\delta^l_n + F_{n\cdot}^{\cdot l}) L^n_{mk} - (\delta^r_m + F_{m\cdot}^{\cdot r}) \Gamma^l_{rk} \quad (3.5)$$

and use the obvious relation (condition of integrability of Eq. (3.5))

$$\partial_s [(\delta^l_n + F_{n\cdot}^{\cdot l}) L^n_{mk} - (\delta^r_m + F_{m\cdot}^{\cdot r}) \Gamma^l_{rk}] - \partial_k [(\delta^l_n + F_{n\cdot}^{\cdot l}) L^n_{ms} - (\delta^r_m + F_{m\cdot}^{\cdot r}) \Gamma^l_{rs}] = 0.$$

After differentiating and again using Eq. (3.5), we obtain

$$(\delta^l_n + F_{n\cdot}^{\cdot l}) B^n_{msk} - (\delta^r_m + F_{m\cdot}^{\cdot r}) R^l_{rsk} = 0,$$

where $R^l_{rsk}$, constructed from the Riemannian $\Gamma^i_{jk}$, is the familiar curvature tensor of general relativity. Hence, finally

$$B^n_{msk} = (\delta^r_m + F_{m\cdot}^{\cdot r}) (\delta^n_l + F^{n}_{\cdot\cdot l}) R^l_{rsk}. \quad (3.6)$$

With this result at hand, we can vary the Lagrangian (2.8) with respect to the $g^{mn}$ to obtain the gravitational field equations. Let us rewrite the Lagrangian (2.8) in the form

$$\mathcal{L} = \sqrt{g}\, (\delta^k_i + F_{i\cdot}^{\cdot k}) (\delta^j_s + F_{\cdot s}^{j\cdot}) g^{sl} B^i_{jlk}.$$

The tensor $B^i_{jkl}$ depends only on the $L^i_{jk}$, and only the factor preceding it needs to be considered in the variation of the $g^{mn}$. Then, taking into account (3.6) and (2.6), we find

$$R_{mn} - \frac{1}{2} g_{mn} R + \frac{\partial F_{i\cdot}^{\cdot\alpha}}{\partial g^{mn}} (\delta^i_\beta + F^{i}_{\cdot\beta}) g^{\beta\mu} g^{sl} (R_{\mu sl\alpha} - R_{s\alpha\mu l}) = 0; \quad (3.7)$$

where $R_{mn} = R^i_{min}$, $R = g^{mn} R_{mn}$, $R_{\mu sl\alpha} = g_{\mu\beta} R^\beta_{sl\alpha}$. The last term in Eq. (3.7) is zero because of the symmetry properties of the tensor $R_{\mu sl\alpha}$[1]. Equation (3.7) thus reduces to the Einstein gravitational field equations

$$R_{mn} - {}^1/_2 g_{mn} R = 0 \quad (3.8)$$

This result is easily understood since the Lagrangian (2.8), taking account of Eq. (3.6) and the subsidiary conditions of Eq. (2.6), numerically coincides with the Lagrangian of general relativity:

$$\mathcal{L} = \sqrt{g} R \quad (3.9)$$

This last result is an indication of the consistency of the considerations on which our approach is based; indeed, we can obtain the result (3.8) in two ways. The first method developed in the text need not be recapitulated; the second method would be to assume the form of the affinity (2.11) from the very outset, obtain the Lagrangian (3.9), which depends only on the metric tensor and its derivatives, and then proceed to Eq. (3.8) via the familiar variation with respect to the $g^{mn}$. The fact that our Lagrangian (2.8) actually turns out to be the well-known Lagrangian (3.9) of gravitational theory seems very important. We have shown that the usual Lagrangian is not only compatible with the Riemannian affinity $\Gamma^i_{jk}$ but also with the non-Riemannian affinity (2.11) which corresponds to a space of non-zero torsion describing both gravitational and electromagnetic fields. Thus we find that in the general theory of relativity there exists an arbitrariness due to which the electromagnetic field can be introduced and given a geometrical meaning.

To conclude this section we note that variation of the Lagrangian (2.8) with respect to $F_{mn}$ leads to the identity 0 = 0. Hence, to obtain the electromagnetic field equations we must introduce a free-field electromagnetic Lagrangian whose possible form we shall consider below.

## 4. THE BIANCHI IDENTITIES AND THE VECTOR POTENTIAL OF THE ELECTROMAGNETIC FIELD

As is well-known[4], the curvature tensor $B^i_{jkl}$ and torsion tensor $\Omega^i_{jk}$ satisfy the Bianchi identities which are conditions of integrability of differential geometry. These identifies have the form

$$(B^k_{h\lambda\mu;\,\nu} + B^k_{h\lambda\rho} \Omega^\rho_{\mu\nu}) \varepsilon^{\alpha\lambda\mu\nu} = 0, \quad (4.1)$$

$$(B^k_{\lambda\mu\nu} + \Omega^k_{\lambda\mu;\,\nu} - \Omega^k_{\lambda\rho} \Omega^\rho_{\mu\nu}) \varepsilon^{\alpha\lambda\mu\nu} = 0, \quad (4.2)$$

where $\epsilon^{\alpha\lambda\mu\nu}$ denotes the totally antisymmetric pseudotensor. If we substitute Eq. (3.6) in Eq. (4.1) we obtain

$$R^k_{h\lambda\mu|\nu} \varepsilon^{\alpha\lambda\mu\nu} = 0 \quad (4.3)$$

where the notation $|\nu$ denotes covariant differentiation with respect to the Riemannian affinity $\Gamma^i_{jk}$. Hence, the first Bianchi identity is imposed only on the gravitational field and yields nothing new compared with the original Einstein theory.

The second identity (4.2), however, is more informative and, as we shall see, will permit us to find additional conditions allowing us to connect the tensor $F_{mn}$ with the electromagnetic field tensor $E_{mn}$. Since $F_{mn}$ has only 6 independent components (cf. Eq. (2.6)) the sought-for connection must be unique.

The second set of Maxwell equations is

$$\varepsilon^{\alpha\mu\nu\sigma} \partial_\mu E_{\nu\sigma} = 0. \quad (4.4)$$

We look for a generalization of the form

$$\varepsilon^{\alpha\mu\nu\sigma} T_{\mu\nu\sigma} = 0, \quad (4.5)$$

where $T_{\mu\nu\sigma}$ is a certain tensor to be constructed from geometrical quantities. We note that the basic geometrical quantity with three indices is the torsion tensor $\Omega^\rho_{\mu\nu}$. In order to introduce the torsion tensor somehow into a relation of the form (4.5) we have to lower its contravariant index by means of some second-rank

[3]Further on, we shall study the relation of the tensor $F_{mn}$ to the electromagnetic field.

covariant tensor. Let us assume that (4.5) has the form

$$\varepsilon^{\alpha\mu\nu\sigma}(g_{\rho\mu} + aF_{\rho\mu})\,\Omega^{\rho}_{\nu\sigma} = 0, \tag{4.6}$$

where a is a number which we shall determine from the conditions imposed by the second Bianchi identity (4.2). Using (4.2) we calculate the divergence of (4.6), viz.,

$$\partial_\alpha[\varepsilon^{\alpha\mu\nu\sigma}(g_{\rho\mu} + aF_{\rho\mu})\,\Omega^{\rho}_{\nu\sigma}] = -\varepsilon^{\alpha\mu\nu\sigma}(\delta^{\varkappa}_{\nu} + F^{\cdot\varkappa}_{\nu\cdot})\,R_{\mu\varkappa\sigma\alpha} + (a-1)\,\varepsilon^{\alpha\mu\nu\sigma}[F^{\cdot k}_{\mu\cdot}F^{\cdot r}_{\nu\cdot}R_{kr\sigma\alpha} + \Omega^{\varkappa}_{\nu\sigma}g_{\mu\rho}(L^{\rho}_{\varkappa\alpha} - \Gamma^{\rho}_{\varkappa\alpha})]. \tag{4.7}$$

The right side of this equation is zero in view of (4.6). The first term on the right hand side of (4.7) is zero because of the properties of $R_{\mu\kappa\sigma\alpha}$, and in the second term the factor proportional to a − 1 is zero only in the uninteresting case of $F_{\mu\nu} = 0$. Hence, the consistency of the assumed conditions (4.6) requires that a = 1.

It turns out that the conditions (4.6) can be integrated in explicit form: We substitute Eq. (3.4) in Eq. (4.6) and, after some rearrangement of terms, obtain

$$\varepsilon^{\alpha\mu\nu\sigma}[\partial_\sigma F_{\nu\mu} + {}^1\!/_2 F^{\cdot\lambda}_{\cdot\nu}\,(2g_{\lambda\delta}g_{\mu\varkappa}\partial_\sigma g^{\varkappa\delta} + \partial_\lambda g_{\mu\sigma} + \partial_\sigma g_{\mu\lambda} - \partial_\mu g_{\lambda\sigma})] = 0.$$

It is easy to see that the bracketed second term is symmetrical in σ and μ and vanishes, leaving

$$\varepsilon^{\alpha\mu\nu\sigma}\partial_\sigma f_{\mu\nu} = 0, \tag{4.8}$$

where $f_{\mu\nu} = (F_{\mu\nu} - F_{\nu\mu})/2$, the antisymmetric part of $F_{\mu\nu}$.

Equation (4.8) is literally identical with Maxwell's equations (4.4) and has the obvious solution

$$f_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \tag{4.9}$$

where $A_\mu$ is an arbitrary vector which it is natural to relate to the electromagnetic vector potential. Since the electromagnetic field $E_{\mu\nu}$ also appears in the form (4.9), we shall assume that the two tensors are simply proportional:

$$f_{\mu\nu} = \pm\, l_0^2 E_{\mu\nu}/e \tag{4.10}$$

(e is the electron charge). The new constant $l_0$, with the dimension of length, must be determined from experiment. Further on, we shall discuss the possible meaning of $l_0$ and the limitations imposed by experiment on its magnitude.

## 5. THE FREE-FIELD ELECTROMAGNETIC LAGRANGIAN

We shall study the possible form of the free-field electromagnetic Lagrangian $\mathscr{L}_F$; our choice for $\mathscr{L}_F$ will be guided by the following considerations:

(1) $\mathscr{L}_F$ must not depend on $L^i_{jk}$ since in the contrary case we would have to take $\mathscr{L}_F$ into account under variations with respect to $L^i_{jk}$. Thus, we shall assume that $\mathscr{L}_F$ contains only the tensors $F_{mn}$ and $g_{mn}$, but not their derivatives.

(2) The expansion of $\mathscr{L}_F$ in powers of $(l_0^2/e)$ must begin with the term $(e^2/4l_0^4)f_{\mu\nu}f_{\mu'\nu'}g^{\mu\nu'}g^{\mu'\nu}$ in order that we obtain Maxwell's equations in first approximation.

If to these considerations we add that of maximum simplicity (of form), we may, for example, choose $\mathscr{L}_F$ in the form:

$$\mathscr{L}_F = \frac{e^2}{2l_0^4}\sqrt{g}\,g^{mn}F_{mn} = \frac{e^2}{2l_0^4}\sqrt{g}\,\bar{s}. \tag{5.1}$$

It is not difficult to express the invariant $\bar{s} = g^{mn}F_{mn}$ in terms of the invariants

$$s = f_{ij}f_{i'j'}g^{ij'}g^{ji'}, \qquad d = \sqrt{g}\,\varepsilon^{ijkl}f_{ij}f_{kl}. \tag{5.2}$$

Following the procedures of our earlier work I, we find:

$$\bar{s} = 2\left[1 + \frac{s}{4} + \left[\left(\frac{s}{4}\right)^2 + \left(\frac{d}{4}\right)^2\right]^{1/2}\right.$$
$$\left. + 2\left[1 + \frac{s}{4} - \left[\left(\frac{s}{4}\right)^2 + \left(\frac{d}{4}\right)^2\right]^{1/2}\right]^{1/2} - 4. \tag{5.3}$$

For completeness, we also write down the symmetric part of the tensor $F_{mn}$ in terms of $f_{mn}$ and invariants:

$$\frac{1}{2}(F_{mn} + F_{nm}) = \frac{1}{4}\left(\bar{s} - \frac{2s}{\bar{s}+4}\right)g_{mn} + \frac{2}{\bar{s}+4}f_{mk}f_{ln}g^{kl}. \tag{5.4}$$

Thus, $\mathscr{L}_F$ only depends on the $f_{ij}$ and $g_{ij}$. Taking account of Eq. (4.9), we find the electromagnetic field equations in the form

$$\partial_\mu \frac{\partial \mathscr{L}_F}{\partial f_{\mu\nu}} = 0. \tag{5.5}$$

Clearly, the Lagrangian $\mathscr{L}_F$ is essentially nonlinear in the invariants s and d. We shall, here, obtain its expansion to second-order terms for a flat space ($g_{ij} = \delta_{ij}$). Recognizing that

$$s = \frac{l_0^4}{e^2}E_{ij}E^{ji} = 2\frac{l_0^4}{e^2}(E^2 - H^2),$$
$$d = \frac{l_0^4}{e^2}\varepsilon^{ijkl}E_{ij}E_{kl} = 4\frac{l_0^4}{e^2}(\mathbf{EH}),$$

where **E** and **H** have their usual meaning, we have

$$\mathscr{L}_F = \frac{E^2 - H^2}{2} - \frac{l_0^4}{4e^2}\left[\frac{(E^2 - H^2)^2}{2} + (\mathbf{EH})^2\right] + \ldots \tag{5.6}$$

To conclude this section, we remark again that there is an arbitrariness to the choice of $L_F$ and our Lagrangian (5.1) is notable relative to other possibilities only for its simplicity. Nevertheless, as will be seen below, the explicit form of the Lagrangian is not vital to the experimental study of the question of whether or not the electromagnetic field is related to the geometry of space.

## 6. MOTION OF A TEST BODY. EXPERIMENTAL EFFECTS

The essential consideration of our scheme appears in the expression for the affine-connection (2.11) which determines the geometry of space-time.

The coefficients of the affine-connection determine the geodesic lines of the space through the usual equations:

$$\frac{d^2x^i}{ds^2} + L^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{6.1}$$

The geodesic line has an important characteristic: parallel-transfer of a vector, from point to point, along a geodesic always maintains the tangency of the vector if it has this property at the original point. The velocity, in any case, is a vector tangent to the trajectory so that motion along a geodesic intrinsically describes inertial motion.

For the case of a space with both curvature and torsion we propose to generalize the Galilean law of inertia to the following second basic postulate:

In the presence of gravitational and electromagnetic fields only (no other fields) a neutral test body moves along a geodesic defined by Eq. (6.1) with the affinity given by Eq. (2.11). It should be understood that our term "neutral body" means a body without charge or any other electromagnetic properties such as electric- or magnetic dipole moment, etc. Let us also note that in our case the geodesic is not, generally speaking, the shortest line.

The foregoing statement permits us to estimate the limits placed on the length $l_0$ by astronomical data. Indeed, for motion in the neighborhood of the Sun, there will appear corrections to the consequences of the general theory of relativity, due to the electric and magnetic fields of the Sun. At the present time there are data according to which the Sun possesses a dipole magnetic field whose intensity, at the Sun's surface, is of the order of $H \approx 1$ G[5].

Let us examine, first, the effect of the perihelion precession of Mercury. The order of the effect is given in general relativity by the ratio $(r/R) \approx 4 \times 10^{-8}$, where r denotes the Schwarzschild radius of the Sun and R the distance between the Sun and Mercury. The order of accuracy of agreement between theory and experiment is 0.5%[6]. Hence, we must consider that the smallness parameter $\xi$, which appears upon taking into account the effect of the Sun's magnetic field on the motion of Mercury, is limited by the condition $\xi \lesssim 10^{-10}$. This parameter equals

$$\xi = \frac{l_0^2}{e} H' = \frac{l_0^2}{e} H \left(\frac{R_\odot}{R}\right)^3, \qquad (6.2)$$

where $R_\odot$ is the radius of the Sun and $H'$ its magnetic field calculated at the position of Mercury. Inserting the appropriate numbers we find

$$l_0 \lesssim 10^{-7}\,\text{cm} \qquad (6.3)$$

A somewhat better estimate is obtained from the effect of bending of light rays in the field of the Sun. Here, the order of experimental accuracy is comparable with the effect itself, but the magnetic field intensity is greater. The smallness parameter in this case is limited by $\xi' \lesssim 10^{-6}$, and for $l_0$ we find

$$l_0 \lesssim 10^{-8}\,\text{cm} \qquad (6.4)$$

Equations (6.1), obviously, will give the effects due to the presence of even an electromagnetic field alone. In a flat space and using Cartesian coordinates we find[4)] as the first term in the expansion in powers of $(l_0^2/e)$

$$\frac{d^2x^i}{ds^2} + \frac{l_0^2}{e}\frac{\partial E_{j\cdot}^{\cdot i}}{\partial x^k}\frac{dx^j}{\partial s}\frac{dx^k}{\partial s} = 0. \qquad (6.5)$$

Let us examine the limiting nonrelativistic case where we can neglect the space-like part of the velocity 4-vector (i.e., $dx/ds = 0$, $dx^0/ds \neq 0$). Then the time-like component of (6.5) gives

$$x^0 = ks + r,$$

where k and r are constants and the space-like part gives (taking $x^0 = ct$)

$$\frac{d^2\mathbf{x}}{dt^2} = c\frac{l_0^2}{e}\frac{\partial \mathbf{E}}{\partial t}. \qquad (6.6)$$

Let us remark two important corollaries of Eq. (6.6): In the first place, the force

$$\mathbf{F} = mc\frac{l_0^2}{e}\frac{\partial \mathbf{E}}{\partial t}$$

which acts on a neutral body is universal in the sense that it is proportional to the mass just as for the gravitational force. In the second place, Eq. (6.6) is not invariant with respect to reversal of the time T. (It is scarcely necessary to remark that this is due to the presence of a first time derivative, since the acceleration and electric field are unaffected by the operation). Hence, if experiment convinces us of the correctness of (6.6), then we shall have proved both the geometrical nature of the electromagnetic field and the connection between the geometrization of electromagnetism and violation of T-invariance[5)] (and CP-invariance). In the Appendix we shall discuss an idealized experimental scheme for testing (6.6).

In our earlier work[3,7] we have already discussed the connection between geometrization of the electromagnetic field and violation of T- and CP-invariance. Under certain additional assumptions this connection leads to a weak-electromagnetic variant of the violation of CP-invariance. In this case, agreement with experiments on neutral K-meson decay requires that the quantity $l_0$ have a magnitude of order:

$$l_0 \sim 10^{-17} - 10^{-18}\,\text{cm} \qquad (6.7)$$

As is evident from the preceding text, such a value does not contradict existing data. If experiments discussed elsewhere[7,8] confirm a weak-electromagnetic variant of the violation of CP-invariance this will be an indication, albeit indirect, of the correctness of our proposed theory.

There is another possible indirect test of our theory, viz., searching for nonlinear effects in the free-electromagnetic field Lagrangian (cf. Eq. (5.6)). The effects which might result from such a nonlinearity are discussed in[9], and lead to the conclusion that existing data limit $l_0$ to

$$l_0 \lesssim 10^{-14}\,\text{cm}$$

Thus, the theory here proposed, leads to a number of experimental consequences which will permit a decision as to its validity.

The author wishes to express sincere thanks to A. T. Filippov and O. A. Khrustalev for many fruitful discussions.

## APPENDIX

Here, in a schematic way, we shall estimate what order of magnitude for $l_0$ might be arrived at in a laboratory experiment aimed at the search for the universal force implied by the equation

4)In Eqs. (6.5) and (6.6) we arbitrarily use the plus sign, keeping in mind that it could also be minus.

5)The possibility of violation of T-invariance in a unified field theory was already noted by Einstein (cf. [1], p. 176).

$$\frac{d^2\mathbf{x}}{dt^2} = c\frac{l_0^2}{e}\frac{\partial \mathbf{E}}{\partial t}. \tag{A.1}$$

Let us imagine the following ideal experimental scheme: A mechanical system (e.g., a torsion pendulum) with natural frequency $\omega$ and a small damping constant $\gamma$ is placed within an electrical system which can pump energy into the pendulum via the sought-for (universal) force, related to an appropriate (time-) varying electric field. For example, both arms of the torsion pendulum, let us say, composed of a neutral dielectric, can be placed between capacitor plates; the field therein must be highly uniform in order to exclude the effects of the force acting on the dipole-moment induced by polarization. Let the field between the plates be given by

$$E = E_0 \sin \omega t.$$

where $\omega$ is the same as above. Then, according to our discussion, the force will be

$$F = mc\frac{l_0^2}{e}\omega E_0 \cos \omega t = F_0 \cos \omega t.$$

At resonance the (mechanical) amplitude is given by

$$x = \frac{F_0}{m\omega\gamma} = \frac{cl_0^2}{e\gamma}E_0, \tag{A.2}$$

whence

$$l_0^2 = \frac{e\gamma}{cE_0}x = 1.6\cdot 10^{-20}\frac{x\gamma}{E_0} \tag{A.3}$$

with $E_0$ in esu. We emphasize that in Eq. (A.2) the (sought-for) universality is evidenced by the absence of dependence on the mass and the violation of T-invariance by the absence of dependence on the frequency, all other things being equal.

To get an impression of the order of magnitude of the quantities involved, let us assume the following values for the experimental parameters: let the equilibrium time be of the order of a day, i.e., $\gamma$ of the order $\gamma \approx 10^{-4}$ sec$^{-1}$, the accuracy of measurement of the amplitude x of the order $x \sim 10^{-4}$ cm, and $E_0 \approx 10^3$ esu. Then, absence of the effect implies

$$l_0 \lesssim 4\cdot 10^{-16}\,\text{cm}$$

This estimate is considerably better than those obtained otherwise in the text, prior to Eq. (6.7), and is not far from the value suggested by that equation.


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Translated by J. G. Adashko
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