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INSTABILITY OF NONLINEAR PERIODIC WAVES IN A DISPERSIVE MEDIUM

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Instability of solutions of the nonlinear periodic wave type is considered in the presence of a perturbation which may be due to an inhomogeneity or to nonstationarity of the medium. The model of ion sound and the Korteweg-de Vries equation are investigated as examples. It is shown that an instability of the stochastic type exists for periodic waves sufficiently "close" to a solitary wave. This instability signifies time mixing of the wave phase and quasi-random variation of the wave velocity, which in real problems leads to overturning. The time during which wave overturning occurs is estimated for the ion-sound model.

INTRODUCTION

A characteristic feature of nonlinear media with dispersion is the existence of exact solutions describing nonlinear periodic waves. An analysis of the stability of such waves is connected with certain technical difficulties, and the first results in this direction were obtained only recently (see, for example[1-3]).

The most general and best developed method of investigating the evolution of nonlinear periodic waves, based on the use of the Lagrangian formalism and a certain averaging operation, was proposed by Whitham[4].

We propose below another approach for the investigation of the stability of nonlinear waves, based on an essentially nonlinear analysis of the influence of small but finite perturbations. It will be shown that under definite conditions, the phase of the nonlinear wave begins to "become muddled" under the influence of the perturbation, and the law governing its time variation has a nearly random character. This circumstance leads to turbulization of the wave, the end result of which is overturning and a transition to a multistream motion.

Although the method developed below for the investigation of stability is applicable to arbitrary nonlinear periodic waves, we shall nonetheless use for convenience the concrete model of ion sound[5], and also the Korteweg-de Vries equation, which describes nonlinear steady-state waves for a large class of physical problems.

In Sec. 1 we present the fundamental equation, derive a number of relations necessary for the subsequent analysis, and formulate concretely the stability problem. In Sec. 2 we introduce an instability criterion leading to randomization of the phase of the nonlinear wave. In Sec. 3 we consider the process of diffusion of the wave parameters and estimate the time during which the overturning takes place. We also present there a more detailed physical analysis of the obtained instability.

1. FORMULATION OF PROBLEM

One-dimensional motion of a plasma with "cold" ions is described by the system

$$\frac{\partial V}{\partial t} = -\frac{e}{M}\frac{\partial \varphi}{\partial x} - V\frac{\partial V}{\partial x},$$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}(nV),$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -4\pi e(n - n_e e^{e\varphi/T}), \qquad (1.1)$$

where n_e is the electron density, which in a nonstationary inhomogeneous medium is generally speaking a certain function of the time and of the coordinate. We write

$$n_e = n_0 + \varepsilon n_1(x, t), \qquad (1.2)$$

where n_0 = const, and the dependence on (x, t) is represented in the form of the perturbation n_1, while ϵ is a dimensionless small parameter.

Let us dwell on certain properties of the unperturbed system (1.1) when $\epsilon = 0$. In this case we can construct a solution that depends only on the variable

$$\xi = (1/r_d)(x - Ut),$$

where r_d is the Debye radius and U is an arbitrary parameter. The system (1.1) reduces to a single integrable equation

$$v'^2(u-v)^2 = u^2(1 - v/u) + \exp\{uv - v^2/2\} - (1+u^2) - 2C;$$

$$v \equiv \frac{V}{c}, \quad u \equiv \frac{U}{c}, \quad v' \equiv \frac{dv}{d\xi}, \quad c = \sqrt{\frac{T}{M}}, \qquad (1.3)$$

where C is the integration constant.

The equation of motion (1.3) is best analyzed on the phase plane (v', v), and we present two families of curves. The first (Fig. 1) represents phase trajectories at different values of the parameter C, which has the meaning of the Hamiltonian. Closed trajectories,

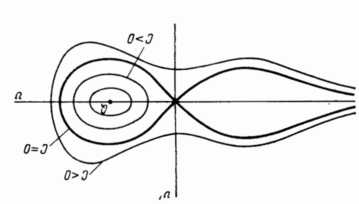

FIG. 1.

describing steady-state nonlinear oscillations, correspond to $C > 0$. The separatrix ($C = 0$) describes a solitary wave (soliton), and when $C < 0$ the motion is not finite. The second family (Fig. 2) of phase trajectories pertains to a fixed $C > 0$, but to different values of the wave velocity u. With increasing u, the angles on the phase curve become more acute, and at a critical value $u_c \approx 1.6$ the overturning takes place. Multistream motion then sets in, and for its description it is necessary already to alter the initial system (1.1).

To avoid too cumbersome and inconvenient calculations, we simplify the system (1.1), assuming all quantities to be sufficiently small but finite. This approximation corresponds to a low wave velocity u, differing little from unity:

$$\alpha \equiv u - 1 \ll 1 \quad (\alpha > 0), \tag{1.4}$$

and the system (1.1) reduces to the Korteweg-de Vries equation with a right-hand side

$$\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3} = \varepsilon F(y\ \tau),$$
$$F(y,\tau) \approx -\frac{1}{2n_0}\left(\frac{\partial n_1}{\partial \tau} + v\frac{\partial n_1}{\partial y}\right),$$
$$y = x / r_d, \quad \tau = ct / r_d. \tag{1.5}$$

Equation (1.3) goes over into

$$v'^2 = \alpha v^2 - {}^1/_3 v^3 - 2C. \tag{1.6}$$

We note that the points Q on Figs. 1 and 2 correspond to the following relation between parameters

$$\alpha_0 = ({}^3/_2 C_0)^{1/3},$$

at which the periodic solution of the unperturbed motion (1.5) collapses.

We shall now investigate Eq. (1.5) under the assumption that it is exact. As will be shown subsequently, the obtained analysis is perfectly adequate to draw conclusions concerning the stability of the system (1.1), and the refinements necessitated by the fact that (1.5) has nevertheless a limited region of applicability will be made in the appropriate place.

We should expect intuitively (as will be justified later) that the strongest influence of the perturbation $F(y, \tau)$ will occur near the separatrix, i.e., on the unperturbed steady-state oscillations, which are "close" to the soliton and have consequently a very long period. Such a limit corresponds to the inequality

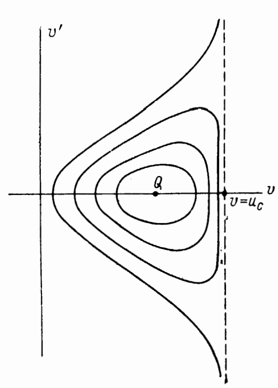

FIG. 2.

$$C \ll \alpha^3. \tag{1.7}$$

In this case

$$\max v \approx 3\alpha, \quad \min v \approx \sqrt{2C/\alpha}$$

and from (1.6) we get for the unperturbed wave

$$v \approx 3\alpha\,\mathrm{cn}^2\left[\frac{\sqrt{3\alpha}}{2}\left(1 + \sqrt{\frac{C}{18\alpha^3}}\right)\xi; \varkappa\right] + O\left(\sqrt{\frac{C}{\alpha^3}}\right),$$
$$\varkappa \approx 1 - \sqrt{2C/9\alpha^3} \tag{1.8}$$

and, in addition, the wave number

$$k = \frac{\pi}{4}\sqrt{3\alpha}\left(\ln\frac{4}{(8C/9\alpha^3)^{1/4}}\right)^{-1} \equiv \frac{2\pi}{L}, \tag{1.9}$$

where L is the period of the oscillations (1.8).

Let us consider now a variational principle for (1.5) in the absence of perturbation ($\epsilon = 0$). Expanding the unperturbed solution v in a Fourier series

$$v(y,\tau) = \sum_{n=-\infty}^{\infty} a_n e^{ikn(y-u\tau)}$$
$$= \sum_{n=-\infty}^{\infty} a_n e^{ikny} e^{-in\vartheta} = \sum_{n=-\infty}^{\infty} v_n(\tau) e^{ikny}, \tag{1.10}$$

we get the Hamiltonian[6] in the phase space of the harmonics (v_n)

$$H = \lim_{\mathscr{L}\to\infty} \frac{1}{2\mathscr{L}} \int_{-\mathscr{L}}^{\mathscr{L}} \left[\frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 - \frac{1}{6}v^3 - \frac{1}{2}v^2\right] dy$$
$$= -\frac{1}{2}\sum_n (ikn)^2 v_n v_{-n} - \frac{1}{6}\sum_{n_1+n_2+n_3=0} v_{n_1} v_{n_2} v_{n_3} - \frac{1}{2}\sum_n v_n v_{-n} \tag{1.11}$$

and the canonical equations of motion

$$\frac{dv_n}{d\tau} = \frac{ikn}{2\pi}\frac{\partial H}{\partial v_{-n}}, \quad \frac{dv_{-n}}{d\tau} = -\frac{ikn}{2\pi}\frac{\partial H}{\partial v_n}. \tag{1.12}$$

We obtain a functional relation between the Hamiltonian H in (v_n) space and the Hamiltonian C in (v, v') space. From (1.6) and (1.11) we get

$$H = \bar{C} + kI - \frac{1}{2\pi} ku \int_{-L/2}^{L/2} v^2(\xi)\, d\xi,$$

$$\bar{C} = \frac{1}{L}\int_{-L/2}^{L/2} C\, dy,$$

$$I = \frac{1}{2\pi}\int_{-L/2}^{L/2}\left(\frac{\partial v}{\partial y}\right)^2 dy = \frac{1}{2\pi}\int_{-L/2}^{L/2}\left(\frac{\partial v(\xi)}{\partial \xi}\right)^2 d\xi$$
$$= \frac{1}{2\pi}\oint v'(C, v)\, dv. \tag{1.13}$$

Here I has the meaning of the action in the phase (v, v') space, and $\bar{C}$ coincides with C in the unperturbed problem.

We now formulate the investigated problem in (v_n) space. We assume that the unperturbed motion is a nonlinear steady-state oscillation (1.8) with a fixed period L (or wave number k). This determines the integral of motion $H = H(k)$ and the next equations of motion, equivalent to (1.5) when $\epsilon = 0$:

$$\frac{dH}{d\tau} = 0, \quad \frac{d\vartheta}{d\tau} = \omega(H) = ku(H). \tag{1.14}$$

The second equation determines the phase ϑ, and relation (1.13) for a specified k establishes a unique connection between H and u, i.e., between the Hamiltonian and the frequency ω.

At sufficiently small perturbations of the system (1.4), it is possible to construct an approximate integral of motion, which represents the unperturbed Hamiltonian H plus a correction of order ϵ. However, as will be shown in the next section, at sufficiently large ϵ the phases begin to vary in a quasi-stochastic manner with time under the influence of the perturbation, and this leads to a Brownian motion of the quantity H. Thus, the type of instability considered by us is a stochastic disintegration of the integral of motion, and the growth of H in the course of time is similar to the acceleration of a Brownian particle.

2. CRITERION OF STOCHASTIC INSTABILITY

We expand Eq. (1.5) in a Fourier integral with respect to y:

$$\frac{dv_q}{d\tau}+iqv_q+(iq)^3v_q+\frac{1}{2}iq\int dq_1\,v_{q_1}v_{q-q_1}=\varepsilon F_{q},$$

$$v_q(\tau)=\frac{1}{2\pi}\int e^{-iqy}\,v(\tau,y)\,dy,\qquad v_{-q}=v_q^*,$$

$$F_q(\tau)=\frac{1}{2\pi}\int e^{-iqy}\,F(\tau,y)\,dy,\qquad F_{-q}=F_q^*. \tag{2.1}$$

We now assume that H is defined as before by relation (1.11), but now the velocity v in this equation is the solution of the perturbed problem (1.5). In this case we get from (1.11) and (2.1), in analogy with (1.12),

$$\frac{dH}{d\tau}=\int dq\left(\frac{\partial H}{\partial v_q}\frac{dv_q}{d\tau}+\frac{\partial H}{\partial v_{-q}}\frac{dv_{-q}}{d\tau}\right)$$
$$=\varepsilon\int\frac{dq}{iq}\left(\frac{dv_{-q}}{d\tau}F_q+\frac{dv_q}{d\tau}F_{-q}\right). \tag{2.2}$$

If we use in the right side of (2.2) the zeroth approximation for v_q in accordance with (1.10), we get

$$\frac{dH}{d\tau}=\frac{\varepsilon}{ik}\sum_n\frac{1}{n}\left(\frac{dv_{-n}}{d\tau}F_n-\frac{dv_n}{d\tau}F_{-n}\right)$$
$$=\varepsilon u\sum_n\left(v_{-n}F_n+v_nF_{-n}\right). \tag{2.3}$$

The terms written out in formula (1.5) for $F(\tau, y)$, may turn out to be, generally speaking, of different orders of magnitude, depending on the values of the derivatives of the perturbation n_1 with respect to x and t. It is convenient to start the investigation with the case when the term containing the derivative with respect to x is small and can be neglected.

We consider now the simplest case of the time dependence of the perturbation

$$F_n=\Phi_n e^{i\nu t}$$

Substituting F_n in the equation for v_n from (1.10) in (2.3), we get

$$\frac{dH}{d\tau}=-\varepsilon u\sum_n\left[a_n\Phi_{-n}e^{i(n\vartheta-\nu t)}+a_{-n}\Phi_ne^{-i(n\vartheta-\nu t)}\right]. \tag{2.4}$$

We obtain the equation for ϑ in the following manner: we retain the functional relation (1.13) for the perturbed quantities. In view of the uniqueness of the connection between ω and H, we can write

$$d\vartheta\,/\,d\tau=\omega(H)=ku(H), \tag{2.5}$$

where H is now the function of the time in accordance with (2.4). A justification and an estimate of the accuracy of the approximation (2.5) will be given in Sec. 3.

The obtained system (2.4) and (2.5) is analogous to the problem investigated in[7], and we shall henceforth use the same reasoning in what follows. It is simplest to interpret the system (2.4) and (2.5) as the motion of a nonlinear oscillator with frequency $\omega(H)$ and energy H under the influence of an external force.

Let now the following resonance condition be satisfied for a certain value $H = H_m$

$$m\omega(H_m)=\nu. \tag{2.6}$$

The next nearest resonance is determined by the condition

$$(m+1)\,\omega(H_{m+1})=\nu.$$

Thus, when $m \gg 1$, the distance between the nearest resonant frequencies is

$$\Omega=\omega(H_m)-\omega(H_{m+1})\approx\nu\,/\,m^2=\omega^2\,/\,\nu. \tag{2.7}$$

If condition (2.6) is satisfied, we can retain in the right side of (2.4) only the resonant term, and we can estimate the maximum change of energy δH as the result of the resonance. In view of the nonlinearity, the frequency $\omega(H)$ is changed thereby by an amount

$$\delta\omega\approx\frac{d\omega(H)}{dH}\delta H. \tag{2.8}$$

We now introduce the parameter

$$K=(\delta\omega\,/\,\Omega)^2 \tag{2.9}$$

characterizing the ratio of the resonance width $\delta\omega$ to the distance between resonances Ω. As shown in[8,9], when $K \gg 1$ and the resonances overlap, the oscillator motion becomes of the intermixing type, and the phase ϑ becomes a random function of time. The phase correlation is split:

$$\int_0^{2\pi}e^{i\vartheta(\tau'+\tau)}\,e^{-i\vartheta(\tau')}\,d\vartheta(\tau')\sim e^{-\tau/\tau_c}$$

after a time τ_c equal to[7]

$$\tau_c=\frac{\nu}{\omega^2}\frac{1}{\ln K}. \tag{2.10}$$

To the contrary, when $K \ll 1$, the motion is stable in the sense defined at the end of Sec. 1. The value $K \sim 1$ can be regarded of the boundary of the transition from the stable motion to the stochastic instability.

Let us estimate the parameter K. From (2.4) we get

$$\delta H\sim\frac{\varepsilon u\Phi_m a_m}{m\delta\omega}\sim\frac{\varepsilon u\omega\Phi_m a_m}{\nu\,(d\omega/dH)\,\delta H},$$

whence

$$\delta H\sim\left(\frac{\varepsilon u\omega\Phi_m a_m}{\nu d\omega/dH}\right)^{1/2}. \tag{2.11}$$

The amplitude a_m can be found by using the expansion

$$\mathrm{sn}^2 z=2\left[\frac{\pi}{kK(\pi/2,\varkappa)}\right]^2\sum_{n=1}\left[\frac{q^n}{(1-q^n)^2}-\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi z}{K(\pi/2,\varkappa)}\right],$$

$$q=\exp\left\{-\frac{\pi K(\pi/2,\sqrt{1-\varkappa^2})}{K(\pi/2,\varkappa)}\right\}, \tag{2.12}$$

where $K(\pi/2, \kappa)$ is a complete elliptic integral of the first kind and κ is its modulus. Substituting the written-out expression in (1.8) and using the expansion near the separatrix (i.e., the fact that $1 - \kappa \ll 1$), we get

$$a_m \sim \alpha / N, \quad N = K(\pi/2, \varkappa). \tag{2.13}$$

N determines here the characteristic number of harmonics in the spectrum. An estimate of the value of $d\omega/dH$ is more complicated. We note by way of introduction that at a fixed value of k the Hamiltonian $\overline{C}$ in (v, v') space is uniquely connected with u by relation (1.9). From (1.3) we have

$$\frac{dH}{d\omega} = \frac{1}{k}\frac{dH}{du} = \frac{1}{k}\frac{d\overline{C}}{d\alpha} + \frac{\partial I}{\partial \overline{C}}\frac{d\overline{C}}{d\alpha} - \frac{\partial I}{\partial \alpha}$$
$$-\frac{1}{2\pi}\int_{-L/2}^{L/2} v^2(\xi)\,d\xi - \frac{u}{2\pi}\left(\frac{\partial}{\partial \alpha} + \frac{d\overline{C}}{d\alpha}\frac{\partial}{\partial \overline{C}}\right)\int_{-L/2}^{L/2} v^2(\xi)\,d\xi. \tag{2.14}$$

Taking into consideration the relation

$$\partial \overline{C} / \partial I = -k$$

and using (1.8) and (1.9), we can obtain from (1.14), accurate to terms of higher order of smallness,

$$dH / d\omega \sim u\sqrt{\alpha}. \tag{2.15}$$

Combining (2.9), (2.7), (2.8), (2.11), (2.13), and (2.15), we arrive at the following condition for the decay of the integral of motion

$$K \sim \varepsilon \nu \Phi / k^2 u^3 \gtrsim 1 \quad (\Phi \equiv \Phi_{m=\nu/\omega}). \tag{2.16}$$

Postponing a detailed analysis of the criterion (2.16) and of the ensuing consequences to the next section, we call attention here only to the fact that when $\epsilon \to 0$ and the other parameters in (2.16) are fixed, the criterion is no longer satisfied and stability is obtained, at least for a sufficiently long time. The resultant correction to the unperturbed Hamiltonian H can be readily determined. Indeed, from (2.4), in the presence of a single resonance at the harmonic numbered n, we have:

$$dH / d\tau = \psi(H)\cos(n\vartheta - \nu\tau),$$
$$\psi(H) = 2\varepsilon u(H) a_n(H) \Phi_n(H),$$

where the phases of the amplitude a_n and Φ_n have been omitted for simplicity. From this, with (2.5) taken into account, we have the exact integral

$$\int^{H} dH \frac{n\omega(H) - \nu}{\psi(H)} = \sin(n\vartheta - \nu\tau) + \text{const.}$$

Expanding $\omega(H)$ near the resonance at $H = H_n$, we have

$$n\frac{d\omega(H)}{dH}(H - H_n)^2 = 2\psi(H_n)\sin(n\vartheta - \nu\tau) + \text{const}$$

The obtained expression shows that as a result of the resonance a weak modulation appears on the nonlinear wave (compare the obtained formula with (2.11)).

To the contrary, at all arbitrarily small ϵ, there always exists the decay of the integral of motion, provided k is sufficiently small, i.e., the unperturbed periodic wave is sufficiently "close" to the separatrix (solitary wave). The latter circumstance, however, calls for a refinement, which will also be discussed in Sec. 3.

In the course of the derivation of the criterion (2.16), we have made the following three simplifications: 1) in lieu of the initial (1.1) we consider the approximation (1.4), when the Mach number differs little from unity; 2) only the first term in the perturbation of $F(\tau, y)$ was considered; 3) the perturbation $F(\tau, y)$ contained only one harmonic with respect to the time τ.

The first simplification denotes that the investigation of the stability is carried out sufficiently far from the values of the parameters at which the overturning of the wave takes place. In this case α is the only parameter that varies substantially from a value much smaller than unity to a value $\alpha_c \approx 0.6$. The latter means that the inequality (2.16) can be retained for a very rough estimate near the overturning regime.

Allowance for the second term in $F(\tau, y)$ does not lead to any fundamental changes; to save space, it will not be considered here.

Let us stop to discuss in greater detail the third simplification. It is obvious that for an effective development of the stochastic instability it is necessary that the right side of (2.4) contain a sufficiently large number of terms, which may turn out to be resonant at definite instants of time, owing to changes of H. In the case considered above, this means that a perturbation containing only one harmonic with respect to time should have many harmonics with respect to the coordinate. We can consider also another limiting case, when the perturbation has only one harmonic with respect to the coordinate:

$$F_n(\tau) = F_{n_0}(\tau)\delta_{nn_0},$$

expanding $F_{n_0}(\tau)$ in a Fourier time series, we obtain in lieu of (2.4)

$$\frac{dH}{d\tau} = \varepsilon u\left\{a_{n_0} e^{in_0\vartheta} \sum_m F_{-n_0, m} e^{-im\nu t} + a_{-n_0} \bar{e}^{in_0\vartheta} \sum_m F_{n_0, -m} e^{im\nu t}\right\}. \tag{2.17}$$

In this case, a large number of resonant terms in the right side can be ensured by a sufficiently broad temporal spectrum of the perturbation. The case (2.17) can be analyzed in analogy with the preceding case, the distances between resonances now being

$$\Omega = \nu / n_0 \tag{2.18}$$

and the overlap condition (2.9) can always be satisfied at sufficiently small ν.

3. EVOLUTION OF NONLINEAR PERIODIC WAVE

As already noted, the system (2.4) can be interpreted as the motion of a nonlinear oscillator with energy H and frequency $\omega(H)$. When the criterion (2.16) is satisfied, the phase of such an equivalent oscillator becomes randomized within a time (2.10), and the entire motion can be regarded as random wandering. The average value of H increases, and with it also ω and u. Thus, the resultant Brownian motion causes the solution of (1.5) to be given by an expression that coincides functionally with a nonlinear wave in which the velocity u is a random function of the time. As follows from (2.16), the criterion becomes worse with increasing u, and the region of stochastic motion is bounded on the phase plane by a certain maximum u_{max}, at which (2.16) is no longer satisfied. If it now turns out that $u_{max} > u_c$, this means that the considered instability leads, in final analysis, to an overturning of the wave.

Another limitation on the growth of u may be connected with the fact that when $\nu \lesssim$ = ku the resonances (2.4) are impossible. Finally, the third limitation is connected with the width of the spectrum with respect to the coordinate of the perturbation $F(\tau, y)$, since there are no resonances outside the region of the nonzero harmonics $F_n(\tau)$.

We now assume for simplicity that none of these limitations hold, and describe with the aid of the kinetic equation the evolution of the wave instability in analogy with the description in[7].

We note first that the right side in (2.4) is proportional roughly speaking to $v(\tau, y)$. According to (1.8), $v(\tau, y)$ is a sequence of very narrow pulses which differ greatly from zero in the interval $\tau \sim 1$ ($t \sim r_d/c$) and which follow each other periodically with a frequency $\omega = ku \ll 1$. Each resonance acts on the equivalent oscillator as a jolt (collision) with interval between the jolts

$$\Delta\tau \sim 1/\Omega = \nu/\omega^2. \tag{3.1}$$

This information is sufficient to write down the Fokker-Planck equation for the distribution function f(H):

$$\frac{\partial f}{\partial \tau} = -\frac{\partial}{\partial H}\left[\frac{\langle \Delta H\rangle}{\Delta\tau} f\right] + \frac{1}{2}\frac{\partial^2}{\partial H^2}\left[\frac{\langle(\Delta H)^2\rangle}{\Delta\tau} f\right], \tag{3.2}$$

where the angle brackets denote averaging over the random phase ϑ. Let us calculate $\langle \Delta H\rangle$ and $\langle(\Delta H)^2\rangle$. Taking into account the remarks made above, we get from (2.1) and (2.5), by integrating over the small time interval $\tau \sim 1$ in the vicinity of the jolt (resonance):

$$v_n \approx a_n e^{in\vartheta} + \varepsilon F_n + O(\varepsilon^2). \tag{3.3}$$

Analogously, we get from (2.4) and (3.3)

$$\Delta H \approx 2\varepsilon u |a_n \Phi_n| \cos(n\vartheta - \nu\tau + \vartheta_0) + 2\varepsilon^2 u |\Phi_n|^2 \cos^2 \nu t + O(\varepsilon^3), \tag{3.4}$$

where ϑ_0 is the phase of the quantity $(a_n \Phi_{-n})$. From (3.4) we obtain the necessary expressions for the moments, taking (2.13) into account

$$\begin{gathered}\langle \Delta H\rangle \approx 2\varepsilon^2 u |\Phi|^2 \cos^2 \nu t + O(\varepsilon^4),\\ \langle(\Delta H)^2\rangle \approx 2\varepsilon^2 u^2 |\Phi|^2 |a_{\nu/\omega}|^2 \approx 2\varepsilon^2 k^2 u^2 \alpha |\Phi|^2.\end{gathered} \tag{3.5}$$

Substitution of (3.5) and (3.1) in (3.2) and the change of variables

$$f(H) = f(u)\frac{du}{dH} = \frac{1}{k} f(u) \frac{d\omega}{dH} = \frac{f(u)}{ku\sqrt{\alpha}}$$

yields ultimately

$$\frac{\partial f(u)}{d\tau} = -2\varepsilon^2 k |\Phi|^2 \frac{\cos^2\nu t}{\nu}\frac{\partial}{\partial u}\left(\frac{u^2}{\sqrt{u-1}} f(u)\right) + \varepsilon^2\frac{k^2}{\nu}|\Phi|^2 \frac{\partial}{\partial u}\left[\frac{1}{u\sqrt{u-1}}\frac{\partial}{\partial u}(u^3\sqrt{u-1}\, f(u))\right]. \tag{3.6}$$

From (3.6) follows an estimate for the characteristic diffusion time:

$$\frac{1}{\tau_D} \sim \frac{\varepsilon^2 k^2}{\nu}|\Phi|^2 \frac{u^2}{(\Delta u)^2},$$

where Δu is the characteristic change of the wave velocity during the diffusion time. Since u_c is also of the order of unity in the case of overturning, the time T during which the instability in question leads to the formation of a multistream motion is

$$T \sim \nu / \varepsilon^2 k^2 |\Phi|^2. \tag{3.7}$$

We proceed now to investigate the evolution of the spectrum of a nonlinear periodic wave under the influence of a perturbation. Under the influence of the resonances, the amplitudes on the harmonics change, and as the result of the nonlinearity of the problem a change takes place, generally speaking, in the frequencies of all the harmonics $\omega_n = n\omega$. So far we have taken this change into account by putting

$$\delta\omega_n = n\delta\omega. \tag{3.8}$$

This meant that the resultant corrections to the frequencies did not change the dispersion law and no account was taken of the additional spreading of the packet (1.8) as the result of this change.

We now present more accurate estimates. We consider the unperturbed equation (2.1):

$$\frac{dv_n}{d\tau} = -ink\left[(1-k^2n^2)v_n + \frac{1}{2}\sum_{n_1+n_2=n} v_{n_1} v_{n_2}\right],$$

or after substituting

$$v_n = a_n e^{-in\vartheta}$$

we get

$$\omega_n a_n = nk\left[(1-k^2n^2)a_n + \frac{1}{2}\sum_{n_1+n_2=n} a_{n_1} a_{n_2}\right].$$

From the foregoing expression we obtain

$$\frac{\partial\omega_n}{\partial a_m} = -\frac{kn}{a_n}[\alpha + (kn)^2]\delta_{nm} + kn\frac{a_{n-m}}{a_n}, \tag{3.9}$$

or for any $n \neq m$ and $n, m < N$

$$\frac{\partial\omega_n}{\partial a_m} = kn\frac{a_{n-m}}{a_n} \approx kn. \tag{3.9'}$$

On the other hand, according to (2.13), (2.15), and (1.11) we previously had

$$\frac{\partial\omega_n}{\partial a_m} = n\frac{\partial\omega}{\partial a_m} = n\frac{d\omega}{dH}\frac{\partial H}{\partial a_m} \approx n\frac{d\omega}{dH} u a_m \approx kn,$$

which coincides with (3.9′). The agreement demonstrates that the perturbed motion of the wave can actually be regarded as the motion of a nonlinear oscillator with energy H and phase ϑ. The fact that the phase of the oscillation becomes randomized means that all the harmonics of the packet (more accurately, the harmonics of the fundamental part of the spectrum with numbers from zero to N) have random phases that are not correlated with one another.

The process of stochastic decay of a nonlinear wave can proceed also in a different manner in the case (2.17), when the external perturbation has many temporal harmonics. It becomes possible to separate two limiting cases. The first is analogous to that considered and correspond to the conditions when only one or several temporal harmonics of the perturbation contribute to the resonances. The second case corresponds to the following mechanism: Each harmonic of the wave turns out to be in resonant interaction with a large number of temporal harmonics of the perturbation. In this case the change of the frequency of the harmonic is determined with the aid of (3.9) with n = m, as the distance between resonances is given by (2.18). Each harmonic executes Brownian motion independently of the others, and the number of degrees of freedom in the problem becomes very large. Such a mechanism calls for a special analysis, and will not be discussed here.

In conclusion we note that when $k \to 0$ and at a specified number of harmonics m_0 of the perturbation F, the third limitation begins to exert a strong influence (see the start of this section) on the evolution of the wave. The relative number of resonance harmonics $m_0/N \to 0$ (since $N \to \infty$ when $k \to 0$), and a different approach must be used to investigate the stability of a solitary wave.

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