POLARIZATION OF NORMAL WAVES AND SYNCHROTRON RADIATION TRANSFER IN A RELATIVISTIC PLASMA

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The dielectric tensor of a magnetoactive relativistic plasma is calculated for the high frequencies that are of interest from the viewpoint of relativistic electron synchrotron radiation. The polarization of the normal waves is determined. In contrast to the case of a nonrelativistic plasma the polarization is found to be linear in the case of a monoenergetic distribution. For a power-law energy distribution, the polarization may be arbitrary and, in particular, circular. It is pointed out that the latter circumstance should be taken into account in interpreting radioastronomy observation.

THE change of the parameters of radiation during its transfer is determined by the dielectric tensor of the medium. In this article we calculate the dielectric tensor of a relativistic electronic magnetoactive plasma. The obtained expressions make it possible to consider the problem of polarization of normal waves and the transfer of synchrotron radiation of relativistic electrons in the case when the properties of the medium in which the radiation propagates are governed by the relativistic electrons themselves, and the nonrelativistic ("cold") plasma is nonexistent or its density is sufficiently small.

The dielectric tensor will be calculated by the kinetic-equation method. We note in this connection that in the investigation of the transfer of synchrotron radiation by the method of Einstein coefficients it is necessary to know beforehand the polarization in the normal wave. This requirement is connected with the fact that in the Einstein-coefficient method one uses wave intensities, i.e., quantities quadratic in the field, but not the field themselves. Therefore the use of the indicated method is possible if the polarization of the normal waves for which the absorption coefficients are calculated is known (for details see, for example [1]). The polarization, however, depends in turn on the absorption coefficients. It is clear therefore that to solve the problem it is necessary to go outside the framework of the Einstein-coefficient method and to find the polarization of the normal waves in an independent manner, say by the kinetic-equation method. If we do not resort to the Einstein-coefficient method and use the transport equation for the radiation polarization tensor, then the polarization characteristics of the radiation are taken into account automatically.

1. As already noted, the radiation transfer is determined by the dielectric tensor

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + 4\pi\varkappa_{ij}(\omega, \mathbf{k}),$$

where κ_{ij} is the dielectric susceptibility tensor. When the conditions

$$4\pi|\varkappa_{ij}| \ll 1, \qquad i, j = 1, 2, 3, \tag{1}$$

are satisfied, the electric field in the high-frequency electromagnetic wave propagating in a magnetoactive plasma can be regarded as transverse. We introduce a right-hand system of axes 1-2-3, directing the 3 axis along the wave vector **k** and the 2 axis parallel to the projection of the magnetic field **H** on the plane perpendicular to **k**.

In this system, the equation for the Fourier component of the electric field of the wave

$$(c^2k^2 - \omega^2\varepsilon)E_\alpha = \omega^2 4\pi\varkappa_{\alpha\beta}E_\beta, \qquad \alpha, \beta = 1, 2, \tag{2}$$

and the transport equation for the radiation polarization tensor $I_{\alpha\beta}$ is:

$$\left(\frac{\partial}{\partial t} + c\frac{\mathbf{k}}{k}\frac{\partial}{\partial \mathbf{r}}\right) I_{\alpha\beta} = cP_{\alpha\beta} + i2\pi\omega\left[\varkappa_{\alpha\sigma}\delta_{\beta\tau} - \delta_{\alpha\sigma}\varkappa_{\beta\tau}^*\right] I_{\sigma\tau}, \tag{3}$$

where $P_{\alpha\beta}$ is the tensor of the volume power of the spontaneous radiation. Definitions of $I_{\alpha\beta}$ and $P_{\alpha\beta}$ and a derivation of Eqs. (2) and (3) can be found in [2].

The Greek indices α, β, σ, and τ denote the axes 1 and 2 in the plane perpendicular to the wave vector **k**. The substitution $\kappa_{\alpha\beta} \to a\delta_{\alpha\beta} + \kappa_{\alpha\beta}$, where a is real and $|a| \ll 1$, does not change Eq. (3). It is therefore convenient to assume that the sum of the diagonal elements of the tensor $\kappa_{\alpha\beta}$ is a pure imaginary quantity, and the real part of Sp $\kappa_{\alpha\beta}$ defines the quantity ϵ (see [2]):

$$\varepsilon = 1 + 4\pi \operatorname{Re} {}^1\!/_2 \operatorname{Sp} \varkappa_{\alpha\beta} = 1 - (\omega_0/\omega)^2, \tag{4}$$

Here ω_0 is the plasma frequency of the medium.

To analyze Eqs. (2) and (3) (see Secs. 3 and 4 below), it is necessary to calculate the tensor $\kappa_{\alpha\beta}$ (Secs. 1 and 2). General formulas for $\kappa_{\alpha\beta}$ can be readily obtained by solving the self-consistent system consisting of the linearized kinetic equation and Maxwell's equations. For a magnetoactive plasma, such a procedure yields, for example for κ_{12} (see [3]):

$$\varkappa_{12} = -2\pi i\frac{e^2}{\omega}\int_0^\infty dp\, p^2 \frac{v}{\tilde{\omega}_H}\frac{E}{mc^2}\int_0^\pi d\theta \sin^2\theta \sum_{n=-\infty}^{+\infty}\frac{J_n'(x)J_n(x)}{n-q-i\delta} \times\left[N_p'\left(\sin\varphi\cos\theta - \frac{n}{x}\cos\varphi\sin\theta\right) + \frac{N_\theta'}{p}\left(\frac{n}{x}\frac{kv}{\omega} - \left(\frac{n}{x}\cos\varphi\cos\theta + \sin\varphi\sin\theta\right)\right)\right]. \tag{5}$$

Here p, v, and E are the momentum, velocity, and total energy of the electron, θ is the angle between **p** and **H**,

φ is the angle between **k** and **H**,

$$x = \frac{kv}{\tilde{\omega}_H}\frac{E}{mc^2}\sin\varphi\sin\theta, \quad q = \frac{\omega}{\tilde{\omega}_H}\frac{E}{mc^2}\left(1 - \frac{kv}{\omega}\cos\varphi\cos\theta\right),$$

$\tilde{\omega}_H = |e|H/mc$ is the gyrofrequency of the nonrelativistic electrons, J_n is the Bessel function, and $N(p, \theta)$ is the electron distribution function.

In the calculation[1)] of $\kappa_{\alpha\beta}$ we confine ourselves to the region of sufficiently high frequencies, when

$$\omega \gg \tilde{\omega}_H, \tag{6a}$$

and in addition we assume that the following inequalities hold

$$\eta = \sqrt{2\left(1 - \frac{kv}{\omega}\right)} \approx \sqrt{\left(\frac{mc^2}{E}\right)^2 + \left(\frac{\omega_0}{\omega}\right)^2} \ll 1; \tag{6b}$$

$$\varphi \gg \eta, \qquad \pi - \varphi \gg \eta. \tag{6c}$$

Assuming that the conditions (6) are satisfied, we sum in (5) over n and integrate over $d\theta$ for the isotropic function[2)] $N(p, \theta) = N(p)$. As a result we obtain $\kappa_{12}^{(r)}$—that part of the component κ_{12} which is due to the interaction between the radiation and the relativistic electrons:

$$\varkappa_{12}^{(r)} = -\varkappa_{21}^{(r)} = \frac{-i}{2}\frac{e^2mc^4}{\omega^2}\frac{\tilde{\omega}_H\cos\varphi}{\omega}\cdot$$

$$\times \int dE\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]E\left(2\ln\frac{1}{\eta} - f_{12}(z)\right), \tag{7a}$$

$$f_{12}(z) = \int_z^\infty dy\left[i\int_0^\infty dv\exp\left\{ivy + i\frac{v^3}{3}\right\} + \frac{1}{y}\right]$$

$$+ \frac{2}{3} + i\frac{2z}{3}\int_0^\infty dv\exp\left\{ivz + i\frac{v^3}{3}\right\} - 2\ln(2\sin\varphi), \tag{7b}$$

where

$$z = \eta^2\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi}\frac{E}{mc^2}\right)^{2/3} \equiv \left(\frac{3}{2}\frac{\omega}{\omega_c}\right)^{2/3},$$

$$\omega_c = \frac{3\sin\varphi}{2}\tilde{\omega}_H\frac{mc^2}{E}\eta^{-3}.$$

The frequency ω_c has been introduced in connection with the fact that the intensity of the synchrotron radiation of the electron with energy E has a maximum at a frequency $\omega \sim \omega_c$, i.e., when $z \sim 1$; N(E) is the energy distribution function of the relativistic electrons and is connected with N(p) by

$$4\pi N(p)p^2dp = N(E)dE, \qquad E = cp. \tag{8}$$

With increasing $\psi = \theta - \varphi$ (the angle between the direction of radiation and the electron momentum), the contribution of the electrons to the antihermitian part of $\kappa_{12}^{(r)}$ decreases rapidly like $\exp[-|\psi/\eta|^3]$, and the contribution to the Hermitian part decreases in accordance with the power law $\sim 1/|\psi|$ when $|\psi| \gg \eta$, leading to the appearance of a term containing $\ln(1/\eta)$ in Im $\kappa_{22}^{(r)}$.

For the components $\kappa_{11,22}^{(r)}$, analogous calculations yield

$$\varkappa_{11,22}^{(r)} = -\frac{1}{4}\frac{e^2m^2c^6}{\omega^2}\left(\frac{\tilde{\omega}_H\sin\varphi}{\omega}\right)^2\int dE\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]\eta^{-4}f_{11,22}(z),$$

$$f_{11,22}(z) = \frac{z^3}{2}\int_{-\infty}^{+\infty}\frac{dv}{v}\exp\left\{ivz + i\frac{v^3}{3}\right\} + z^2\left(1 \pm \frac{1}{2}\right)\int_{-\infty}^{+\infty}dv\cdot v \tag{9a}$$

$$\times\exp\left\{ivz + i\frac{v^3}{3}\right\} \pm z^2\int_0^\infty dv\cdot v\cos\left\{vz + \frac{v^3}{3}\right\}. \tag{9b}$$

The anti-Hermitian part of $\kappa_{\alpha\beta}^{(r)}$(Im $\kappa_{11,22}^{(r)}$, Re $\kappa_{12,21}^{(r)}$) describes reabsorption (stimulated emission and absorption) of the radiation by the relativistic electrons. We can verify that the anti-Hermitian part is expressed in terms of the Macdonald functions $\mathscr{K}_{1/3}$, $\mathscr{K}_{2/3}$, $\mathscr{K}_{5/3}$ of the argument ω/ω_c and coincides with the corresponding expressions obtained in [2] by a somewhat different method. The Hermitian part of $\kappa_{\alpha\beta}^{(r)}$(Re $\kappa_{11,22}^{(r)}$, Im $\kappa_{12,21}^{(r)}$) corresponds to the role played by the relativistic electrons in a magnetic field as a transparent anisotropic medium.

2. We shall use formulas (7) and (9) to derive expressions for $\kappa_{\alpha\beta}^{(r)}$ in the case of various distribution functions N(E).

A. Let us consider the monoenergetic spectrum:

$$N(E)dE = N_e\delta(E - E_0)dE = N_e\delta(\mathscr{E} - \mathscr{E}_0)d\mathscr{E}. \tag{10}$$

Here N_e is the concentration of the relativistic electrons, and $\mathscr{E} = E/mc^2$ is the dimensionless energy. Obviously, $\mathscr{E}\eta \geq 1$.

Substituting (10) in (7) and (9) and integrating by parts, we obtain[3)]

$$\varkappa_{12}^{(r)} = -\varkappa_{21}^{(r)} = i\cos\varphi\frac{N_ee^2}{m\omega^2}\frac{\tilde{\omega}_H}{\omega}\frac{1}{\mathscr{E}^2}$$

$$\times\left\{\ln\frac{1}{\eta} + \frac{1}{\eta^3\mathscr{E}^3} - \frac{1}{2}f_{12}(z) - \frac{z}{3}f_{12}'(z)\left(1 - \frac{6}{\eta^2\mathscr{E}^2}\right)\right\},$$

$$\varkappa_{11,22}^{(r)} = \frac{\sin^2\varphi}{2}\frac{N_ee^2}{m\omega^2}\left(\frac{\tilde{\omega}_H}{\omega}\right)^2\frac{1}{\eta}$$

$$\times\left\{\frac{2}{\eta^5\mathscr{E}^5}f_{11,22}(z) + \frac{z}{3\eta^3\mathscr{E}^3}f_{11,22}'(z)\left(1 - \frac{6}{\eta^2\mathscr{E}^2}\right)\right\}. \tag{11}$$

Greatest interest attaches to the frequency region in which

$$\omega/\omega_c \sim 1, \qquad z \sim 1. \tag{12}$$

Under the condition (12), the modulus of the complex functions $f_{11,22}(z)$ and $f_{12}(z)$ will be of the order of unity. It follows therefore that in the region of frequencies (12) the ratio of the nondiagonal components to the diagonal ones is (see (6))

$$\frac{|\varkappa_{12,21}^{(r)}|}{|\varkappa_{11,22}^{(r)}|} \approx \frac{|\mathrm{Im}\,\varkappa_{12,21}^{(r)}|}{|\varkappa_{11,22}^{(r)}|} \sim \eta\ln\frac{1}{\eta}\operatorname{ctg}\varphi \ll 1, \tag{13}$$

and in this sense the tensor $\kappa_{\alpha\beta}^{(r)}$ is diagonal. The inequality (13) allows us to draw a conclusion that the normal waves are linearly polarized in a relativistic magnetoactive plasma. This conclusion, incidentally, is quite natural. For details see [1] and Sec. 3 below. We note here that the tensor $\kappa_{\alpha\beta}^{(r)}$, describing the interaction of the radiation with the nonrelativistic ("cold") plasma, has in the frequency region (6a) the form[4]

1)The anti-Hermitian part of the tensor $\kappa_{\alpha\beta}$ was calculated in [2] subject to the conditions (6). It is impossible to use the dispersion relations to reconstruct the Hermitian part from the anti-Hermitian part, since the anti-Hermitian part is known only in the frequency region (6a).

2)These calculations are performed in the Appendix, where the case $N'_\theta \neq 0$ is also discussed.

3)All the energy-dependent quantities are taken in this section at $E = E_0$; for brevity, we shall henceforth omit the index ^{0}.

$$\varkappa_{12}^{(c)}=-\varkappa_{21}^{(c)}=i\cos\varphi\frac{n_e e^2}{m\omega^2}\frac{\tilde{\omega}_H}{\omega}\left(1-2i\frac{\nu}{\omega}\right),$$

$$\varkappa_{11,22}^{(c)}=\mp\frac{\sin^2\varphi}{2}\frac{n_e e^2}{m\omega^2}\left(\frac{\tilde{\omega}_H}{\omega}\right)^2$$

$$+i\frac{\nu}{\omega}\frac{n_e e^2}{m\omega^2}\left[1-\left(\frac{\tilde{\omega}_H}{\omega}\right)^2\left(1+\frac{3\sin^2\varphi}{2}\mp\frac{3\sin^2\varphi}{2}\right)\right], \quad (14)$$

where n_e is the concentration of the nonrelativistic electrons and ν is the effective collision frequency ($\nu \ll \omega$). In this case the following inequality is satisfied

$$\frac{|\varkappa_{12,21}^{(c)}|}{|\varkappa_{11,22}^{(c)}|}\approx\frac{2\cos\varphi}{\sin^2\varphi}\frac{\omega}{\tilde{\omega}_H}\gg 1, \quad (15)$$

provided the angle ω is not too close to $\pi/2$. The inequality (15) shows that the tensor $\kappa_{\alpha\beta}^{(c)}$ is antidiagonal. By virtue of (14) and (15), the normal waves in the cold plasma are circularly polarized.

B. Let us take the power-law spectrum

$$N_e(E)\,dE=\begin{cases}\tilde{N}_e\mathscr{E}^{-\gamma}d\mathscr{E} & \text{for } \mathscr{E}_{min}<\mathscr{E}<\mathscr{E}_{max}\\ 0 & \text{for } \mathscr{E}<\mathscr{E}_{min} \text{ for } \mathscr{E}>\mathscr{E}_{max}\end{cases}. \quad (16)$$

Here $\mathscr{E}_{min,max} = E_{min,max}/mc^2$ and $\tilde{N}_e$ is a parameter with the dimension of concentration; $\tilde{N}_e$ differs from the actual concentration N_e by a numerical factor which may be large:[4)]

$$N_e=\tilde{N}_e\frac{1}{1-\gamma}\left[\mathscr{E}_{max}^{1-\gamma}-\mathscr{E}_{min}^{1-\gamma}\right].$$

We shall assume that $\mathscr{E}_{max} \gg \mathscr{E}_{min} \gg 1$; in addition, we put

$$\eta\approx mc^2/E=1/\mathscr{E}. \quad (17)$$

As is well known (see, for example,[1]), formula (17) can be used when $\omega \gg \omega_0^2/\tilde{\omega}_H\sin\varphi$.

Greatest interest attaches to the frequency region in which

$$z_{min}\equiv\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi}\right)^{2/3}\mathscr{E}_{max}^{-4/3}\ll 1,$$

$$z_{max}\equiv\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi}\right)^{2/3}\mathscr{E}_{min}^{-4/3}\gg 1. \quad (18)$$

Assuming that the conditions (17)–(18) are satisfied, we substitute (16) in (7) and (9). Carrying out in (7) and (9) integration for the anti-Hermitian part of $\kappa_{\alpha\beta}^{(r)}$, we can put $\mathscr{E}_{min} = 0$ and $\mathscr{E}_{max} = \infty$, since the integrals converge rapidly at the indicated limits. As to the Hermitian part, it depends little on the upper limit of the energy spectrum, and depends strongly on its lower limit; we can therefore use $\mathscr{E}_{max} = \infty$ when calculating the Hermitian part in (7) and (9), but we must retain $\mathscr{E}_{min}$. The main contribution to the integral, which determines $\mathrm{Im}\,\kappa_{12}^{(r)}$, is made by the term containing $\ln(1/\eta)$; discarding the other terms of $\mathrm{Im}\,\kappa_{12}^{(r)}$, we obtain

$$\varkappa_{12}^{(r)}=-\varkappa_{21}^{(r)}=i\cos\varphi\frac{N_e e^2}{m\omega^2}\frac{\tilde{\omega}_H}{\omega}\left[\frac{\ln\mathscr{E}_{min}}{\mathscr{E}_{min}^{\gamma+1}}\frac{1}{\gamma+1}\right. \quad (19)$$

$$\left.-i\frac{2+\gamma}{8\sqrt{3}}\frac{\gamma+3}{\gamma+1}3^{\gamma/2}\Gamma\left(\frac{3\gamma+7}{12}\right)\Gamma\left(\frac{3\gamma+11}{12}\right)\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi}\right)^{-(\gamma+1)/2}\right]$$

Calculating $\mathrm{Re}\,\kappa_{11,22}^{(r)}$ in (9), we expand the integrand in powers of z^{-1} and integrate with respect to dz, with limits 1 and z_{max}. As a result we obtain

$$\varkappa_{11,22}^{(r)}=\frac{\sin^2\varphi}{2}\frac{N_e e^2}{m\omega^2}\left(\frac{\tilde{\omega}_H}{\omega}\right)^2\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi}\right)^{-(\gamma-2)/2}$$

$$\times\left\{\mp\frac{2}{\gamma-2}\left(\left(\frac{\omega}{\tilde{\omega}_H\sin\varphi\mathscr{E}_{min}^2}\right)^{-(\gamma-2)/2}-1\right)+i\frac{3^{(\gamma+1)/2}}{16}\right.$$

$$\left.\times\Gamma\left(\frac{3\gamma+2}{12}\right)\Gamma\left(\frac{3\gamma+10}{12}\right)\left[\gamma+\frac{10}{3}\pm(\gamma+2)\right]\right\}. \quad (20)$$

The described operations are valid for $\gamma \geq 2$. When $\gamma < 2$, the first term in the curly brackets in (20) can be taken in the form $\mp 2/(\gamma-2)$ (see (18)). Then $\mathrm{Re}\,\kappa_{11,22}^{(r)}$ differs from the more accurate expression by a numerical factor of the order of unity. To calculate $\mathrm{Re}\,\kappa_{12,21}^{(r)}$ and $\mathrm{Im}\,\kappa_{11,22}^{(r)}$, we used the integral representations of the Macdonald functions (see the Appendix of [5] and also formula 6.561 (16) in [6]).

For a power-law spectrum the ratio $|\kappa_{12,21}^{(r)}|/|\kappa_{11,22}^{(r)}|$ for $\gamma < 2$ is equal to

$$\frac{|\varkappa_{12,21}^{(r)}|}{|\varkappa_{11,22}^{(r)}|}\sim\left(\frac{\omega}{\omega_{min}}\right)^{\gamma/2}\frac{\ln\mathscr{E}_{min}}{\mathscr{E}_{min}}\operatorname{ctg}\varphi, \quad (21a)$$

and for $\gamma > 2$

$$\frac{|\varkappa_{12,21}^{(r)}|}{|\varkappa_{11,22}^{(r)}|}\sim\left(\frac{\omega}{\omega_{min}}\right)\frac{\ln\mathscr{E}_{min}}{\mathscr{E}_{min}}\operatorname{ctg}\varphi. \quad (21b)$$

We have introduced the frequency $\omega_{min} = \tilde{\omega}_H\sin\varphi\,\mathscr{E}_{min}^2$, which is the characteristic frequency of the synchrotron radiation of the electrons at the lower limit of the spectrum. As follows from (18), $\omega/\omega_{min} \gg 1$.

Unlike the case of the monoenergetic spectrum (see (13)), expressions (21) can be either much smaller or much larger than unity. This means that the normal waves in a relativistic plasma with a power-law electron spectrum can be polarized either linearly[5)] or circularly. Of course, an intermediate case of elliptical polarization is also possible. At a fixed value of ω/ω_{min}, the polarization of the normal waves depends on the quantity $\omega_{min} = E_{min}/mc^2$: if ω_{min} is sufficiently large, the polarization is linear, and at moderate values of ω_{min} it is circular (see below).

3. We have thus calculated $\kappa_{\alpha\beta}^{(r)}$, the relativistic part of the dielectric susceptibility tensor (see (11) or (19), (20)).

We shall now use Eq. (2) to obtain expressions for the polarization vectors and the absorption coefficient of the transverse normal waves (see [4,8]). Putting in (2) $k = k_0 + \Delta k$, where $k_0 = \omega\sqrt{\epsilon}/c \approx \omega/c$ and $|\Delta k| \ll k_0$, we get

$$4\pi\frac{\omega}{c}\varkappa_{\alpha\beta}E_\beta=2E_\alpha\Delta k. \quad (22)$$

The total tensor $\kappa_{\alpha\beta}$ which enters in (2) and (22) is obviously given by $\kappa_{\alpha\beta} = \kappa_{\alpha\beta}^{(r)} + \kappa_{\alpha\beta}^{(c)}$, where $\kappa_{\alpha\beta}^{(c)}$ is the tensor of the cold plasma. Separating the Hermitian and anti-Hermitian parts of $\kappa_{\alpha\beta}$, we introduce the notation

$$4\pi\frac{\omega}{c}\varkappa_{\alpha\beta}=\begin{pmatrix}+h & +if\\ -if & -h\end{pmatrix}+i\begin{pmatrix}\mu+\lambda & +i\rho\\ -i\rho & \mu-\lambda\end{pmatrix}. \quad (23)$$

4)The spectrum (16) is frequently written in the form $K_eE^{-\gamma}dE$; obviously, $N_e = (mc^2)^{1-\gamma}K_e$.

5)Linear polarization is implied in [2,7]. The need for this refinement was called to our attention by V. V. Zheleznyakov.

The contribution of the cold plasma in the expressions for h, f, μ, etc. can be easily obtained by comparing (23) with (14); the contribution of the relativistic electrons is determined by formulas (11) or (19) and (20).

The polarization of the normal waves depends strongly on the ratio of the components f and ρ to the components h and λ in (23). Let us ascertain under which conditions the tensor (23) can be regarded as diagonal. In the high frequency region $\omega \gg \tilde{\omega}_H$, the relativistic anti-Hermitian part of (23) is diagonal: $|\rho^{(r)}| \ll |\lambda^{(r)}|$. Therefore the tensor (23) will be diagonal under the condition

$$|f| \ll |h| \text{ and } |f| \ll |\lambda|. \quad (24)$$

For the monoenergetic spectrum (10), in the frequency region $\omega \sim \omega_c$, the conditions (24) reduce to $|\operatorname{Im} \kappa_{12}^{(c)}| \ll |\kappa_{11,22}^{(r)}|$, which yields

$$n_e \ll N_e \eta \operatorname{tg} \varphi = N_e[(mc^2/E)^2 + (\omega_0/\omega)^2]^{1/2} \operatorname{tg} \varphi. \quad (25)$$

For the power-law spectrum (16), the limitation on n_e is insufficient. The conditions (24) reduce to $|\operatorname{Im} \kappa_{12}^{(c)}| \ll \kappa_{11\,22}^{(r)}|$ and to $|\operatorname{Im} \kappa_{12}^{(r)}| \ll |\kappa_{11\,22}^{(r)}|$. When $\gamma < 2$ we obtain respectively

$$n_e \ll \tilde{N}_e \left(\frac{\omega}{\omega_{min}}\right)^{-\gamma/2} \mathscr{E}_{min}^{-\gamma} \operatorname{tg} \varphi, \quad (26a)$$

$$\left(\frac{\omega}{\omega_{min}}\right)^{\gamma/2} \frac{\ln \mathscr{E}_{min}}{\mathscr{E}_{min}} \operatorname{ctg} \varphi \ll 1; \quad (26b)$$

and when $\gamma > 2$

$$n_e \ll \tilde{N}_e \left(\frac{\omega}{\omega_{min}}\right)^{-1} \mathscr{E}_{min}^{-1-\gamma/2} \operatorname{tg} \varphi, \quad (27a)$$

$$\left(\frac{\omega}{\omega_{min}}\right) \frac{\ln \mathscr{E}_{min}}{\mathscr{E}_{min}} \operatorname{ctg} \varphi \ll 1. \quad (27b)$$

We recall that

$$\omega_{min} = \tilde{\omega}_H \mathscr{E}_{min} \sin \varphi = \tilde{\omega}_H (E_{min}/mc^2)^2 \sin \varphi,$$

and that we have confined ourselves to the frequency region $\omega \gg \omega_{min}$. The condition (26a) in a somewhat different form was derived in [1].

Let the conditions (24) and the ensuing conditions (25)–(27) be satisfied, so that the tensor (23) can be regarded as diagonal. Substituting (23) in (22), we get $e_\alpha^{(1)}$ and $e_\alpha^{(2)}$, the polarization vectors of the normal waves, which are the normalized eigenvectors of the operator $4\pi\omega\kappa_{\alpha\beta}/c$, and also $\Delta k^{(1)}$ and $\Delta k^{(2)}$. It is obvious that $\operatorname{Im} \Delta k^{(1,2)}$ is the amplitude damping coefficient of the corresponding normal wave. Retaining only the lower powers of the small quantities f and ρ, we obtain

$$2\Delta k^{(1,2)} = i\mu \pm (h + i\lambda),$$
$$e_1^{(1)} = 1, \quad e_2^{(1)} = -\frac{i}{2}\frac{f + i\rho}{h + i\lambda}, \quad |e_2^{(1)}| \ll |e_1^{(1)}|,$$
$$e_1^{(2)} = -\frac{i}{2}\frac{f + i\rho}{h + i\lambda}, \quad e_2^{(2)} = 1, \quad |e_1^{(2)}| \ll |e_2^{(2)}|. \quad (28)$$

Formulas (28), confirm the previously made conclusion that the first and second normal waves are polarized almost linearly, across and along the projection **H** on the plane perpendicular to **k**.

When the inequalities (26a) or (27a) are satisfied, but the inequalities (26b) or (27b) have an opposite sign, the influence of the cold plasma can be neglected, but nevertheless the normal waves are circularly polarized. Indeed, under the indicated conditions, the quantity f greatly exceeds all the other quantities in (22) (see (21)). Retaining only the lowest powers of the small quantity f^{-1}, we obtain from (22)

$$2\Delta k^{(1,2)} = i\mu \pm (f + i\rho),$$
$$e_1^{(1)} = \frac{1}{\sqrt{2}}, \quad e_2^{(1)} = -\frac{i}{\sqrt{2}}\left(1 - \frac{h + i\lambda}{f}\right),$$
$$e_1^{(2)} = \frac{i}{\sqrt{2}}\left(1 - \frac{h + i\lambda}{f}\right), \quad e_2^{(2)} = \frac{1}{\sqrt{2}}. \quad (29)$$

Formulas (29) show that the first and second normal waves are almost circularly polarized. The circular polarization in the relativistic plasma is possible for the following reason: the main contribution to the Hermitian part of the tensor $\kappa_{\alpha\beta}^{(r)}(\omega)$ is made by the nonrelativistic electrons with energy near the lower boundary of the spectrum. The frequency of the synchrotron radiation of these electrons is much lower than the frequency under consideration, $\omega_{min} \ll \omega$, and in this sense electrons with $\mathscr{E} \sim \mathscr{E}_{min}$ differ little from nonrelativistic electrons. Therefore if $\mathscr{E}_{min}$ is small compared with ω/ω_{min}, the polarization of the normal waves can be the same as in a nonrelativistic plasma. It must be emphasized that in this case f^{-1} is the characteristic distance through which the plane of the linear polarization is rotated, and is much smaller than μ^{-1}, which is characteristic distance over which the wave damping takes place. Indeed, if inequalities opposite to (26b) or (27b) are satisfied, we obtain from (21)

$$\frac{f}{\mu} \sim \left(\frac{\omega}{\omega_{min}}\right)^{\gamma/2} \frac{\ln \mathscr{E}_{min}}{\mathscr{E}_{min}} \operatorname{ctg} \varphi \gg 1 \quad (\gamma < 2),$$
$$\frac{f}{\mu} \sim \left(\frac{\omega}{\omega_{min}}\right) \frac{\ln \mathscr{E}_{min}}{\mathscr{E}_{min}} \operatorname{ctg} \varphi \gg 1 \quad (\gamma > 2).$$

It follows therefore that in a relativistic plasma, under the conditions indicated above, a strong depolarization of the radiation and rotation of the plane of polarization without noticeable absorption are possible. In this connection, we note the estimate of the cold-plasma concentration, made in [9] on the basis of the observations of [10]. The possibility of rotation of the plane of polarization in a relativistic plasma was not taken into account there, and this may alter the results. For more details see [11].

4. A curious situation arises when the Hermitian and anti-Hermitian parts of the tensor $4\pi c^{-1}\omega\kappa_{\alpha\beta}$ are equal in order of magnitude, for example when $n_e \sim N_e \eta \tan \varphi$ (see (25)). In this case the normal waves may not be orthogonal to each other:

$$|e_\alpha^{(1)} e_\alpha^{(2)*}| \sim 1, \quad (30)$$

where the asterisk denotes the complex conjugate.

Let us imagine an observer who has registered a certain polarization and measures at each point the intensity of the radiation with this polarization. When the radiation is transferred in a medium in which (30) takes place, the observer, having fixed a normal polarization $e_\alpha^{(j)}$, notices that the intensity, generally speaking, does not satisfy the equation

$$\left(\frac{\partial}{\partial t} + c\frac{\mathbf{k}}{k}\frac{\partial}{\partial \mathbf{r}}\right) I^{(j)} = c(P^{(j)} - \mu^{(j)} I^{(j)}), \quad j = 1, 2; \quad (31)$$

here the absorption coefficient is $\mu^{(j)} = 2 \operatorname{Im} \Delta k^{(j)}$.

Indeed, assume that the medium contains, besides the radiation with polarization, say, $e_\alpha^{(1)}$, also radiation with polarization $e_\alpha^{(2)}$; the electric field of the latter, under condition (30), will be projected on the direction of $e_\alpha^{(1)}$. If at the same time $\mu^{(1)} \neq \mu^{(2)}$, then Eq. (31) will not be satisfied.

This raises the question: what should the polarization direction be in order for the intensity to vary, in the case of transfer, in accordance with (31) regardless of the presence of radiation with another polarization. It is almost obvious that polarizations possessing this property (these can be called transfer polarizations, $t_\alpha^{(j)}$, exist and are orthogonal to the normal polarizations

$$t_\alpha^{(1)} e_\alpha^{(2)*} = t_\alpha^{(2)} e_\alpha^{(1)*} = 0. \tag{32}$$

Indeed, $I^{(j)} = t_\alpha^{(j)*} t_\beta^{(j)} I_{\alpha\beta}$ is the intensity of the radiation with polarization $t_\alpha^{(j)}$, separated from the total radiation.[12] Multiplying Eq. (3) by $t_\alpha^{(j)*} t_\beta^{(j)}$, we observe that (3) takes on the form (31) when

$$4\pi \frac{\omega}{c} \tilde{\varkappa}_{\alpha\beta} t_\beta^{(j)} = 2\Delta k^{(j)*} t_\alpha^{(j)}, \tag{33}$$

where $\tilde{\kappa}_{\alpha\beta} = \kappa^*_{\beta\alpha}$ (compare with (22)). From (33) we get immediately (32) if $\Delta k^{(1)} \neq \Delta k^{(2)}$. Thus, the vectors $t_\alpha^{(j)}$ are the eigenvectors of the operator $4\pi\omega\tilde{\kappa}_{\alpha\beta}/c$.

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APPENDIX

Let us calculate the integral with respect to $d\theta$ in (5), henceforth denoted S_{12} (see also the calculations in [2,13], which has many features in common with the present calculations). Using the integral representation of the Bessel functions, we write

$$J_n(x)J_n'(x) = \frac{-i}{4\pi^2} \int_{-\pi}^{+\pi} d\alpha \int_{-\pi}^{+\pi} d\beta \sin\beta \exp\{i(\alpha+\beta)n - ix(\sin\alpha + \sin\beta)\}$$

$$= \frac{-i}{4\pi^2} \int_{-2\pi}^{+2\pi} d\sigma \int_0^{2\pi-|\sigma|} d\tau \sin\frac{\sigma}{2}\cos\frac{\tau}{2} \exp\left\{in\sigma - 2ix\sin\frac{\sigma}{2}\cos\frac{\tau}{2}\right\}. \tag{A.1}$$

Here $\sigma = \alpha + \beta$ and $\tau = \alpha - \beta$. Substituting (A.1) in (5), we average S_{12} over a certain frequency interval $\Delta\omega$, such that $\tilde{\omega}_H \ll \Delta\omega \ll \omega$ (see (6a)). When ω is varied in the interval $\Delta\omega$, the change of the quantities q and x will be of the order of

$$\frac{\omega\Delta}{\tilde{\omega}_H} \frac{E}{mc^2} \gg 1,$$

and therefore the discrete character of the summation in n is smoothed out to a considerable degree and it can be replaced by integration with respect to dn. Putting

$$\frac{1}{n-q-i\delta} = \mathrm{P}\frac{1}{n-q} + i\pi\delta(n-q),$$

where P is the symbol of the principal value, and integrating in (5) with respect to dn, we obtain

$$\sum_n \frac{n}{x} \frac{J_n(x)J_n'(x)}{n-q-i\delta} = \frac{-i}{4\pi^2 x} \int_{-2\pi}^{+2\pi} d\sigma \int_0^{2\pi-|\sigma|} d\tau \times \sin\frac{\sigma}{2}\cos\frac{\tau}{2} \exp\left\{iq\sigma - 2ix\sin\frac{\sigma}{2}\cos\frac{\tau}{2}\right\}$$

$$[i\pi(\operatorname{sign}\sigma + 1) + 2\pi\delta(\sigma)] = \frac{1}{2\pi}\frac{q}{x} \int_0^{2\pi} d\tau \int_0^{2\pi-\tau} d\sigma \times \sin\frac{\sigma}{2}\cos\frac{\tau}{2} \exp\left\{iq\sigma - 2ix\sin\frac{\sigma}{2}\cos\frac{\tau}{2}\right\};$$

$$\sum_n \frac{J_n(x)J_n'(x)}{n-q-i\delta} = \frac{1}{2\pi} \int_0^{2\pi} d\tau \int_0^{2\pi-\tau} d\sigma \sin\frac{\sigma}{2}\cos\frac{\tau}{2} \cdot \times \exp\left\{iq\sigma - 2ix\sin\frac{\sigma}{2}\cos\frac{\tau}{2}\right\}.$$

Substituting the latter formulas in (5), we write S_{12} in the following form:

$$S_{12} = \frac{1}{2\pi} \int_0^{\pi} d\theta \sin^2\theta \left[N_p' \sin\varphi\cos\theta - \frac{N_\theta'}{p}\sin\varphi\sin\theta + \frac{q}{x}\left(\frac{N_\theta'}{p}\left(\frac{kv}{\omega} - \cos\varphi\cos\theta\right) - N_p'\cos\varphi\sin\theta\right)\right]$$
$$\times \int_0^{2\pi} \left\{\left\langle \int_0^{2\pi-\tau} \sin\frac{\sigma}{2}\cos\frac{\tau}{2}\exp\left\{iq\sigma - 2ix\sin\frac{\sigma}{2}\cos\frac{\tau}{2}\right\} d\sigma + \frac{1}{2}\frac{\cos(\tau/2)}{(q - x\cos(\tau/2))^2}\right\rangle - \frac{1}{2}\frac{\cos(\tau/2)}{(q - x\cos(\tau/2))^2}\right\} d\tau. \tag{A.2}$$

In the integration of the term in the round brackets in (A.2), the region $\sigma, \tau, \psi \equiv \theta - \varphi \gg \eta$ is insignificant. Extending the integration with respect to $d\psi$ and $d\tau$ from $-\infty$ to $+\infty$, and with respect to $d\sigma \equiv 2(x\cos(\tau/2))^{-1/3} d\nu$ from 0 to $+\infty$, we obtain, after expanding all the quantities in powers of σ, τ, and ψ and retaining the first nonvanishing term

$$\frac{\eta^6}{2z^3\sin\varphi}\left(\frac{N_\varphi'}{p} - N_{p\varphi}''\right) \int_z^\infty dy \left(i\int_0^\infty d\nu \exp\left\{i\nu y + i\frac{\nu^3}{3}\right\} + \frac{1}{y}\right)$$
$$- \frac{\eta^6}{z^3\sin\varphi} N_p' \operatorname{ctg}\varphi \left[\frac{2}{3} + \frac{2i}{3} z \int_0^\infty d\nu \exp\left\{i\nu z + i\frac{\nu^3}{3}\right\} + \int_z^\infty dy\left(i\int_0^\infty d\nu \exp\left\{i\nu y + \frac{\nu^3}{3}\right\} + \frac{1}{y}\right)\right]. \tag{A.3}$$

In the integration of the last term in the curly brackets in (A.2), we have confined ourselves to the case of an isotropic function N(p). The integral with respect to $d\theta$ and $d\tau$ reduces in this case to a tabulated integral, and we obtain

$$2N_p'\cos\varphi\left(\frac{\tilde{\omega}_H}{\omega}\right)^2\left(\frac{mc^2}{E}\right)^2 \ln\frac{2\sin\varphi}{\eta}. \tag{A.4}$$

For a non-isotropic function, the last term in (A.2) can be integrated with logarithmic accuracy, and we obtain in addition to (A.4)

$$-\left(\frac{N_\varphi'}{p} - N_{p\varphi}'\right)\sin\varphi\left(\frac{\tilde{\omega}_H}{\omega}\right)^2\left(\frac{mc^2}{E}\right)^2 \ln\frac{1}{\eta}. \tag{A.5}$$

Of course, if expressions (A.4) and (A.5) are added, it is necessary to make in (A.4) the substitution

$$\ln\frac{2\sin\varphi}{\eta} \to \ln\frac{1}{\eta}.$$

The sum of (A.3) and (A.4) yields S_{12}. Substituting S_{12} in (5), we obtain (7). The calculation of (9) follows the same procedure; expressions for $\kappa_{11,22}$ analogous to (5), can be found, for example, in [3].

[1] V. L. Ginzburg and S. I. Syrovatsky, Ann. Rev. Astron. and Astroph. 7 (1969).
[2] V. N. Sazonov and V. N. Tsytovich, Radiofizika 11, 1287 (1968).

[3] A. A. Andronov, V. V. Zheleznyakov, and M. I. Petelin, Radiofizika **7**, 251 (1964).
[4] V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in a Plasma) Fizmatgiz, 1967 [Addison-Wesley, 1964].
[5] D. S. Kuznetsov, Spetsial'nye funktsii (Special Functions) Vysshaya Shkola, 1965.
[6] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products) Fizmatgiz, 1962 [Academic, 1966].
[7] V. L. Ginzburg, V. N. Sazonov, and S. I. Syrovatskiĭ, Usp. Fiz. Nauk **94**, 63 (1968) [Sov. Phys.-Usp. **11**, 34 (1968)].
[8] V. M. Agranovich and V. L. Ginzburg, Kristallooptika s uchetom prostranstvennoĭ dispersii i teorii éksitonov (Spatial Dispersion in Crystal Optics and Exciton Theory) Nauka, 1965 [Wiley, 1966].
[9] M. Rees and M. Simon, Astroph. J. **152**, L145 (1968).
[10] H. D. Aller and F. T. Haddok, Astroph. J. **147**, 833 (1967).
[11] V. N. Sazonov, Astron. Zh. **46**, (1969) [Sov. Astron.-AJ **13** (1969)].
[12] A. I. Akhiezer and V. V. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics), Fizmatgiz, 1959 [Interscience, 1963].
[13] K. S. Westfold, Astroph. J. **130**, 241 (1959).

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