# ON THE SPECTRAL THEORY OF RADIATION TRANSPORT EQUATIONS

Yu. N. BARABANENKOV

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The problem of normal incidence of a plane wave on a plane scattering layer is considered. The Dyson and Bethe-Saltpeter equations are employed. Equations for the spectral field densities within and outside the scattering medium are derived in the approximation of weak nonlocality of the mass and intensity operators. Conditions are investigated under which the spectral densities as functions of the wave-vector modulus are concentrated in the close vicinity of the energy surface and the obtained equations can be reduced to the phenomenological transport equation. It is found that the transport equation describes only a part of the spectrum of a random field, which has been termed the Fraunhofer part. The contribution of the extra-Fraunhofer part of the spectrum (which cannot be described by the transport equation) to the field longitudinal correlation function is estimated.

A statistical derivation of the equation of radiation transfer in scattering media has been the subject of a number of papers. Under the most general assumptions with respect to the properties of the scattering medium, a transport equation was obtained in a joint paper by the author and V. M. Finkel'berg[1] 1). In that paper, the initial equations were those of Dyson and Bethe-Salpeter. On the basis of a paper by Finkel'berg[2], the Dyson equation was solved in neglecting spatial dispersion of the waves. The solution of the Bethe-Salpeter equation was represented in the form of an iteration series, each term of which was simplified in the Fraunhofer approximation. It was assumed here that the mass operator and the intensity operator are characterized by finite effective nonlocality radii, i.e., they tend sufficiently rapidly to zero when their arguments move apart. Subsequently, this property of the mass operator and of the intensity operator, also called compactness, was more definitely formulated by the author[3]. From an intuitive point of view, the nonlocality radii of the mass operator and of the intensity operator play the role of the dimensions of the effective inhomogeneities of the medium.

The transformation of the Bethe-Salpeter equation in the Fraunhofer approximation has one shortcoming, namely, it is impossible to impart to this transformation an asymptotic character to make it possible to analyze with sufficient detail the conditions of its applicability, and consequently also the conditions for the applicability of the transport equation.

In this paper we propose a principally new method of transforming the Bethe-Salpeter equation; this method has an asymptotic character. It is based on the concept of spectral density of a random wave field. The method makes it possible, in particular, to formulate the conditions for the applicability of the aforementioned Fraunhofer approximation and to present a rather complete analysis of the conditions of applicability of the transport equation.

1)Unfortunately, some of the formulas in the cited papers contain misprints: in formula (24), a factor $k_0^2$ was left out in front of the integral; the argument of the exponential in formula (27) should have a plus sign.

## 1. GENERALIZED TRANSPORT EQUATION

As in the cited papers[1,3] we start from the Dyson equation for the average field $\langle \psi(\mathbf{r})\rangle$ and the Bethe-Salpeter equation for the average bilinear combination of the field $\langle \psi(\mathbf{r}_1)\overline{\psi}(\mathbf{r}_2)\rangle$. The Bethe-Salpeter equation can be rewritten in a form quite similar to the transport equation, by using the concept of the spectral density of the field. This concept was used by Dolin[4] and also by Kalashnikov and Ryazanov[5].

The spectral density of the field $f(\mathbf{R}, \mathbf{p})$ is defined as the Fourier transform of the average bilinear field combination $\langle \psi(\mathbf{r}_1)\overline{\psi}(\mathbf{r}_2)\rangle$ with respect to the difference $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ of the coordinates of the observation points. Thus, we can write

$$f(\mathbf{R},\mathbf{p})=(2\pi)^{-3}\int \exp(-i\mathbf{pr})\,d^3\mathbf{r}\left\langle \psi\left(\mathbf{R}+\frac{1}{2}\mathbf{r}\right)\overline{\psi}\left(\mathbf{R}-\frac{1}{2}\mathbf{r}\right)\right\rangle, \quad (1)$$

where $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ denotes the coordinates of the "center of gravity" of the observation points. In analogy with (1), we introduce also the Fourier transforms of the bilinear combinations of the average field $\langle \psi(\mathbf{r}_1)\rangle\langle \overline{\psi}(\mathbf{r}_2)\rangle$ and of the average Green's function $\langle G(\mathbf{r}_1, \mathbf{r}_1')\rangle\langle \overline{G}(\mathbf{r}_2, \mathbf{r}_2')\rangle$, with respect to the difference of the observation-point coordinates, putting

$$f_0(\mathbf{R},\mathbf{p})=(2\pi)^{-3}\int \exp(-i\mathbf{pr})\,d^3\mathbf{r}\left\langle \psi\left(\mathbf{R}+\frac{1}{2}\mathbf{r}\right)\right\rangle\left\langle \overline{\psi}\left(\mathbf{R}-\frac{1}{2}\mathbf{r}\right)\right\rangle, \quad (2)$$

$$\mathcal{F}(\mathbf{R},\mathbf{p};\mathbf{R}',\mathbf{p}')=(2\pi)^{-6}\int \exp(-i\mathbf{pr}+i\mathbf{p}'\mathbf{r}')\,d^3\mathbf{r}\,d^3\mathbf{r}'$$
$$\times\left\langle G\left(\mathbf{R}+\frac{1}{2}\mathbf{r},\mathbf{R}'+\frac{1}{2}\mathbf{r}'\right)\right\rangle\left\langle \overline{G}\left(\mathbf{R}-\frac{1}{2}\mathbf{r},\mathbf{R}'-\frac{1}{2}\mathbf{r}'\right)\right\rangle. \quad (3)$$

The functions $f_0(\mathbf{R}, \mathbf{p})$ and $\mathcal{F}(\mathbf{R}, \mathbf{p}; \mathbf{R}', \mathbf{p}')$ will be called the spectral densities corresponding to the average field and the average Green's function.

It is not particularly difficult to write the Bethe-Salpeter equation in terms of the introduced spectral densities (1), (2), and (3). Omitting the intermediate transformations, we present the final result. The Bethe-Salpeter equation in terms of the spectral densities is given by

$$f(\mathbf{R},\mathbf{p})=f_0(\mathbf{R},\mathbf{p})+\int \mathcal{F}(\mathbf{R},\mathbf{p};\mathbf{R}'',\mathbf{p}'')\,d^3\mathbf{R}''\,d^3\mathbf{p}''Q(\mathbf{R}'',\mathbf{p}'';\mathbf{R}',\mathbf{p}')$$
$$\times\, d^3\mathbf{R}'\,d^3\mathbf{p}' f(\mathbf{R}',\mathbf{p}'). \quad (4)$$

The kernel of equation $Q(\mathbf{R}, \mathbf{p}; \mathbf{R}', \mathbf{p}')$ is expressed in terms of the Fourier transform $\tilde{K}(\mathbf{p}, \mathbf{p}'; \mathbf{q}, \mathbf{q}')$ of the intensity operator $K(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$ by means of the relation

$$Q(\mathbf{R}, \mathbf{p}; \mathbf{R}', \mathbf{p}') = (2\pi)^{-6} \int \exp(i\mathbf{q}\mathbf{R} - i\mathbf{q}'\mathbf{R}')\, d^3\mathbf{q}\, d^3\mathbf{q}' \times \tilde{K}\left(\mathbf{p} + \frac{1}{2}\mathbf{q}, \mathbf{p}' + \frac{1}{2}\mathbf{q}'; \mathbf{p} - \frac{1}{2}\mathbf{q}, \mathbf{p}' - \frac{1}{2}\mathbf{q}'\right). \quad (5)$$

We shall call Eq. (4) the generalized transport equation.

Let us analyze the content of Eq. (4) from the physical point of view. We note first that this equation takes into account cooperative effects in multiple scattering of waves[1], specifically, cooperative effects are taken into account with the aid of the mass operator M and the intensity operator K. They consist in the fact that the scattering properties of the individual effective inhomogeneities of the discrete medium do not coincide with the scattering properties of its individual microscopic scatterers.

The spectral field density $f(\mathbf{R}, \mathbf{p})$ as a function of the coordinates $\mathbf{R}$ is characterized by a certain scale of spatial inhomogeneity $L_f$. Equation (4), generally speaking, imposes no limitation on the ratio of the scale $L_f$ to the nonlocality radius $l$ of the mass operator and of the intensity operator[2)]. In particular, if this ratio is of the order of unity or is small, i.e., $L_f \lesssim l$, then the spectral density of the field experiences in space a noticeable change over the dimension of the effective inhomogeneities of the medium.

If the medium is statistically homogeneous in the mean, then the mass operator M, the intensity operator K, the kernel of Eq. (4) Q, and the spectral density $\mathscr{F}$ corresponding to the average Green's function, all have the property of spatial translational invariance. It follows therefore, in particular, that the Fourier transform of the intensity operator is

$$\tilde{K}(\mathbf{p}, \mathbf{p}'; \mathbf{q}, \mathbf{q}') = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}' - \mathbf{q} + \mathbf{q}') \tilde{K}_0(\mathbf{p}, \mathbf{p}'; \mathbf{q}, \mathbf{q}').$$

When the medium is not statistically homogeneous in the mean, the aforementioned translational invariance is lost. One of the manifestations of the inhomogeneity of the medium is its boundedness. The presence of a boundary leads to certain physical effects. First, strictly speaking, the boundary of a scattered medium is not a smooth geometrical surface, but is subject to fluctuations of the order of the dimension of the effective scattering inhomogeneities $l$. Second, the presence of a boundary leads to the occurrence of mean-field waves reflected from the boundary. Third, the boundedness of the medium is manifest purely geometrically, when it becomes necessary in the calculation of the Fourier integrals to integrate not over the entire unbounded space, but only over a region that is bounded at least in one dimension.

We have listed the main physical phenomena that can be described by Eq. (4). Our main task is to determine the conditions under which this equation goes over into the phenomenological transport equation. We shall show that such a transition is realized by neglecting or approximately describing certain of the aforementioned physical phenomena. This will explain why Eq. (4) can be meaningfully called a generalized transport equation.

## 2. GENERALIZED TRANSPORT EQUATION IN THE APPROXIMATION OF WEAK NONLOCALITY OF THE MASS OPERATOR AND INTENSITY OPERATOR

The main assumption which we make with an aim at simplifying the generalized transport equation (4) is to neglect the spatial variation of the spectral field density over the dimension of the effective inhomogeneities of the medium. In other words, we assume that the spatial scale $L_f \gg l$. This is the approximation of weak nonlocality of the mass operator and intensity operator, which we used in the cited paper[3] in terms of the ray amplitude of the field. Since the radius of the spatial nonlocality of the kernel $Q(\mathbf{R}'', \mathbf{p}''; \mathbf{R}', \mathbf{p}')$ is of the order of the radius of nonlocality of the intensity operator $l$, the transition to the approximation of weak nonlocality of the intensity operator in Eq. (4) is realized by replacing the argument of the spectral density of the field $\mathbf{R}'$ by $\mathbf{R}''$:

$$f(\mathbf{R}', \mathbf{p}') \to f(\mathbf{R}'', \mathbf{p}') \quad \text{if} \quad L_f \gg l. \quad (6)$$

Let us simplify Eq. (4) further. We neglect the deviation of the properties of the medium from three-dimensional homogeneity in the mean, and also the fluctuations of the interface. Formally this is done by representing the kernel Q as follows:

$$Q(\mathbf{R}'', \mathbf{p}''; \mathbf{R}', \mathbf{p}') = \chi(\mathbf{R}'') Q(\mathbf{R}'' - \mathbf{R}', \mathbf{p}'', \mathbf{p}'), \quad (7)$$

where $\chi(\mathbf{R}'')$ is the characteristic function of the scattering region and $Q(\mathbf{R}'' - \mathbf{R}', \mathbf{p}'', \mathbf{p}')$ is the kernel of Eq. (4) for an unbounded medium that is homogeneous in the mean.

Once the fluctuations of the interface are neglected, Eq. (4) admits of further simplification. This is realized by introducing the spectral densities of the field inside and outside the scattering medium. We denote them by $f_i(\mathbf{R}, \mathbf{p})$ and $f_e(\mathbf{R}, \mathbf{p})$. The spectral density of the field $f_i$ inside ($f_e$ outside) of the medium is determined by the same formula (1) as the "total" spectral field density f, the only difference being that the integration over the difference of the coordinates $\mathbf{r}$ is carried out between such limits that both observation points $\mathbf{R} + \mathbf{r}/2$ and $\mathbf{R} - \mathbf{r}/2$ are located inside (outside) the medium. Consequently, the introduced spectral densities make it possible to calculate with the aid of the inverse Fourier transformation the average bilinear combination of the field $\langle \psi(\mathbf{r}_1) \overline{\psi}(\mathbf{r}_2) \rangle$, when both observation points $\mathbf{r}_1$ and $\mathbf{r}_2$ lie inside or outside the medium. We introduce analogously the spectral density inside and outside the medium, $f_i^0(\mathbf{R}, \mathbf{p})$ and $f_e^0(\mathbf{R}, \mathbf{p})$, corresponding to the average field. For the average Green's function we introduce spectral densities with two lower indices: $\mathscr{F}_{ii}(\mathbf{R}, \mathbf{p}; \mathbf{R}', \mathbf{p}')$ and $\mathscr{F}_{ei}(\mathbf{R}, \mathbf{p}; \mathbf{R}', \mathbf{p}')$. In the first, with index ii, the spatial coordinates $\mathbf{R}$ and $\mathbf{R}'$ correspond to points of space located inside the medium; in the second, with index ei, the point $\mathbf{R}$ is located outside the medium and $\mathbf{R}'$ inside. Whereas the spectral density $\mathscr{F}_{ii}$ characterizes radiation propagating inside the medium, $\mathscr{F}_{ei}$ characterizes radiation emerging from the medium.

A distinguishing feature of the spectral densities inside and outside the medium is that they vanish if their spatial arguments correspond to points located on the

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[2)] As follows from the optical theorem [6], the nonlocality radii of the mass operator and of the intensity operator are of the same order of magnitude.

boundary of the medium. However, the integrals of the aforementioned spectral densities with respect to the components of the wave vector **p**, parallel to the normal to the interface, differ from zero.

We can write Eq. (4) in terms of the introduced spectral densities inside and outside the medium. It breaks up into two equations: the integral equation for the spectral density of the field inside the medium and the integral relation for the spectral density outside the medium.

We limit the subsequent analysis to the case of planar symmetry, assuming that the scattering medium has the form of a plane layer of thickness L occupying the region of space $0 < z < L$. Assume that a plane wave $\psi_0(z) = \exp(ik_0 z)$ is normally incident on the boundary $z = 0$ from the free region of space $z < 0$. Then, in the approximation of weak nonlocality of the intensity operator (6), and neglecting the fluctuations of the interface (7), we obtain the following equations for the spectral densities of the field inside and outside the medium:

$$f_i(Z,\mathbf{p}) = f_i^0(Z,\mathbf{p}) + \int \mathcal{F}_{ii}(Z,\mathbf{p}_\perp,p_z;Z'',\mathbf{p}_\perp,p_z'')\,dZ''\,dp_z'' \times \tilde{K}_0(\mathbf{p}_\perp,p_z'',\mathbf{p}';\mathbf{p}_\perp,p_z'',\mathbf{p}')\,d^3\mathbf{p}' f_i(Z'',\mathbf{p}'), \tag{8}$$

$$f_e(Z,\mathbf{p}) = f_e^0(Z,\mathbf{p}) + \int \mathcal{F}_{ei}(Z,\mathbf{p}_\perp,p_z;Z'',\mathbf{p}_\perp,p_z'')\,dZ''\,dp_z'' \times \tilde{K}_0(\mathbf{p}_\perp,p_z'',\mathbf{p}';\mathbf{p}_\perp,p_z'',\mathbf{p}')\,d^3\mathbf{p}' f_i(Z'',\mathbf{p}'). \tag{9}$$

The capital letters Z and Z″ denote here the projections on the z axis of the vectors **R** and **R**″, and $\mathbf{p}_\perp$ denotes the wave-vector component of **p** perpendicular to the z axis.

Let us turn to Dyson's equations for the average field and average Green's function. We replace approximately this equation by the Helmholtz wave equation with effective complex wave number k. The conditions under which such a replacement is possible are analyzed in our paper[7]. The effective wave number k, neglecting spatial dispersion of the waves, is determined approximately by the relation

$$k^2 \cong \varkappa_0^2 - \tilde{M}(\varkappa_0). \tag{10}$$

$\tilde{M}(p)$ denotes the Fourier transform of the mass operator of an unbounded homogeneous-in-the-mean medium; $\kappa_0$ is the renormalized real wave number of "free" space[8], defined by the condition $\mathrm{Re}\,\tilde{M}(\kappa_0) = 0$.

No difficulties are involved in solving the Helmholtz equation with complex wave number for the mean field; the solution can be found in a number of sources. It is convenient to represent the solution of this equation for the average Green's function in the form of a Fourier integral. We write

$$\langle G(\mathbf{r},\mathbf{r}')\rangle = \int \exp[i\mathbf{p}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')]\,d^2\mathbf{p}_\perp g(p_\perp;z,z'). \tag{11}$$

The integrand functions $g(p_\perp; z, z')$ were calculated in our cited paper[7]. We present them here in simplified form, neglecting multiple reflection of the mean field from the boundary of the medium:

$$g(p_\perp;z,z') = \frac{1}{2(2\pi)^2 i}\frac{1}{a}[\exp(ia|z-z'|) + V\exp[ia(z+z')] + V\exp[ia(2L-z-z')]]$$
$$\text{if } 0<z<L, \quad 0<z'<L; \tag{12}$$

$$g(p_\perp;z,z') = \frac{1}{2(2\pi)^2 i}\frac{1}{a}[\exp(iaz') + V\exp[ia(2L-z')]](1+V)\exp(ia_0|z|) \quad \text{if } z<0<z'<L. \tag{13}$$

By V we denote here the coefficient of reflection of a plane wave from the interface when propagating from the medium into free space; this coefficient is equal to

$$V = (a-a_0)/(a+a_0). \tag{14}$$

The values of a and $a_0$ are:

$$a = a' + ia'' = \sqrt{k^2 - p_\perp^2}, \quad a_0 = a_0' + ia_0'' = \sqrt{k_0^2 - p_\perp^2}.$$

The first term in (12) represents a wave propagating in the medium from the point z′ to the point z without reflection from the boundaries. The second and third terms represent waves experiencing reflection from the boundary. In formula (13), the first term corresponds to a wave emerging from the medium to the free region of space in front of the layer ($z < 0$); the wave experiences refraction at the boundary $z = 0$. The second term corresponds to an emerging wave experiencing additional reflection from the boundary $z = L$. It is easy to write out a formula similar to (13) for a wave emerging from the medium to the free space behind the layer ($z > L$).

We call attention to the fact that the terms in (12) and (13), representing reflected waves and proportional to the reflection coefficient V, have the form of inhomogeneous plane waves that attenuate exponentially with increasing distance from the interface in the interior of the medium. To estimate the role of the reflected wave, let us consider the behavior of the reflection coefficient V as a function of the attenuation exponent a″.

The real and imaginary parts a′ and a″ of a are determined by

$$2a'^2 = \varkappa_0^2 - p_\perp^2 + \sqrt{(\varkappa_0^2 - p_\perp^2)^2 + (\varkappa_0/d)^2},$$
$$a'' = \varkappa_0/2a'd, \tag{15}$$

where d denotes the extinction length of the wave, whose reciprocal is $1/d = -\mathrm{Im}\,\tilde{M}(\kappa_0)/\kappa_0$. Instead of the attenuation coefficient a″ of the reflected waves it is convenient to use the dimensionless quantity $u = 2a''d$. As the modulus of the transverse wave vector $p_\perp$ increases from $p_\perp = 0$ to $p_\perp = \infty$, the variable u increases monotonically from a value $u \cong 1$ to $u = \infty$.

The behavior of the reflection coefficient V is essentially determined by the ratio of the next two dimensionless parameters:

$$\delta = (\varkappa_0^2 - k_0^2)/\varkappa_0^2, \quad \varepsilon = 1/\varkappa_0 d. \tag{16}$$

According to the cited paper by Finkel'berg[2], neglecting the spatial dispersion of the waves, both parameters are small compared with unity: $|\delta| \ll 1$ and $\epsilon \ll 1$.

An investigation has shown that the reflection coefficient V assumes values that are close to unity in absolute value ($|V| \sim 1$) when the parameter u is large compared with unity ($u \gg 1$) and is approximately equal to $u \cong u_0$. By $u_0$ we denote the value of the parameter u corresponding to the transverse wave vector $p_\perp = k_0$. The approximate value of $u_0^2$ is[3)]

$$u_0^2 \cong 2/\varepsilon \quad \text{if} \quad \varepsilon \gg \delta, \tag{17}$$

$$u_0^2 \cong 1/\delta \quad \text{if} \quad \varepsilon \ll \delta. \tag{18}$$

From the described behavior of the reflection coeffi-

3)We confine ourselves to the case $\delta > 0$, when the medium is optically denser than the free space.

cient it follows that the reflected inhomogeneous waves, whose amplitude is of the order of unity, are concentrated in a narrow region near the boundary of the medium, with a thickness $\Delta L$ of the order of magnitude $\Delta L \sim d/u_0$. We shall henceforth assume that the layer thickness L greatly exceeds $\Delta L : L \gg \Delta L$, neglecting the reflected waves.

## 3. GENERALIZED TRANSPORT EQUATION NEGLECTING REFLECTION AND REFRACTION OF THE AVERAGE FIELD ON THE INTERFACE

We proceed to calculate the spectral densities inside and outside the medium, corresponding to the average field and the average Green's function. We shall neglect not only reflection but also refraction of the wave.

We first calculate the spectral densities $f_i^0(Z, \mathbf{p})$ and $f_e^0(Z, \mathbf{p})$ corresponding to the average field. The spectral density inside the scattering layer $f_i^0(Z, \mathbf{p})$ is

$$f_i^0(Z, \mathbf{p}) \cong \exp(-Z/d)\,\frac{1}{\pi}\,\frac{\sin[(p_z - \varkappa_0)2Z]}{p_z - \varkappa_0}\,\delta^2(\mathbf{p}_\perp) \tag{19}$$

when $0 < Z < L/2$. If the observation point Z lies in the second half of the layer, where $L/2 < Z < L$, then it is necessary to replace Z by $L - Z$ in the argument of the sine function of formula (19). The spectral density $f_e^0(Z, \mathbf{p})$ in front of the layer is

$$f_e^0(Z, \mathbf{p}) \cong \frac{1}{\pi}\,\frac{\sin[(p_z - k_0)2|Z|]}{p_z - k_0}\,\delta^2(\mathbf{p}_\perp) \quad \text{if} \quad Z < 0. \tag{20}$$

In order to obtain the spectral density $f_e^0(Z, \mathbf{p})$ behind the layer $(Z > L)$, it is necessary to replace $|Z|$ in (20) by $Z - L$.

Let us calculate the spectral densities $\mathscr{F}_{ii}$ and $\mathscr{F}_{ei}$ corresponding to the average Green's function. The spectral density $\mathscr{F}_{ii}$, which describes radiation propagating inside the layer, has a rather cumbersome form. We shall therefore calculate it only for a semibounded medium, when one of the boundaries of the layer goes off to infinity: $L \to \infty$. In this case

$$\begin{aligned}\mathscr{F}_{ii}(Z, \mathbf{p}_\perp, p_z; Z', \mathbf{p}_\perp, p_z') \cong \mathscr{F}_i(Z - Z', \mathbf{p})\,\frac{1}{\pi}\,\frac{\sin[(p_z - p_z')2Z']}{p_z - p_z'} \\ - (2\pi)^{-4}\frac{1}{|a|^2}\operatorname{Re}\Big\{\exp[2ia'(Z - Z')] \\ \times \frac{\exp[-(ip_z + a'')2Z]}{ip_z + a''}\,\frac{\operatorname{sh}[(ip_z' + a'')2Z']}{ip_z' + a''}\end{aligned} \tag{21}$$

when $Z > Z'$. By $\mathscr{F}_i(Z, \mathbf{p})$ we denote the function

$$\begin{aligned}\mathscr{F}_i(Z, p) = (2\pi)^{-3}\frac{\exp(-2a''Z)}{2|a|^2(p_z^2 + a''^2)}\Big\{(p_z a' + a''^2) \\ \times \frac{\sin[(p_z - a')2Z]}{p_z - a'} + a''\cos[(p_z - a')2Z]\Big\} \quad \text{if} \quad Z > 0.\end{aligned} \tag{22}$$

If $Z < Z'$, then it is necessary to replace in (21) and (22) Z by $Z'$, $Z'$, by Z, $p_z$ by $(-p_z')$, and $p_z'$ by $(-p_z)$.

The spectral density $\mathscr{F}_{ei}$ describing the radiation emerging from the medium into the free region of space ahead of the layer is equal to

$$\begin{aligned}\mathscr{F}_{ei}(Z, p_\perp, p_z; Z', p_\perp, p_z') \cong (4\pi)^{-2}\exp(-2a_0''|Z|) \\ \times \frac{1}{\pi}\,\frac{\sin[(p_z + a_0')2|Z|]}{p_z + a_0'}\,\frac{\exp(-2a''Z')}{|a|^2}\,\frac{1}{\pi}\,\frac{\sin[(p_z' + a')2Z']}{p_z' + a'} \\ \text{if} \quad 0 < Z' < L/2,\ Z < 0.\end{aligned} \tag{23}$$

If the point of the source $Z'$ lies in the second half of the layer $(L/2 < Z' < L)$, then it is necessary to replace $Z'$ by $L - Z'$ in the argument of the second sinusoidal factor. In order to obtain the spectral density $\mathscr{F}_{ei}$ describing the radiation emerging from the medium into the free region of space behind the layer $(Z > L)$, it is necessary to replace in (23) $|Z|$ by $Z - L$ and $a_0'$ by $(-a_0')$, and in the second exponential factor it is necessary to replace $Z'$ by $L - Z'$.

In the derivation of formulas (19)—(23) we have neglected reflection and refraction of the waves. In order to bring these formulas into correspondence with the conservation of the energy flux on the interace, it is necessary to neglect the difference between the renormalized and non-renormalized wave numbers $\kappa_0$ and $k_0$, and also between the real parts of the numbers a and $a_0$, assuming

$$\varkappa_0 \cong k_0, \quad a' \cong a_0'. \tag{24}$$

Formulas (19)—(23) contain characteristic sinusoidal factors, the arguments of which depend on the coordinates of the point of observation Z or the source point $Z'$ (but not on their difference). These factors represent the geometric effect of the boundary. By regarding the Fourier transform of the intensity operator $\widetilde{K}_0(\mathbf{p}, \mathbf{p}'; \mathbf{p}, \mathbf{p}')$ as the kernel of an integral operator, we form the expressions for $\widetilde{K}_0 f_i^0$, $\widetilde{K}_0\mathscr{F}_{ii}\widetilde{K}_0$, and $\mathscr{F}_{ei}\widetilde{K}_0$. Let the point of observation Z inside the layer, as well as the source point $Z'$, be located away from the boundary at a distance exceeding the nonlocality radius $l$ of the intensity operator: $Z \gg l$, $Z' \gg l$. Then the corresponding sinusoidal factors can be replaced by Dirac $\delta$-functions.

If the observation point Z is located outside the layer, as is the case in formulas (20) and (23), then the corresponding sinusoidal factors can be replaced by $\delta$ functions, bearing in mind the fact that this leaves the values of the mean bilinear combination of the field and of the average energy flux outside the layer unchanged.

In formula (21) the geometrical effect of the boundaries is represented also by the peculiar last term. When the source point and the observation point are moved far from the boundary of the medium, to a distance exceeding the nonlocality radius of the intensity operator, this term tends to zero.

Neglecting reflection and refraction of the waves, and also the geometrical effects of the boundary, Eqs. (8) and (9) for the spectral densities of the field inside and outside the medium assume the form:

$$\begin{aligned}f_i(Z, \mathbf{p}) = \exp(-Z/d)\,\delta(p_z - k_0)\,\delta^2(\mathbf{p}_\perp) \\ + \int \mathscr{F}_i(Z - Z', \mathbf{p})\,dZ'\,\widetilde{K}_0(\mathbf{p}, \mathbf{p}'; \mathbf{p}, \mathbf{p}')\,d^3\mathbf{p}' f_i(Z', \mathbf{p}');\end{aligned} \tag{25}$$

$$\begin{aligned}f_e(Z, \mathbf{p}) = \delta(p_z - k_0)\,\delta^2(\mathbf{p}_\perp) + (4\pi)^{-2}\exp(-2a_0''|Z|) \\ \times\delta(p_z + a_0')\,\frac{1}{|a|^2}\,\exp(-2a''Z')\,dZ'\cdot\widetilde{K}_0(\mathbf{p}_\perp, -a', \mathbf{p}'; \mathbf{p}_\perp, -a', \mathbf{p}')\,d^3\mathbf{p}' f_i(Z', \mathbf{p}') \\ \text{if} \quad Z < 0.\end{aligned} \tag{26}$$

The function $\mathscr{F}_i(Z, \mathbf{p})$ which enters in (25) is determined by formula (22) and has the meaning of the spectral density corresponding to the average Green's function in an unbounded medium.

## 4. TRANSITION TO A PHENOMENOLOGICAL TRANSPORT EQUATION

The spectral density inside the medium $f_i(Z, \mathbf{p})$, which satisfies Eq. (25), is concentrated, as a function of the wave-vector component $p_z$, mainly near the two lines $p_z = \pm a'$. The effective width of these lines is of the order of the larger of the two quantities $1/L_f$ and $a''$:

$$|p_z \pm a'| \sim \max(1/L_f, a''). \tag{27}$$

We assume here that the following conditions are also satisfied

$$\varkappa_0 L_f \gg 1, \quad \varkappa_0 \gg a'', \tag{28}$$

making it possible to neglect the remaining part of the spectrum with respect to the longitudinal wave vector $p_z$. Let the effective width of the lines (27) be smaller than the effective width of the spectrum of the intensity operator, being of the order of $1/l$, so that $L_f \gg l$ (this condition has already been employed earlier), and

$$a'' \ll 1/l. \tag{29}$$

Then, regarding again the Fourier transform of the intensity operator $\widetilde{K}_0(\mathbf{p}, \mathbf{p}'; \mathbf{p}, \mathbf{p}')$ as the kernel of an integral operator and setting up the expression for $\widetilde{K}_0\mathscr{F}_i$, we can approximately replace the sinusoidal factor in the first term of (22) by a $\delta$ function, and the second term with the cosine can be omitted. This procedure is equivalent to the following approximate representation:

$$\mathscr{F}_i(Z, \mathbf{p}) \cong (4\pi)^{-2}\frac{\exp(-2a''|Z|)}{|a|^2} \times [\delta(p_z - a')\eta(Z) + \delta(p_z + a')\eta(-Z)], \tag{30}$$

where $\eta(Z)$ is the unit step function.

We must also satisfy the second condition (24), which ensures conservation of the energy flux on the interface. This condition can be satisfied by confining ourselves to that part of the spectrum with respect to the transverse wave vector $p_\perp$, in which the inequalities

$$p_\perp < k_0, \quad k_0^2 - p_\perp^2 \gg k_0/d. \tag{31}$$

are satisfied. When these inequalities are satisfied, as well as the first condition of (24), we obtain approximately from (15)

$$a' \cong \sqrt{\varkappa_0^2 - p_\perp^2} \cong a_0', \quad 2a'' \cong \frac{k_0/d}{\sqrt{k_0^2 - p_\perp^2}}, \quad a''/a' \ll 1. \tag{32}$$

In the approximation (32), in the space of the values of the wave vector $\mathbf{p}$, it is convenient to go over from cylindrical coordinates $p_z$ and $p_\perp$ to spherical coordinates p and $\mu = p_z/p$. Then, as follows from representation (30), the spectrum turns out to be concentrated on the "energy" surface $p = k_0$. The spectral densities inside and outside the medium $f_i(Z, \mathbf{p})$ and $f_e(Z, \mathbf{p})$ can be represented in the form

$$f_{i,e}(Z, \mathbf{p}) \cong \delta(p - k_0)k_0^{-2}I_{i,e}(Z, \mu), \tag{33}$$

where $I_i(Z, \mu)$ and $I_e(Z, \mu)$ are the ray intensities of the radiation inside and outside the medium. Equation (25) reduces to the phenomenological transport equation for the ray intensity $I_i(Z, \mu)$ inside the medium. On the other hand, Eq. (26) is transformed into the boundary condition

$$I_e(0, \mu) = I_i(0, \mu), \tag{34}$$

according to which the ray intensity remains continuous on going through the interface.

## 5. CONDITIONS FOR APPLICABILITY OF THE PHENOMENOLOGICAL TRANSPORT EQUATION

Let us summarize the main simplifying assumptions made in the derivation of the phenomenological transport equation.

According to Sec. 2 and the first inequality of (28), the spectral density as a function of the coordinates should vary smoothly within the scales of the effective inhomogeneities of the medium and the wavelength.

By neglecting in Sec. 3 the reflection and refraction of the waves, we have arrived at the rather stringent limitations (24) that follow from the energy conservation law.

In Sec. 4, wishing the spectrum with respect to the longitudinal wave vector $p_z$ to be concentrated near the sufficiently narrow lines (27), we have arrived at two limitations (28) and (29) on the exponent $a''$, i.e., on the spectrum with respect to the transverse wave vector $p_\perp$. The third limitation of this spectrum was obtained in the form of inequalities (31), satisfying the condition (24) for the conservation of the energy flux on the boundary. The limitations (28), (29), and (31) on the spectrum with respect to the transverse wave vector are best represented in the form of the following three inequalities for the cosine of the angle $\mu$ between the wave vector $\mathbf{p}$ and the z axis:

$$\mu \gg \frac{1}{k_0 d}, \quad \mu \gg \frac{l}{d}, \quad \mu \gg \frac{1}{\sqrt{k_0 d}}. \tag{35}$$

Comparison of the inequalities (35) shows that when $k_0 l^2 \ll d$ the strongest inequality is the third.

Thus, it turns out as a result that the phenomenological transport equation describes the bounded part of the spectrum, satisfying the inequalities (31). It is appropriate to call this part of the spectrum the Fraunhofer part[4)].

It is of interest to estimate the contribution that can be made to the value of the different physical quantities by the Fraunhofer part of the spectrum, which does not satisfy the inequalities (31). We shall make such an estimate by solving Eq. (9) for the spectral density of the field outside the medium in the single-scattering approximation. We shall assume the medium to be semibounded ($L \to \infty$) and consisting of pointlike isotropic scatterers ($\widetilde{K}_0 = \text{const}$).

It turns out that the part of the spectrum outside the Fraunhofer region makes the most appreciable contribution to the value of the longitudinal field correlation function. Accurate to a constant factor, which is immaterial to us at present, the longitudinal field correlation function $B_\psi(z)$ is represented by an integral of the form

$$B_\psi(z) \cong \int_1^{u_0} \frac{du}{u}\frac{|1+V|^2}{1+u}\exp\left(-i\frac{a_0}{\varkappa_0}\varkappa_0 z\right). \tag{36}$$

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[4)] We note in connection with this term that Eq. (30) with $a'$ and $a''$ in the form of (32) can be obtained directly from expression (3), if we substitute in it the average Green's function for an unbounded medium in the Fraunhofer approximation.

By z we denote the difference of the coordinates of the observation points outside the medium. The transition to the phenomenological transport equation is equivalent to the approximation $u \ll u_0$, when the upper limit of the integral is replaced by infinity ($u_0 \to \infty$), the reflection coefficient is replaced by 0 ($V \to 0$), and the phase function $a_0/\kappa_0$ is assumed equal to $a_0/\kappa_0 \cong 1/u$.

Let us estimate the value of the integral (36) in the case when the distance between the observation points is large compared with the wavelength: $\kappa_0|z| \gg 1$. The estimate can be performed by means of the stationary phase method. The contribution made to the integral by the vicinity of the lower limit of integration $u = 1$, corresponding to the Fraunhofer part of the spectrum, is of the order of $B_1 \sim 1/\kappa_0|z|$. The contribution to the integral by the vicinity of the upper integration limit $u = u_0$, corresponding to the part of the spectrum outside the Fraunhofer region, is of the order of $B_{u_0} \sim u_0/(\kappa_0|z|)^2$. Since, according to (17) and (18), $u_0$ is much larger than unity, it follows that under the condition $1 \ll \kappa_0|z| \ll u_0$ the value of the function of the longitudinal correlation of the field is determined mainly by the part of the spectrum outside the Fraunhofer region.

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