MAGNETIC SCATTERING OF NEUTRONS

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The small-angle behavior of the amplitude describing magnetic scattering of neutrons by electrons is investigated. It is shown that the statement that the forward elastic scattering amplitude vanishes when the electron polarization vector is parallel to the neutron momentum is incorrect. A new expression for the forward scattering amplitude is derived.

SCHWINGER[1] and Halpern and Jonson[2] found that the asymptotic expression for the wave describing elastic magnetic scattering of a neutron by an electron in the Born approximation is of the form

$$\psi_{sc} = \frac{2m}{\hbar^2}\left[\mu_1\mu_2 - \frac{(\mu_1\mathbf{q})(\mu_2\mathbf{q})}{q^2}\right]\frac{e^{ikr}}{r}, \qquad (1)$$

where μ_1 is the operator of the magnetic moment of the neutron, μ_2 is the operator of the magnetic moment of the electron, $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, $\mathbf{k}$ is the wave vector of the incident wave, $\mathbf{k}' = k\mathbf{r}/r$ is the wave vector of the scattered wave, and $k = |\mathbf{k}|$. It follows from (1) that the amplitude of the elastic scattering is not unique in the case of scattering at zero angle; the indicated nonuniqueness occurs both for collision processes with spin flip and for collision processes without flipping of the spins of the particles. The appearance of this nonuniqueness is connected with the long-range nature of the magnetic dipole-dipole interaction. In order to explain the form of the forward scattered wave, Ekstein[3] considered the general expression for the wave function ψ_{sc} in the Born approximation:

$$\psi_{sc} = -\frac{m}{\pi^2\hbar^2}\lim_{\delta\to 0}\int d^3p\,\frac{e^{i\mathbf{pr}}}{p^2-(\mathbf{k}+i\delta)^2}\left[\frac{(\mu_1(\mathbf{k}-\mathbf{p}))(\mu_2(\mathbf{k}-\mathbf{p}))}{(\mathbf{k}-\mathbf{p})^2} - \mu_1\mu_2\right]. \qquad (2)$$

As a result [having represented (2) in the form of a series in reciprocal powers of r and discarding terms of the order of $1/r^2, 1/r^3, \ldots$] he concluded that the coherent forward scattering amplitude vanishes when the polarization vector of the electrons is directed along $\mathbf{k}$.

It should, however, be noted that the discarded terms contain integrals over p which have singularities on the real axis at the points $\pm k$, a fact which invalidates the above expansion. We also note that in[4] an attempt is made to justify the result obtained in[3]. Referring to the vector-field theorem[5] it is stated that

$$\lim_{V\to\infty}\int_V d^3r\,\mu_1\mathbf{H}(\mathbf{r}),$$

which is related in the Born approximation with the forward scattering amplitude, vanishes [$\mathbf{H}(\mathbf{r})$ is the magnetic field produced by the electron]. However, the requirement that $\lim_{\mathbf{r}\to\infty} r\mu_1 \to 0$ follows directly from the conditions of applicability of the above theorem. Clearly, in our case it is not satisfied.

Additional considerations are thus required to establish the form of the elastic magnetic forward scattering amplitude.

It turns out that expression (2) for the scattered wave can be obtained in explicit form. For convenience we direct the z coordinate axis along the vector $\mathbf{k}$. We integrate (2), first over the variable p_z, taking into account the fact that the integral under consideration has poles at the points $\pm[\sqrt{k^2 - p_\perp^2} + i\delta]$ and $k \pm ip_\perp$ where $p_\perp = (p_x^2 + p_y^2)^{1/2}$. As a result we obtain

$$\psi_{sc} = -\frac{im}{\pi\hbar^2}\iint_{-\infty}^{+\infty} dp_x\,dp_y\exp(i\mathbf{p}_\perp\mathbf{r}_\perp)\left\{\frac{\exp(\pm i\sqrt{k^2-p_\perp^2}\,r_z)}{\sqrt{k^2-p_\perp^2}}\right.$$
$$\times\left[-\mu_1\mu_2 + \left(\frac{1}{2}\mp\frac{\sqrt{k^2-p_\perp^2}}{2k}\right)\mu_1^z\mu_2^z - \frac{1}{2k}(\mu_1^z(\mu_2\mathbf{p}_\perp) + \mu_2^z(\mu_1\mathbf{p}_\perp))\right.$$
$$\left.+\left(\frac{1}{2}\pm\frac{\sqrt{k^2-p_\perp^2}}{2k}\right)\frac{(\mu_1\mathbf{p}_\perp)(\mu_2\mathbf{p}_\perp)}{p_\perp^2}\right] + \exp(ikr_z)\exp(\mp p_\perp r_z)$$
$$\left.\times\frac{1}{2k}\left[\pm\mu_1^z\mu_2^z - i\left(\mu_1^z\frac{\mu_2\mathbf{p}_\perp}{p_\perp} + \mu_2^z\frac{\mu_1\mathbf{p}_\perp}{p_\perp}\right)\mp\frac{(\mu_1\mathbf{p}_\perp)(\mu_2\mathbf{p}_\perp)}{p_\perp^2}\right]\right\}, \qquad (3)$$

where the upper sign refers to values $r_z > 0$, the lower to $r_z < 0$, $\mathbf{p}_\perp$ is a vector with components $(p_x, p_y, 0)$, and $\mathbf{r}_\perp = (r_x, r_y, 0)$. The obtained expression can be presented in the form

$$\psi_{sc} = -\frac{im}{\pi\hbar^2}\left\{\left[-\mu_1\mu_2 + \mu_1^z\mu_2^z\left(\frac{1}{2}+\frac{i}{2k}\frac{\partial}{\partial r_z}\right)\right.\right.$$
$$\left.+\frac{i}{2k}(\mu_1^z(\mu_2\nabla_\perp) + \mu_2^z(\mu_1\nabla_\perp))\right]\int_0^\infty p_\perp dp_\perp\frac{\exp(i\sqrt{k^2-p_\perp^2}|r_z|)}{\sqrt{k^2-p_\perp^2}}$$
$$\times\int_0^{2\pi} d\varphi\exp(ip_\perp r_\perp\cos(\varphi-\psi)) + \left(\frac{1}{2}-\frac{i}{2k}\frac{\partial}{\partial r_z}\right)$$
$$\times\int_0^\infty p_\perp dp_\perp\frac{\exp(i\sqrt{k^2-p_\perp^2}|r_z|)}{\sqrt{k^2-p_\perp^2}}\int_0^{2\pi} d\varphi\exp(ip_\perp r_\perp\cos(\varphi-\psi))$$
$$\times\frac{(\mu_1\mathbf{p}_\perp)(\mu_2\mathbf{p}_\perp)}{p_\perp^2} - \frac{e^{ikr_z}}{2k}\left[\mu_1^z\mu_2^z\frac{\partial}{\partial r_z} + \mu_1^z(\mu_2\nabla_\perp) + \mu_2^z(\mu_1\nabla_\perp)\right]$$
$$\times\int_0^\infty dp_\perp\exp(-p_\perp|r_z|)\int_0^{2\pi} d\varphi\exp(ip_\perp r_\perp\cos(\varphi-\psi)) + \frac{e^{ikr_z}}{2k}\frac{\partial}{\partial r_z}$$
$$\times\int_0^\infty dp_\perp\exp(-p_\perp|r_z|)\int_0^{2\pi} d\varphi\exp(ip_\perp r_\perp\cos(\varphi-\psi))\frac{(\mu_1\mathbf{p}_\perp)(\mu_2\mathbf{p}_\perp)}{p_\perp^2} \qquad (4)$$

where φ is the angle between the vector $\mathbf{p}_\perp$ and the x axis, and ψ is the angle between $\mathbf{r}_\perp$ and the x axis.

Using the expansion of $\exp[ip_\perp r_\perp \cos(\varphi - \psi)]$ in Bessel functions and taking into account the recurrence relations between them, we find that the integrals over the variable φ are equal to

$$\int_0^{2\pi} d\varphi \exp(ip_\perp r_\perp \cos(\varphi - \psi)) = 2\pi J_0(p_\perp r_\perp), \tag{5}$$

$$\int_0^{2\pi} d\varphi \exp(ip_\perp r_\perp \cos(\varphi - \psi)) \frac{(\boldsymbol{\mu}_1 \mathbf{p}_\perp)(\boldsymbol{\mu}_2 \mathbf{p}_\perp)}{p_\perp^2}$$
$$= 2\pi \frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2} J_0(p_\perp r_\perp)$$
$$+ 2\pi\left[2\frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2} - \boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \mu_1^z\mu_2^z\right]\frac{1}{r_\perp^2 p_\perp}\frac{\partial}{\partial p_\perp} J_0(r_\perp p_\perp). \tag{6}$$

Substituting equalities (5) and (6) in expression (4), we carry out the remaining integration over the variable $p_\perp$ with account of the fact that[6]

$$\int_0^\infty p_\perp dp_\perp J_0(p_\perp r_\perp) \frac{\exp(i\sqrt{k^2 - p_\perp^2}|r_z|)}{\sqrt{k^2 - p_\perp^2}} = -i\frac{e^{ikr}}{r}. \tag{7}$$

The integrals containing derivatives of the Bessel function $(\partial/\partial p_\perp)J_0(p_\perp r_\perp)$ are obtained by parts. As a result we obtain

$$\psi_{sc} = -\frac{2m}{\hbar^2}\left\{\left[-\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \mu_1^z\mu_2^z \cdot \frac{1}{2}\left(1 - \frac{r_z}{r}\right) - \frac{1}{2}\left(\mu_1^z \frac{\boldsymbol{\mu}_2\mathbf{r}_\perp}{r} + \mu_2^z \frac{\boldsymbol{\mu}_1\mathbf{r}_\perp}{r}\right) + \frac{1}{2}\frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2}\left(1 + \frac{r_z}{r}\right)\right]\frac{e^{ikr}}{r}\right.$$
$$+ \left[\frac{1}{2}\mu_1^z\mu_2^z - \frac{1}{2}\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2}\right]\frac{i}{kr_\perp^2}\left(1 + \frac{r_z}{r}\right)$$
$$\cdot (e^{ikr} - e^{ikr_z}) - \left[\mu_1^z\mu_2^z\frac{r_z}{r} + \mu_1^z\frac{\boldsymbol{\mu}_2\mathbf{r}_\perp}{r} + \mu_2^z\frac{\boldsymbol{\mu}_1\mathbf{r}_\perp}{r}\right.$$
$$\left.\left. - \frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2}\frac{r_z}{r}\right] \cdot \frac{i}{2kr^2}(e^{ikr} - e^{ikr_z})\right\}. \tag{8}$$

Finally the scattered wave can be represented in the following form:

$$\psi_{sc} = -\frac{2m}{\hbar^2}\left\{\left[-\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \frac{(\boldsymbol{\mu}_1 \mathbf{q})(\boldsymbol{\mu}_2 \mathbf{q})}{q^2}\right]\frac{e^{ikr}}{r} + \left[\frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2}\right.\right.$$
$$\left. - \frac{1}{2}\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \frac{1}{2}\mu^z{}_1\mu_2^z\right] e^{ikr}\frac{e^{-ikr(1-\cos\theta)} - 1}{ikr^2(1 - \cos\theta)} - \left[\mu_1^z\mu_2^z\frac{r_z}{r}\right.$$
$$\left.\left. + \mu_1^z\frac{\boldsymbol{\mu}_2\mathbf{r}_\perp}{r} + \mu_2^z\frac{\boldsymbol{\mu}_1\mathbf{r}_\perp}{r} - \frac{(\boldsymbol{\mu}_1 \mathbf{r}_\perp)(\boldsymbol{\mu}_2 \mathbf{r}_\perp)}{r_\perp^2}\frac{r_z}{r}\right] e^{ikr}\frac{e^{-ikr(1-\cos\theta)} - 1}{2ikr^2}\right\}, \tag{9}$$

where $\mathbf{q} = \mathbf{k} - k\mathbf{r}/r$, $\mathbf{k} = (0, 0, k)$, $\cos\theta = (\mathbf{k}\cdot\mathbf{r})/kr$, and θ is the scattering angle.

Expression (9) for ψ_{sc} differs from (1) by a whole series of additional terms. Let us first consider the third term. We note that it vanishes for forward scattering. In addition, it behaves like $1/r^2$ and need not be considered at large distances $(kr \gg 1)$[1)].

At first sight the second term in the scattered wave (9) also behaves like $1/r^2$ and it would seem that it need not be taken into account. Let us note, however, that because of the presence in the numerator of this term of the factor $(1 - \cos\theta)$ that situation is somewhat more complicated. Let us consider in detail the coefficient

$$\frac{1 - e^{-ikr(1-\cos\theta)}}{ikr^2(1 - \cos\theta)},$$

which appears at the bracket of this term. For arbitrary fixed r its limit for $\theta \to 0$ is $1/r$. Consequently, for zero scattering angles the second term behaves like $1/r$ and must not be discarded. Thus, in estimating the contribution of this term to the scattered wave at large distances $(kr \gg 1)$ one must differentiate between two regions of scattering angles: $\theta \gg 1/\sqrt{kr}$ and $\theta \ll 1/\sqrt{kr}$. For θ satisfying the condition $\theta \gg 1/\sqrt{kr}$ one need not take into account the second term and the asymptotic behavior of the wave function (9) will coincide with the behavior of (1). For angles $\theta \ll 1/\sqrt{kr}$ one must not discard the second term. In the case of thermal neutrons at a distance of the order of 1 cm from the scatterer the range of angles in which one must take into account the additional term, $\theta \lesssim 10^{-4}$. From what has been said above it also follows that in the case of magnetic scattering the usual asymptotic behavior of $f(\mathbf{q})e^{ikr}/r$ for the scattered wave is not valid. In fact at large distances $(kr \gg 1)$ in scattering by a small non-zero angle it is as if we had two "scattering amplitudes." In fact, at distances $r \gg 1/k\theta^2$ the second term behaves like $1/r^2$ and the "scattering amplitude" coincides with the scattering amplitude given by expression (1). On the other hand, at distances $r \ll 1/k\theta^2$ but for which nevertheless $kr \gg 1$ we should take into account the second term and the scattering will consequently be described by the other "amplitude." Such a situation means in essence that in the given case $\mathbf{r}$ enters into the group of quantities which constitute the invariants of the scattering amplitude, i.e., $f = f(\mathbf{q}, \mathbf{S}_1, \mathbf{S}_2\, \mathbf{r})$; and $\mathbf{S}_2$ are the spin operators of the particles[2)].

If we consider the limit of expression (9) for $\theta \to 0$, we find that the forward-scattered wave is

$$\psi_{sc}\,(\theta = 0) = \frac{m}{\hbar^2}[\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \mu_1^z\mu_2^z]\frac{e^{ikr}}{r}. \tag{10}$$

Thus, the forward scattering amplitude contains no indeterminacy and is of the following form:

$$f(0) = \frac{m}{\hbar^2}[\boldsymbol{\mu}_1\boldsymbol{\mu}_2 + \mu_1^z\mu_2^z]. \tag{11}$$

It is seen that account of the additional terms in the wave (9) removes the nonuniqueness at zero angles which existed in expression (1). It also follows from (11) that the coherent forward amplitude of scattering of a neutron by an electron whose polarization vector is directed along $\mathbf{k}$ differs from zero and is equal to

$$f_\pm(0) = \pm\frac{2m}{\hbar^2}\mu_1\mu_2,$$

where the + (−) sign refers to a neutron with spin

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1)We draw attention to the fact that at distances satisfying the condition $kr \leqslant 1$ this term is comparable with the terms of expression (9) which behave like $1/r$. Since for slow neutrons the wavelength is of the order of the interatomic distance, the last term must, generally speaking, be taken into account in investigating corrections for coherent rescattering, for example in investigating corrections to the scattering amplitude of molecules and multi-electron atoms.

2)Such a situation does not mean that the total flux of scattered particles is not conserved but is due to the fact that the anisotropic, long-range nature of the investigated interactions leads to the fact that in the fact that in the region of small, nonzero angles the lines of current do not point strictly along the radius even at large distances from the scatterer.

parallel (antiparallel) to the electron spin.

It should also be noted that because of the indeterminacy of the wave function (1) the differential scattering cross section at zero angle was also not unique. Clearly, when use is made of the scattered wave (9) to determine the differential cross section there is no indeterminacy[3)]. For instance, the differential forward scattering cross section of a neutron by an electron is in this case given by the following expression:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0} = \frac{2m}{\hbar^4}\mu_1^2\mu_2^2[3I - 4(2\mathbf{S}_1\mathbf{S}_2 - S_1^zS_2^z)], \tag{12}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are spin operators of the particles.

In conclusion we note that, as appears from the appropriate arguments, the additional terms in ψ_{sc} do not make any further contribution to the total scattering cross section which can thus be calculated by the use of expression (1).

The properties of small-angle scattering considered above are experimentally most conveniently observed by investigating the process of collision of polarized neutrons with electrons with spin flip because in this instance one can separate the scattered beam from the incident beam.

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[3)]It is interesting that defining the differential cross section as the ratio of the fluxes of the incident and scattered waves and considering in it the term due to the spin part of the current

$$d\sigma / d\Omega \sim (r^2/2k)\,\mathrm{rot}_r(\psi_{\mathrm{sc}}^{+}\mathbf{S}_1\psi_{\mathrm{sc}}),$$

we find that the contribution of this term to the differential cross section calculated with the wave function (1) gives for small angles a divergence of the form

$$\left\{\frac{S_1^y r_x - S_1^x r_y}{r^2\sin^2\theta} - \frac{S_2^y r_x - S_2^x r_y}{r^2\sin^2\theta}\right\},$$

which does not appear when one uses the wave function (9).

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147