PARAMETRIC EXCITATION OF A QUANTUM OSCILLATOR

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An exact solution is obtained for the problem of a quantum oscillator with a time-dependent frequency $\omega(t)$ whose law of variation is arbitrary. The probabilities w_{mn} are found for quantum transitions between states of the discrete spectrum which are stationary as $t \to \pm\infty$. The variation of the adiabatic invariant $I = E/\omega$ under the action of the variable frequency $\omega(t)$ is calculated for an arbitrary initial state.

1. As a rule nonstationary problems in quantum mechanics are solved by approximate methods (time-dependent perturbation theory, the adiabatic approximation, the method of sudden perturbations, etc.[1,2]). Only in rare cases is it possible to solve a nonstationary problem exactly. Such a solution is always of interest if only from the point of view that it enables one to clarify the accuracy of various approximate methods. As an example one can point out the problem of an oscillator acted upon by an external force $f(t)$; the solution of this problem was obtained in articles[3-5].

The present article is devoted to an examination of a quantum oscillator whose frequency $\omega(t)$ is an arbitrary function of the time (one can easily reduce the general case when both $m(t)$ and $\omega(t)$ are time-dependent to this case with the aid of the following substitutions: $t' = \int^t dt/m(t)$, $\omega' = m\omega$). The principal possibility of solving the Schrödinger equation (1) now follows from a well-known result of Feynman.[6] As shown in[6], the formula $\psi(x, t) = \exp\{iS(x, t)\}$ is exact if the potential $V(x, t)$ does not contain x to a power higher than the second (S is the classical action). Many aspects of this problem were considered by Husimi;[5] in particular he found an explicit expression for the Green's function. However, the formulas given in[5] for the transition probabilities w_{mn} between stationary states are complicated in form and yield to analysis with difficulty. In that important case when the frequency $\omega(t)$ changes adiabatically, the quantum transition probabilities w_{mn} are calculated in the article by Dykhne.[7]

Let us enumerate the basic results of the present article. The Schrödinger representation is used in Sec. 2. Formula (11) is obtained for the transition probabilities w_{mn}, and various limiting cases are considered. For certain initial states (a coherent state $|\alpha\rangle$ and a Planck distribution) the populations w_n are found as $t \to \infty$. The Heisenberg representation is considered in Sec. 3, a connection with the noncompact group SU(1, 1) is pointed out, and formulas (33) are obtained which give a complete solution of the problem for arbitrary initial conditions. The question of the exactness of the conservation of the adiabatic invariant $I = \omega^{-1}\langle H\rangle$ is discussed in Sec. 4. The relative change $\epsilon = (I_+ - I_-)/I_-$ is given by formulas (39) and (43), which are valid for an arbitrary initial density matrix $\rho(0)$. Several specific examples for the variation of $\omega(t)$ are analyzed in Sec. 5; one of these is of interest in connection with a theory of the parametric amplification of light.[8-10]

2: The Schrödinger equation for a quantum oscillator with a variable frequency $\omega(t)$ has the form ($m = \hbar = 1$)

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\omega^2(t)x^2\psi. \tag{1}$$

With regard to $\omega(t)$ we shall assume that¹⁾

$$\omega(t) = \begin{cases} \omega_- & \text{for} \quad t < 0 \\ \omega_+ & \text{for} \quad t \to +\infty. \end{cases} \tag{2}$$

Under these conditions stationary states exist for $t \to \pm\infty$

$$\psi_n^{(\pm)}(x, t) = \varphi_n(x, \omega_\pm)\exp\left\{-i\left(n + \frac{1}{2}\right)\omega_\pm t\right\},$$
$$\varphi_n(x, \omega) = \left\{\frac{1}{2^n n!}\sqrt{\frac{\omega}{\pi}}\right\}^{1/2}\exp\left\{-\frac{\omega x^2}{2}\right\}H_n(\sqrt{\omega}\,x), \tag{3}$$

and transitions occur between these states. We shall calculate the probability w_{mn} for a transition from the state $\psi_n^{(-)}$ to the state $\psi_m^{(-)}$.

In order to solve this problem we introduce the generating function

$$G(z, x; t) = \sum_{n=0}^{\infty}\psi_n(x, t)\frac{z^n}{\sqrt{n!}}, \tag{4}$$

where z is a subsidiary complex variable,²⁾ and $\psi_n(x, t)$ is that solution of Eq. (1) which goes over into $\psi_n^{(-)}$ as $t \to -\infty$. From (3) we find

$$G(z, x;\ t = 0) = (\omega_-/\pi)^{1/4}\exp\{-\tfrac{1}{2}(\omega_- x^2 - 2\sqrt{2\omega_-}xz + z^2)\}, \tag{5}$$

which is a Gaussian packet with respect to the variable x. Therefore for $t > 0$ one can seek $G(z, x; t)$ in the form[5]

$$G(z, x;\ t) = (\omega_-/\pi)^{1/4}\exp\{-\tfrac{1}{2}(ax^2 - 2bx + c)\}, \tag{6}$$

where a, b, and c are functions of t and z. After

¹⁾The assumption that the frequency $\omega(t)$ starts to change at the moment $t_0 = 0$ is introduced only in order to simplify the calculations. In the final formulas one can regard the moment t_0 of "switching-on" as arbitrary. If the perturbation $\frac{1}{2}[\omega^2(t) - \omega_-^2]x^2$ is switched on adiabatically at $t \to -\infty$, then one should assume $t_0 \to -\infty$.

²⁾The introduction of G(z, x; t) is essentially a transition to the Fock-Bargmann representation for the harmonic oscillator. [11,12]

some calculations we obtain

$$a(t) = -i\frac{\dot{\xi}(t)}{\xi(t)}, \quad b(t) = \frac{\sqrt{2\omega_-}}{\xi(t)}z,$$

$$c(t) = z^2 e^{-2i\gamma(t)} + \ln \xi(t). \tag{7}$$

Here $\xi(t) = |\xi(t)|\, e^{i\gamma(t)}$ is the (complex) solution of the classical equation of motion:

$$\ddot{\xi} + \omega^2(t)\xi = 0, \quad \xi(t) = e^{i\omega_- t} \quad \text{for} \quad t \to -\infty. \tag{8}$$

Expanding (6) in powers of z, we find the form of the wave functions $\psi_n(x, t)$ at any arbitrary moment of time:

$$\psi_n(x,t) = \left\{\frac{1}{2^n n!\,\xi(t)}\sqrt{\frac{\omega_-}{\pi}}\right\}^{1/2} \exp\left[-\left(\frac{ax^2}{2} + in\gamma\right)\right] H_n\left(\frac{x\sqrt{\omega_-}}{|\xi(t)|}\right). \tag{9}$$

We note that $\operatorname{Re} a(t) = \omega_- |\xi(t)|^{-2}$, which guarantees the correct normalization of the wave functions $\psi_n(x, t)$. As $t \to +\infty$

$$\xi(t) = C_1 e^{i\omega_+ t} - C_2 e^{-i\omega_+ t},$$

$$|\xi(t)| = \left[\frac{\omega_-}{\omega_+(1-|s|^2)}\right]^{1/2} |1 - se^{-2i\omega_+ t}|, \quad a(t) = \omega_+\frac{1+se^{-2i\omega_+ t}}{1-se^{-2i\omega_+ t}} \tag{10}$$

($s = C_2/C_1$, $0 \le |s| < 1$). From here it is clear that $\psi_n(x, t)$ oscillates periodically (with a frequency ω_+) even as $t \to \infty$. However the transition probability w_{mn} tends to a constant value:

$$w_{mn} = \lim_{t\to\infty} |\langle \psi_m^+(t)\,|\,\psi_n(t)\rangle|^2 = \frac{n_<!}{n_>!}\sqrt{1-\rho}\,\left|P_{(m+n)/2}^{|m-n|/2}(\sqrt{1-\rho})\right|^2. \tag{11}$$

Here

$$n_< = \min(m, n), \quad n_> = \max(m, n),$$

$$\rho = |s|^2 = |C_2/C_1|^2,$$

and $P_n^m(x)$ is an associated Legendre function. The value of the integral (A.11) (see Appendix A) is used in order to obtain formula (11). We emphasize that in order to determine ρ it is sufficient to solve the classical equation (8) (in regard to other methods for evaluating ρ and certain specific examples, see Sec. 5). In the adiabatic case when $\omega(t)$ is a slowly-varying function of t which is analytic in a certain region $|\operatorname{Im} t| < \epsilon$, the quantity ρ is exponentially small[14,15]

Now let us discuss expression (11) for w_{mn} in more detail.

1) Transitions occur only between states $|n, \omega_-\rangle$ and $|m, \omega_+\rangle$ for which the numbers m and n have the same parity. This is related in an obvious way to the parity of the potential $V(x, t) = \frac{1}{2}\omega^2(t)x^2$.

2) In quantum mechanics the adiabatic invariants are the quantum numbers and also the distribution with respect to the energy levels.[16] The adiabatic case corresponds to $\rho \to 0$; in this connection from (11) we obtain

$$w_{nn} = 1 - \tfrac{1}{2}(n^2 + n + 1)\rho + O(\rho^2);$$

$$w_{mn} = \frac{n_>!}{2^{2k}(k!)^2 n_<!}\rho^k\left[1 - \left(\frac{n_<(n_>+1)}{k+1} + 1\right)\frac{\rho}{2} + \ldots\right], \tag{12}$$

$$k = \tfrac{1}{2}|m-n| = 0, 1, 2, \ldots$$

The main terms of this expansion coincide with the result obtained by Dykhne.[7] The correction is of order $n^2\rho$; therefore the accuracy of formula (12) rapidly deteriorates with increase of the initial excitation n. For $n \gg 1$ a case may be realized in which $\rho \ll 1$ but $n^2\rho \gtrsim 1$. Then in Eq. (12) it is necessary to sum over all the terms in the series, which gives[3)]

$$w_{mn} \approx \frac{n_>!}{(mn)^k n_<!}\,|J_k(\sqrt{mn\rho})|^2 \tag{13}$$

(the conditions for the validity of this formula are: $m, n \gg 1$; $\rho \ll 1$). If $mn\rho \ll 1$ then expression (13) automatically goes over into expression (12).

3) The quantity $\Delta_n = 1 - w_{nn}$ is of special interest (it gives the probability that the oscillator changes its initial state). Graphs of the functions $\Delta_n = \Delta_n(\rho)$ are shown in Fig. 1. From this figure and also from formula (13) it follows that the adiabatic approximation (12) for the transition probabilities w_{mn} is valid only upon fulfillment of the conditions $\rho \ll 1$ and $mn\rho \ll 1$.

4) Expression (11) for w_{mn} simplifies considerably for $n = 0$ and $n = 1$ (in these cases transitions only occur upward, $m \ge n$):

$$w_{2n,0} = \frac{(2n)!}{2^{2n}(n!)^2}(1-\rho)^{1/2}\rho^n, \quad w_{2n+1,1} = \frac{(2n+1)!}{2^{2n}(n!)^2}(1-\rho)^{3/2}\rho^n \tag{14}$$

(in a somewhat different form, these expressions are contained in article[5]).

5) In the opposite case when $n \gg 1$ (and ρ is not too small), in (11) one can substitute the quasiclassical asymptote for the Legendre functions. Assuming m to be a continuous variable, let us transform Eq. (11) to the form

$$w_{mn} = \frac{4\cos^2\Phi_{mn}}{\pi[(m-m_1)(m_2-m)]^{1/2}}, \tag{15}$$

where

$$m_1 = n\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}, \quad m_2 = n\frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}$$

and Φ_{mn} is a rapidly oscillating phase. The distribution of the transition probabilities w_{mn} is primarily concentrated in the region $m_1 \le m \le m_2$; for $m < m_1$ and $m > m_2$ the probabilities w_{mn} decrease exponentially (see Fig. 2). Averaging w_{mn} over the rapid oscillations, we have

$$\overline{w}_{mn} = 2/\pi\sqrt{(m-m_1)(m_2-m)}. \tag{15a}$$

One can obtain this distribution within the framework of classical mechanics (see Appendix B for further details). The oscillations represented by the factor $\cos^2\Phi_{mn}$ are a quantum effect. An explicit expression for the phases Φ_{mn} is given in Appendix B.

The maximum value of the transition probabilities w_{mn} is reached for m close to m_1 and m_2. Compari-

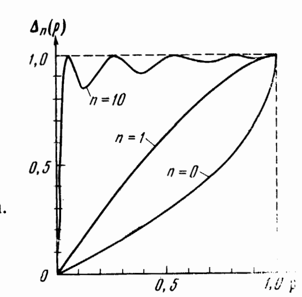

FIG. 1. The quantity $\Delta_n = 1 - w_{nn}$ as a function of ρ for various values of n.

[3)]In the simplest case m = n, formula (13) can be generalized to arbitrary values of ρ. Applying Hilb's asymptotic expression [17] for $P_n(\cos\theta)$, from expression (11) we find

$$w_{nn} \approx \theta\operatorname{ctg}\theta\,|J_0((n+\tfrac{1}{2})\theta)|^2,$$

where $n \gg 1$, but $\rho = \sin^2\theta$ is arbitrary.

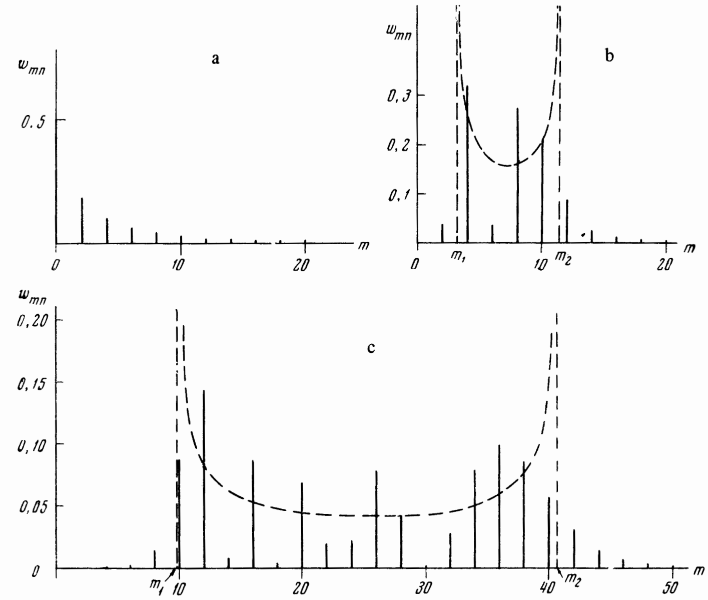

FIG. 2. Probabilities w_{mn} for transitions from an initial state $|n, \omega_-\rangle$ into the state $|m, \omega_+\rangle$ for different values of n and ρ ($\rho = \sin^2\theta$): Fig. 2a is for $n = 0$, $\rho = 0.75$; Fig. 2b is for $n = 6$, $\rho = 1.1$ ($\theta = 18°$); Fig. 2c is for $n = 20$, $\rho = 0.12$ ($\theta = 20°$). The dotted curves indicate the classical envelope $\overline{w}_{mn}$, see formula (15a). The deviations of the probabilities w_{mn} from the dotted curves represent a quantum effect and occur due to the presence of the factor $\cos^2\Phi_{mn}$. The tables given in [13] were used for the construction of these graphs.

son of formulas (14) and (15) indicates a substantial difference in the behavior of w_{mn} for $n \sim 1$ and $n \gg 1$. This difference is clearly evident from Fig. 2.

6) From the general principles of quantum mechanics it follows that the transition probabilities w_{mn} are symmetric with respect to the initial (n) and final (m) states if $\omega(-t) = \omega(t)$. As is clear from (11), the equality $w_{mn} = w_{nm}$ is actually observed for arbitrary dependence $\omega(t)$. One can understand the reason for this additional symmetry if ρ is related to the coefficient for reflection from a one-dimensional barrier (with regard to this reason, see Sec. 5).

We emphasize that only the absolute value of the ratio C_2/C_1 but not its phase enters into formula (11) for w_{mn}. Such will not always be the case. Let us consider, for example, the evolution of a coherent state:[4)]

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{16a}$$

$$\psi_\alpha(x, 0) = \langle x|\alpha\rangle$$
$$= (\omega_-/\pi)^{1/4}\exp\{-{}^1\!/_2[\omega_- x^2 - 2\sqrt{2\omega_-}\,\alpha x + \alpha^2 + |\alpha|^2]\}. \tag{16b}$$

Comparison with (5) shows that, to within a constant factor $\exp(-{}^1\!/_2|\alpha|^2)$, $\psi_\alpha(x, t = 0)$ coincides with the generating function $G(\alpha, x; 0)$. Therefore, for arbitrary $t > 0$

$$\psi_\alpha(x, t) = \exp(-{}^1\!/_2|\alpha|^2)G(\alpha, x; t), \tag{17}$$

where G is given by formula (6). Since $a(t) \neq \omega(t)$ (even for $t \to \infty$, see Eq. (10)), the state $\psi_\alpha(x, t)$ is no longer coherent for $t > 0$. As $t \to +\infty$ the energy level distribution becomes stationary:

$$w_0(\alpha) = \sqrt{1-\rho}\exp\{-[1 - \rho^{1/2}\cos 2(\varphi - \delta)]|\alpha|^2\},$$

$$w_n(\alpha) = w_0(\alpha)\frac{\rho^{n/2}}{2^n n!}\left|H_n\left(\sqrt{\frac{1-\rho}{2\rho^{1/2}}}|\alpha|e^{i(\varphi-\delta)}\right)\right|^2. \tag{18}$$

Here $H_n(z)$ is a Hermite polynomial, $\alpha = |\alpha|e^{i\varphi}$, and $C_2/C_1^* = \sqrt{\rho}e^{2i\delta}$. In contrast to (11) the distribution $w_n(\alpha)$ is not determined by only the single quantity ρ, but also depends on the phase φ of the initial state $|\alpha\rangle$. The quasiclassical nature of the coherent state appears in this.

In the adiabatic region this formula simplifies to:

$$w_n(\alpha) = e^{-\nu}\frac{\nu^n}{n!}\left[1 + \sqrt{\rho}\left(\nu - \frac{n(n-1)}{\nu}\right)\cos 2(\varphi - \delta) + O(\rho)\right]$$
$$(\nu = |\alpha|^2). \tag{19}$$

As an example of the latter let us assume that at $t = 0$ the oscillator is in a state of thermal equilibrium:

$$\rho_{mn}(0) = (1-\xi)\xi^n\delta_{mn}, \quad \xi = \exp(-\hbar\omega_-/kT) \tag{20}$$

(a Planck distribution with temperature T). In this case as $t \to \infty$ the populations w_n are given by

$$w_n = (1-\xi)\xi^n\left(\frac{1-\rho}{1-\rho\xi^2}\right)^{1/2}\left(\frac{1-\rho\xi^{-2}}{1-\rho\xi^2}\right)^{n/2}P_n\left(\frac{1-\rho}{\sqrt{(1-\rho\xi^2)(1-\rho\xi^{-2})}}\right). \tag{21}$$

Since the initial state (20) is an incoherent mixture of n-quantum states, the phase of the ratio C_2/C_1 does not enter into (21).

We note that the generating function

$$G_n(z, t) = \sum_{m=0}^{\infty} w_{mn}(t)z^m,$$

which corresponds to transitions out of the n-quantum state, also has the form (21). In order to obtain $G_n(z, t)$ it is only necessary to omit the factor $1 - \xi$ in (21) and replace the variable ξ by z. With the aid of $G_n(z, t)$ it is not difficult to obtain formulas for the average value $\overline{n}(t)$ and for the dispersion $\overline{\Delta n^2(t)}$.

3. Let us go on to a consideration of the Heisenberg representation. The coordinate operator $\hat{x}(t)$ satis-

[4)]Coherent states were introduced by Glauber [18], and at the present time they are widely used in quantum optics. [19] With regard to their properties, they are nearest to the states of a classical oscillator. If one sets $\alpha = re^{i\varphi}$, then r is related to the amplitude and φ is the phase of the classical oscillation.

fies the equation $\ddot{\hat{x}} + \omega^2(t)\hat{x} = 0$. Therefore

$$\hat{x}(t) = c_{11}\hat{x}(0) + c_{12}\hat{p}(0), \quad \hat{p}(t) = c_{21}\hat{x}(0) + c_{22}\hat{p}(0), \tag{22}$$

where the c_{ij} are real functions of the time. One can express them in terms of the function $\xi(t)$ introduced in (8):

$$c_{11} = \frac{\xi + \xi^*}{2}, \quad c_{12} = \frac{\xi - \xi^*}{2i\omega_-},$$
$$c_{21} = \dot{c}_{11}, \quad c_{22} = \dot{c}_{12}, \quad \det(c_{ij}) = 1. \tag{23}$$

Let us denote by $\hat{a}(t)$ the Heisenberg operator which satisfies the initial condition

$$\hat{a}(t) = a(\omega_-)e^{-i\omega_- t} \quad \text{for} \quad t \to -\infty. \tag{24}$$

(Here $a(\omega)$ (without the caret) is a time-independent operator in the Schrödinger representation: $a(\omega) = (2\omega)^{-1/2}(\omega x + ip)$.) From Eqs. (22) and (24) it follows that

$$\hat{a}(t) = (2\omega_-)^{-1/2}(\omega_-\hat{x}(t) + i\hat{p}(t)). \tag{25}$$

Hence

$$\hat{a}(t) = u'\hat{a}(0) + v'\hat{a}^+(0), \quad \hat{a}^+(t) = v'^*\hat{a}(0) + u'^*\hat{a}^+(0), \tag{26}$$

where

$$u'(t) = {}^1/_2(\xi^* + i\omega_-^{-1}\dot{\xi}^*), \quad v'(t) = {}^1/_2(\xi + i\omega_-^{-1}\dot{\xi}).$$

Since the commutator $[\hat{a}(t), \hat{a}^+(t)] = 1$ does not depend on t, then $|u'|^2 - |v'|^2 = 1$ (which is not difficult to verify by a direct check). From here it follows that the transformation (26) (let us denote it by S(t)) belongs to the group SU(1, 1).

Now let us introduce the transformation T corresponding to a change in the frequency of the oscillator functions: $T\psi_n(\omega_-) = \psi_n(\omega_+)$. It is not difficult to see that it is also contained in the group SU(1, 1):

$$a(\omega_+) = u''a(\omega_-) + v''a^+(\omega_-), \quad a^+(\omega_+) = v''a(\omega_-) + u''a^+(\omega_-), \tag{27}$$

where

$$u'' = \frac{\omega_+ + \omega_-}{2\sqrt{\omega_+\omega_-}}, \quad v'' = \frac{\omega_+ - \omega_-}{2\sqrt{\omega_+\omega_-}}.$$

The matrix element for the transition $|n, \omega_-\rangle \to m, \omega_+\rangle$ now takes the form

$$\langle m, \omega_+|\psi_n(t)\rangle = \langle m, \omega_-|T^+S(t)|n, \omega_-\rangle. \tag{28}$$

Since the states inside the brackets now pertain to the single frequency ω_-, Eq. (28) represents the matrix element of a finite "rotation" for the group SU(1, 1). An arbitrary transformation of SU(1, 1) is determined by the three parameters ψ, θ, and φ and has the form

$$U = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}, \quad u = \operatorname{ch}\frac{\theta}{2}e^{i(\psi+\varphi)/2}, \quad v = \operatorname{sh}\frac{\theta}{2}e^{i(\psi-\varphi)/2} \tag{29}$$

(the range of variation of the parameters is given by: $0 \le \psi, \varphi \le 2\pi$, $-\infty < \theta < \infty$). By multiplying T and S we can determine the values of u and v corresponding to the transformation $U = T^+S$:

$$u = \frac{1}{2}\sqrt{\frac{\omega_+}{\omega_-}}\left(\xi^* + \frac{i}{\omega_+}\dot{\xi}^*\right), \quad v = \frac{1}{2}\sqrt{\frac{\omega_+}{\omega_-}}\left(\xi + \frac{i}{\omega_+}\dot{\xi}\right). \tag{30}$$

In particular, as $t \to \infty$

$$|u| = \sqrt{\frac{\omega_+}{\omega_-}}|C_1| = \frac{1}{\sqrt{1-\rho}}, \quad |v| = \sqrt{\frac{\rho}{1-\rho}}, \quad \operatorname{th}\frac{\theta}{2} = \sqrt{\rho}. \tag{31}$$

As is well known,[20,21] two irreducible representations of the group SU(1, 1) are realized by the wave functions of an oscillator: the states with even n form one of these representations, the other is formed by states with odd n. The irreducible representations of the group SU(1, 1) were investigated in articles[22,23] in which the matrix elements of finite "rotations" were calculated. In terms of Bargmann's notation,[23] the representations we are considering are related to the discrete series D_k^+ with values of the parameter $k = 1/4$ (even n) and $k = 3/4$ (odd n). Therefore the transition probability is $w_{mn} = |\Delta_{mn}^{(k)}(\theta)|^2$, where $\Delta^{(k)}$ is the analogue of the Wigner D-function for the indicated representations of the group SU(1, 1), and the angle θ is given by formula (31). In principle this is the easiest way to obtain formula (11). However, certain transformations are required in order to bring the formula given in[23] for $\Delta_{mn}^{(k)}(\theta)$ to the form (11).

The Heisenberg representation of the operators $\hat{x}$ and $\hat{p}$ is even more important because it enables us to explicitly write a solution of Eq. (1) for an arbitrary initial state. In this connection it is convenient to work with the characteristic function χ and the Wigner quasiprobability W:

$$\chi(\lambda, \mu; t) = \operatorname{Sp}\{\rho(t)\exp[i(\lambda\hat{x}(0) + \mu\hat{p}(0))]\},$$
$$W(x, p; t) = \frac{1}{(2\pi)^2}\int d\lambda\, d\mu\, \chi(\lambda, \mu; t)e^{-i(\lambda x + \mu p)} \tag{32}$$

(the properties of the functions χ and W and, in particular, their relation to the density matrix are set forth, for example, in[9,10,19]). From (22) we immediately obtain

$$\chi(\lambda, \mu; t) = \chi(c_{11}\lambda + c_{21}\mu, c_{12}\lambda + c_{22}\mu; t = 0);$$
$$W(x, p; t) = W(c_{22}x - c_{12}p, -c_{21}x + c_{11}p; t = 0). \tag{33}$$

Thus, the time evolution of the functions χ and W reduces to a linear transformation of their arguments, having exactly the same form as in classical mechanics.[5)]

We emphasize that relations (33) are valid in the most general case when the initial state of the oscillator is a mixed state and is described by a density matrix. The problem of the evolution of an arbitrary initial state is in principle completely solved by the same equations (33). However, the transition from (33) to the populations w_m requires an evaluation of rather complicated integrals.

4. In the quantum case

$$I = \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)|c_n|^2 \quad \left(\text{if } \psi = \sum_{n=0}^{\infty}c_n|n\rangle\right). \tag{34}$$

serves as an analogue of the adiabatic invariant $I_{cl} = E/\omega$. One can determine the change of I as follows. Let us write the density matrix of the initial state in the form of Glauber's P-representation:[18]

$$\hat{\rho}(0) = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|, \tag{35}$$

[5)] Similar relations for a system of N coupled oscillators are derived in article [10]. In contrast to [10] in our case it is possible to obtain explicit expressions for the coefficients $c_{ij}(t)$ – see formula (23).

A transformation of the coordinates $x_j(t)$ and momenta $P_j(t)$ for arbitrary N belongs to the group of real symplectic matrices Sp (2N, R) of order 2N. In the case N = 1, as is clear from Eq. (23), this transformation is contained in the group SL (2, R). This is a characteristic of the elementary case N = 1. As is well known from the theory of groups, all three groups Sp (2, R), SL (2, R), and SU (1, 1) are isomorphic among themselves.

where $P(\alpha)$ is a weight function. Then for arbitrary $t > 0$

$$\hat{\rho}(t) = \int d^2\alpha P(\alpha) |\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|, \tag{36}$$

where $\psi_\alpha(t)$ is determined by formula (17) (in the x-representation). Hence for an arbitrary operator $\hat{A}$ we have

$$\langle A\rangle_t = \mathrm{Sp}\{\hat{\rho}(t)\hat{A}\} = \int d^2\alpha P(\alpha) \langle\psi_\alpha(t)|\hat{A}|\psi_\alpha(t)\rangle. \tag{37}$$

Thus, it is sufficient to average $\hat{A}$ over the states $\psi_\alpha(t)$. After simple calculations we find

$$\langle n_-\rangle = |\alpha|^2,$$
$$\langle n_+\rangle = \frac{1}{1-\rho}[\rho + |\alpha|^2(1+\rho - 2\sqrt{\rho}\cos 2(\varphi-\delta))] \tag{38}$$

(here $\langle n_\pm\rangle$ are the average values of n for the state $\psi_\alpha(t)$ as $t \to \pm\infty$, $\varphi = \arg\alpha$, and δ is defined in Eq. (18)). From Eqs. (37) and (38) it follows that

$$\varepsilon = \frac{I_+ - I_-}{I_-} = \frac{2}{1-\rho}\left[\rho - \sqrt{\rho}\,\mathrm{Re}\left(\frac{A}{B}e^{2i\delta}\right)\right], \tag{39}$$

where

$$A = \langle a^{+2}\rangle_- = \int d^2\alpha P(\alpha)\alpha^{*2},$$
$$B = I_- = \int d^2\alpha P(\alpha)\left(|\alpha|^2 + \frac{1}{2}\right). \tag{39a}$$

Formulas (39) are valid for an arbitrary state.

From the uncertainty relation it follows that $|A| \leq (B^2 - 1/4)^{1/2}$. Therefore the quantity ϵ is always confined within the limits

$$-\frac{2\sqrt{\rho}}{1+\sqrt{\rho}} < \varepsilon < \frac{2\sqrt{\rho}}{1-\sqrt{\rho}}. \tag{40}$$

If the initial state is stationary, then $P(\alpha, t_0) = P(|\alpha|)$, whence $A = 0$; consequently

$$\varepsilon = \frac{I_+ - I_-}{I_-} = \frac{2\rho}{1-\rho} \tag{41}$$

(a result which is independent of the specific form of this state). In particular, formula (41) is valid for all n-quantum states. For n = 0 or 1 it is not difficult to verify this directly from the distributions (14).

As is evident from Eqs. (40) and (41), on the average $I_+ > I_-$. Having considered an oscillator with a slowly-varying frequency, one can easily understand the reason for the increase of the invariant I. Let $\omega = \omega_0 + \Delta\omega$, $\Delta\omega \ll \omega_0$:

$$I = \frac{p^2}{2\omega} + \frac{\omega x^2}{2} = I_0 + \frac{\Delta\omega}{2}\left(x^2 - \frac{p^2}{\omega_0^2}\right) + \left(\frac{\Delta\omega}{\omega_0}\right)^2\frac{p^2}{2\omega_0} + \ldots$$

Averaging this operator with respect to the unperturbed wave functions, we take into consideration that according to the virial theorem, $\langle p^2\rangle = \langle\omega_0^2 x^2\rangle = \omega_0 I_0$, from which it follows that

$$I = I_0\left\{1 + \frac{1}{2}\left(\frac{\Delta\omega}{\omega_0}\right)^2\right\} > I_0.$$

By a similar method one can find the average even for operators which are more complicated than $\bar{n}$. We shall present an expression for the dipersion $\overline{\Delta n_+^2} = \langle n^2 - \bar{n}^2\rangle_{t\to\infty}$. In this connection since the general formula has a rather cumbersome form, let us confine our attention to the case of a stationary state (for $t \to -\infty$). Then

$$\overline{\Delta n_+^2} = \frac{1+4\rho+\rho^2}{(1-\rho)^2}\overline{\Delta n_-^2} + \frac{2\rho}{(1-\rho)^2}(\bar{n}_-^2 + \bar{n}_- + 1). \tag{42}$$

If the variation of the frequency $\omega(t)$ takes place adiabatically, then $\rho \to 0$. In this case formulas (39) and (42) take the simpler form

$$\varepsilon = -2\mathrm{Re}\left(\frac{A}{B}R\right), \quad \overline{\Delta n_+^2} = \overline{\Delta n_-^2} + 2\rho(\bar{n}_-^2 + \bar{n}_- + 1 + 3\overline{\Delta n_-^2}), \tag{43}$$

where $R = \rho^{1/2}e^{2i\delta}$ is the coefficient of reflection (see Eq. (44) below).

The problem of the change of the adiabatic invariant I for a quantum oscillator was considered earlier by Dykhne.[7] The formula for ϵ obtained by him is in agreement with (43) in that case when A = B. This equation is satisfied only for quasiclassical wave packets $\psi = \Sigma c_n|n\rangle$ for which all of the expansion coefficients c_n have the same phase. Thus, if a coherent state $|\alpha\rangle$ is taken as the initial state of the oscillator, then

$$\frac{A}{B} = \frac{2|\alpha|^2}{1+2|\alpha|^2}e^{-2i\varphi} \to e^{-2i\varphi} \text{ for } |\alpha| \gg 1$$
$$(\alpha = |\alpha|e^{i\varphi}).$$

Therefore ϵ is not determined by only the real part of the reflection coefficient R, but it also depends on the phase φ of the initial state.

5. As already mentioned above, the solution of the problem of a quantum oscillator is completely determined by the values of the constants C_1, C_2, and $\rho = |C_2/C_1|^2$. In principle one can find these quantities by directly solving the differential equation (8). There is, however, a more intuitive interpretation of ρ as the coefficient of reflection from a one-dimensional potential barrier.[6)] Namely, out of $\xi(t)$ and $\xi^*(t)$ we form a linear combination $\varphi(t)$ with the following properties:

$$\varphi(t) = \xi(t) + R\xi^*(t) = \begin{cases} e^{i\omega_- t} + Re^{-i\omega_- t} & \text{for } t \to -\infty \\ De^{i\omega_+ t} & \text{for } t \to +\infty \end{cases}. \tag{44}$$

From here it is clear that $\varphi(t)$ coincides with the wave function of the one-dimensional Schrödinger equation if we replace t by x and $\omega(t)$ by k(x). The coefficients R and D are the amplitudes of the reflected and transmitted waves. From a comparison of (44) with (10) we find

$$C_1 = \frac{D}{1-|R|^2}, \quad C_2 = \frac{RD^*}{1-|R|^2}, \quad \rho = |R|^2. \tag{45}$$

Thus, the solution (44) corresponds to a wave incident on the barrier from the left. Time reversal ($t \to -t$, $\omega(t) \to \omega(-t)$) corresponds to the transition to a wave which is incident from the right. Denoting the coefficients of reflection for these two waves by ρ and ρ', we have $w_{mn}(\rho) = w_{nm}(\rho')$. But, as is well known,[1] $\rho' = \rho$; the additional symmetry of the transition probability w_{mn} with respect to the indices m and n, mentioned in Sec. 2, also follows from here.

Since $\omega^2(t) > 0$, the question concerns above-barrier reflection. This analogy enables us to apply methods developed in quantum mechanics in order to evaluate ρ. Thus, if $\omega(t) = \omega_0[1 + \epsilon f(t)]$ ($\epsilon \to 0$) then the formula of perturbation theory[1,2] is valid,

$$\rho = \varepsilon^2\left|\omega_0\int_{-\infty}^{\infty} dt\, f(t)e^{2i\omega_0 t}\right|^2. \tag{46}$$

6)This idea is due to L. P. Pitaevskiǐ, see [7].

If $\omega(t)$ is changing adiabatically, then one can apply the quasiclassical method developed by Pokrovskiĭ and others.[24,25] We shall confine our attention to several examples in which Eq. (8) can be solved exactly, but we shall also consider the especially interesting case of parametric resonance.

1) Let

$$\omega^2(t)=\omega_-^2+\frac{\omega_+^2-\omega_-^2}{1+e^{-2\gamma t}}+\frac{a\gamma^2}{(e^{\gamma t}+e^{-\gamma t})^2} \quad (47)$$

(such a dependence for $\omega(t)$ corresponds to the Eckart potential which is well known from quantum mechanics). Here $a > -(\omega_+ + \omega_-)^2/\gamma^2$ or else $\omega^2(t)$ takes negative values. The function $\omega(t)$ has a minimum for $-a_0 < a < -a_1$, a maximum for $a > a_1$, and for $|a| \le a_1$ it changes monotonically from ω_- to ω_+ (here $a_0 = (\omega_+ + \omega_-)^2/\gamma^2$, $a_1 = |\omega_+^2 - \omega_-^2|/\gamma^2$). The coefficient of reflection for (47) is given by[26]

$$\rho=\frac{\operatorname{ch}(\alpha-\beta)+\cos\lambda}{\operatorname{ch}(\alpha+\beta)+\cos\lambda},$$

$$\alpha=\pi\frac{\omega_-}{\gamma},\quad \beta=\pi\frac{\omega_+}{\gamma},\quad \lambda=\pi\sqrt{a+1}. \quad (48)$$

2) With the aid of (47) one can accomplish a continuous transition from the adiabatic region to an abrupt change in the value of $\omega(t)$. Assuming $a=0$ we have

$$\rho=\left[\frac{\operatorname{sh}(\alpha-\beta)/2}{\operatorname{sh}(\alpha+\beta)/2}\right]^2$$

$$=\begin{cases}\exp\left(-2\pi\dfrac{\omega_<}{\gamma}\right) & \text{for } \gamma\ll\omega_\pm\\ \left(\dfrac{\omega_+-\omega_-}{\omega_++\omega_-}\right)^2\left[1-\dfrac{\pi^2\omega_+\omega_-}{3\gamma^2}+\ldots\right] & \text{for } \gamma\gg\omega_\pm\end{cases} \quad (49)$$

The value $\rho = (\omega_+ - \omega_-)^2/(\omega_+ + \omega_-)^2$ corresponds to an abrupt change in the frequency and may be obtained by the method of sudden perturbations.[2]

3) One can indicate a law of variation for $\omega(t)$ for which, in general, no reflection is present. Namely, let $f(t)$ be an arbitrary function satisfying only the conditions $f(t) \to \omega_\pm$ as $t \to \pm\infty$. Then for

$$\omega^2(t)=f^2+\frac{\ddot f}{2f}-\frac{3}{4}\left(\frac{\dot f}{f}\right)^2 \quad (50)$$

Eq. (8) has the exact solution

$$\xi(t)=\frac{\text{const}}{\sqrt{f(t)}}\exp\left\{i\int_0^t f(t')\,dt'\right\} \quad (51)$$

and $\rho \equiv 0$. Here $w_{mn} = \delta_{mn}$, i.e., the oscillator remains in its initial state. A number of specific examples of such types of $\omega(t)$ are considered in article[27].

4) As is evident from (46) one can expect a noticeable increase of ρ in that case when the spectrum $\omega(t)$ contains a component with the doubled frequency $2\omega_0$ (for a simple explanation of this fact, see[28]). This is the case of parametric resonance. Assuming $\omega^2(t) = \omega_0^2[1 + 2\epsilon \sin(2+\delta)\omega_0 t]$, where $|\epsilon|, |\delta| \ll 1$, we find

$$\rho=\begin{cases}\dfrac{\varepsilon^2\operatorname{sh}^2\tau}{4\mu^2+\varepsilon^2\operatorname{sh}^2\tau} & \text{for } |\varepsilon|>|\delta|\\ \dfrac{\varepsilon^2\sin^2\tau}{4\mu^2+\varepsilon^2\sin^2\tau} & \text{for } |\varepsilon|<|\delta|\end{cases} \quad (52)$$

(here $\mu = \frac{1}{2}\sqrt{|\epsilon^2 - \delta^2|}$ and $\tau = \mu\omega_0 t$). For $|\epsilon| > |\delta|$ (a region of instability for Mathieu's equation) ρ increases monotonically from 0 to 1. For $|\epsilon| < |\delta|$ the solution has an oscillatory character, and its maximum value is given by $\rho_{max} = (\epsilon/\delta)^2 < 1$. In both cases strong excitation of the oscillator ($\rho \sim 1$) is possible for an arbitrarily small value of ϵ. This example may serve as a model for a quantum parametric amplifier (with a single mode). The theory of two-mode amplifiers is considered in articles[8,9].

6. From unitarity it follows that $0 \le w_{mn}(\rho) \le 1$ for arbitrary m, n, and ρ. One can somewhat strengthen the upper bound for w_{mn}. In fact from the inequality

$$|P_l^m(x)| < [(l+m)!/(l-m)!]^{1/2}\quad(-1<x<1)$$

it immediately follows that

$$w_{mn}(\rho)\leqslant w_{00}(\rho)=\sqrt{1-\rho} \quad (53)$$

(for $\rho > 0$ and $m + n > 0$ this inequality is strict). For a given value of ρ the probability $w_{00}(\rho)$ has the maximum value, which is in complete agreement with Fig. 1.

In conclusion we note that inequality (53) enables us to find a simple example of unitary inequivalence of the canonical commutation relations $[a_i, a_j^+] = \delta_{ij}$ for a system with an infinite number of degrees of freedom.[7)] For this purpose let us consider two systems of oscillators with frequencies ω_i^- and ω_i^+ ($i = 1, 2, 3, \ldots$). Transformation (27), which corresponds to a change of the frequency, may be written in the form

$$p_i^+=c_ip_i^-,\qquad q_i^+=c_i^{-1}q_i^-;\qquad c_i=(\omega_i^-/\omega_i^+)^{1/2}. \quad (54)$$

The scalar product of the vacuum vectors Φ_0^- and Φ_0^+ is given by

$$|\langle\Phi_0^+|\Phi_0^-\rangle|^2=\prod_{i=1}^\infty\sqrt{1-\rho_i}=\prod_{i=1}^\infty\frac{2c_i}{1+c_i^2} \quad (55)$$

where $\rho_i = (\omega_i^+ - \omega_i^-)^2/(\omega_i^+ + \omega_i^-)^2$ (this value for ρ_i corresponds to an abrupt change of the frequency from ω_i^- to ω_i^+). Because of the fact that the product in (55) contains an infinite number of factors, it may be equal to zero even if all $c_i \ne 0$. In this case the vacua Φ_0^- and Φ_0^+ are orthogonal to each other. Now let us take two arbitrary basis vectors $\Phi_{\{n\}}^-$ and $\Phi_{\{m\}}^+$ from the Hilbert spaces $\mathscr{H}_-$ and $\mathscr{H}_+$ spanned by Φ_0^- and Φ_0^+:

$$\Phi_{\{n\}}^-=|n_1,\omega_1^-;n_2,\omega_2^-;\ldots\rangle,\quad \Phi_{\{m\}}^+=|m_1,\omega_1^+;m_2,\omega_2^+;\ldots\rangle. \quad (56)$$

With inequality (53) taken into account we find

$$|\langle\Phi_{\{m\}}^+|\Phi_{\{n\}}^-\rangle|^2=\prod_{i=1}^\infty w_{m_in_i}(\rho_i)\leqslant|\langle\Phi_0^+|\Phi_0^-\rangle|^2. \quad (57)$$

Consequently if the vacuum vectors are orthogonal to each other, then any two vectors Φ^+ and Φ^- from the spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ are orthogonal: $\langle\Phi^+|\Phi^-\rangle = 0$, i.e., the representations of the canonical commutation relations which are realizable in the spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ are not unitarily equivalent.[8)]

7)See, for example, articles [29,30] for examples of inequivalent representations. One can find a more detailed discussion and references to the literature in the book [31].

8)The condition $\Pi 2c_i/(1 + c_i^2) = 0$, which is necessary and sufficient for unitary inequivalence of two representations, is not new and is mentioned in [31]. As is evident from the account set forth here, utilization of inequality (53) gives a very simple derivation of this condition.

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APPENDIX A

Let us deduce the values of certain integrals which are required in order to obtain the formulas of Section 2.

1) Let

$$I_n(\alpha,\beta)=\int_{-\infty}^{\infty} e^{-\alpha x^2+2\beta x}H_n(x)\,dx,\quad \operatorname{Re}\alpha>0. \tag{A.1}$$

Let us transform the generating function

$$g(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}I_n(\alpha,\beta)=\sqrt{\frac{\pi}{\alpha}}\exp\left\{\frac{(1-\alpha)t^2+2\beta t+\beta^2}{\alpha}\right\} \tag{A.2}$$

(the generating function for Hermite polynomials is used in order to obtain this formula). Expanding in a series in powers of t we obtain

$$I_n(\alpha,\beta)=\sqrt{\frac{\pi}{\alpha}}e^{\beta^2/\alpha}\left(1-\frac{1}{\alpha}\right)^{n/2}H_n\left(\frac{\beta}{\sqrt{\alpha(\alpha-1)}}\right). \tag{A.3}$$

2) Now let us consider the integrals J_{mn}:

$$J_{mn}(\alpha,\beta_1,\beta_2)=\int_{-\infty}^{\infty} e^{-\alpha x^2}H_m(\beta_1 x)H_n(\beta_2 x)\,dx \tag{A.4}$$

($J_{mn}=0$ if the numbers m and n have different parity). Let us form $h(t_1, t_2)$ according to:

$$h(t_1,t_2)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}J_{mn}t_1{}^m t_2{}^n$$

$$=\sqrt{\frac{\pi}{\alpha}}\exp\left\{\frac{(\beta_1t_1+\beta_2t_2)^2}{\alpha}-(t_1{}^2+t_2{}^2)\right\}. \tag{A.5}$$

Introducing the polynomials $q_{mn}(x)$ which are determined from the expansion

$$\exp(t_1^2+t_2^2+2xt_1t_2)=\sum_{m,n=0}^{\infty}q_{mn}(x)t_1{}^m t_2{}^n, \tag{A.6}$$

from Eq. (A.5) we obtain

$$J_{mn}=m!n!\sqrt{\frac{\pi}{\alpha}}\left(\frac{\beta_1{}^2}{\alpha}-1\right)^{m/2}\left(\frac{\beta_2{}^2}{\alpha}-1\right)^{n/2}\cdot q_{mn}\left(\frac{\beta_1\beta_2}{[(\beta_1{}^2-\alpha)(\beta_2{}^2-\alpha)]^{1/2}}\right). \tag{A.7}$$

One can relate the polynomials $q_{mn}(x)$ to the associated Legendre functions. In fact, from Eq. (A.6) we have

$$q_{mn}(x)=\sum_k (2x)^k/k!\left(\frac{m-k}{2}\right)!\left(\frac{n-k}{2}\right)!$$
$$(k=n_<,\,n_<-2,\,n_<-4,\ldots;\ n_<=\min(m,n)). \tag{A.8}$$

On the other hand

$$(1+x^2)^{l/2}P_l{}^m\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{i^m(l+m)!}{l!}\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+i\cos\varphi)^l\cos m\varphi\,d\varphi. \tag{A.9}$$

Expanding the binomial $(x + i\cos\varphi)^l$, integrating term-by-term, and comparing the result with Eq. (A.8), we arrive at the desired identity:

$$i^{n_>}n_>!\,q_{mn}(-ix)=(2\sqrt{1+x^2})^{(m+n)/2}P_{(m+n)/2}^{|m-n|/2}\left(\frac{x}{\sqrt{1+x^2}}\right). \tag{A.10}$$

Taking this equality into consideration, we transform Eq. (A.7) to its final form:

$$J_{mn}=n_<!\sqrt{\frac{\pi}{\alpha}}\left(\frac{1-\lambda_1}{1-\lambda_2}\right)^{(m-n)/4}(2\sqrt{\lambda_1+\lambda_2-1})^{(m+n)/2}\cdot P_{(m+n)/2}^{|m-n|/2}\left(\sqrt{\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2-1}}\right), \tag{A.11}$$

where $\lambda_1=\beta_1^2/\alpha$ and $\lambda_2=\beta_2^2/\alpha$.

We note that the polynomials $q_{mn}(x)$ can be expressed in terms of a hypergeometric function:

$$q_{mn}(x)=\frac{(2x)^{n_<}}{(^1/_2|m-n|)!\,n_<!}F\left(-\frac{n_<}{2},\frac{1-n_<}{2},1+\frac{|m-n|}{2};\frac{1}{x^2}\right). \tag{A.12}$$

One can verify the validity of this equation if the hypergeometric function is expanded in a series and it is taken into consideration that

$$\left(-\frac{n}{2}\right)_k\left(\frac{1-n}{2}\right)_k=\frac{n!}{2^{2k}(n-2k)!}.$$

Relations (A.10) and (A.11) also follow from here if the expressions for $P_n^m(\cos\theta)$ given in book[32] are used. Finally, by performing a transformation of the argument of the type $z\to z^{-1}$ in Eq. (A.12), we obtain

$$q_{2m,2n}(x)=\frac{1}{m!\,n!}F(-m,-n,{}^1/_2;x^2),$$
$$q_{2m+1,2n+1}(x)=\frac{2x}{m!\,n!}F(-m,-n,{}^3/_2;x^2). \tag{A.13}$$

APPENDIX B

Expression (15a) for a classical oscillator follows from geometrical considerations. The initial state with a random phase uniformly distributed in the interval $0\le\varphi\le 2\pi$ is represented in the phase plane by the ellipsoid $(p^2/\omega_-)+\omega_-x^2=2I_-$. According to Eq. (22) the time evolution of x and p leads to a rotation and an elongation of this ellipse with conservation of its initial area (since $\det(c_{ij})=1$).

The distribution with respect to $I_+=\omega_+^{-1}E$ (for $t\to\infty$) (the adiabatic invariant I plays the role of the number of quanta) has the form

$$w(I_+)=\int_0^{2\pi}\frac{d\varphi}{2\pi}\delta\left(\lambda\omega_+x^2+\frac{p^2}{\lambda\omega_+}-2I_-\right)=\frac{1}{\pi[(I_+-I_1)(I_2-I_+)]^{1/2}}, \tag{B.1}$$

where $\sqrt{\omega_+}\,x=\sqrt{2_+I}\cos\varphi$, $p/\sqrt{\omega_+}=\sqrt{2I_+}\sin\varphi$, $I_1=\lambda I_-$, $I_2=\lambda^{-1}I_-$, and $(\lambda\omega_+/\omega_-)^{1/2}$ is the coefficient of elongation of the ellipse where $\lambda=(1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}$. One can find the distribution $w(I_+)$ in similar fashion even in that case when in addition an external force f(t) acts on the oscillator. In this connection the initial ellipse, besides rotation and elongation, also undergoes a displacement in the phase plane.

Thus, the envelope of the distribution (15) can be found from purely classical considerations. However, in (15) there is also a factor $\cos^2\Phi_{mn}$ which leads to quantum oscillations of the probabilities w_{mn} around their average classical values (these oscillations are clearly shown in Fig. 2).

In order to obtain the phases Φ_{mn} we shall use a quasiclassical asymptotic expression[2] for the associated Legendre functions. In the limit $m, n\gg 1$ we obtain

$$\Phi_{mn}=\int_{\sqrt{1-\rho}}^{x_2}k(x)\,dx=\frac{1}{2}\left\{(m+n)\lambda-|m-n|\operatorname{arc\,ctg}\left(\frac{|m-n|}{m+n}\operatorname{ctg}\lambda\right)\right\}, \tag{B.2}$$

where

$$k(x)=\frac{m+n}{2}\frac{\sqrt{x_0^2-x^2}}{1-x^2},\quad x_0=\frac{2\sqrt{mn}}{m+n}$$

and the parameter λ is introduced according to the equation $\sqrt{1-\rho}=x_0\cos\lambda$ (λ varies within the limits $0\le\lambda\le\sin^{-1}\sqrt{\rho}$; the value $\lambda=\sin^{-1}\sqrt{\rho}$; corresponds to $m=n$). From Eq. (B.2) we find

$$\frac{\partial\Phi_{mn}}{\partial m}=\int\limits_{\sqrt{1-\rho}}^{x_0}\frac{\partial k}{\partial m}dx=\frac{1}{2}\left\{\lambda-\operatorname{sign}(m-n)\operatorname{arc\,ctg}\left(\frac{|m-n|}{m+n}\operatorname{ctg}\lambda\right)\right\}. \quad (B.3)$$

From here it follows that the inequality $0<\partial\Phi/\partial m<\pi/2$ is satisfied for $m_1<m<n$, and for $n<m<m_2$ we have $0>\partial\Phi/\partial m>-\pi/4$. At the point $m=n$ the phase Φ_{mn} as a function of m has a break, where $\Phi_{nn}=n\sin^{-1}\sqrt{\rho}$. Since $\Phi_{mn}=0$ for $m=m_1$, m_2 and $|\partial\Phi_{mn}/\partial m|<\pi/2$ then the number N of zeros of $\cos^2\Phi_{mn}$ is given by $N=2\pi^{-1}\Phi_{nn}<n$. The average distance between the zeros characterizes the period of quantum oscillations of the distribution (15); it is equal to

$$\Delta n=\frac{m_2-m_1}{N}=2\pi\frac{\sqrt{\rho}}{(1-\rho)\arcsin\sqrt{\rho}}. \quad (B.4)$$

Note added in proof (February 28, 1969). We note that the quasi-classical approximation (15) for the transition probabilities w_{mn} is only valid upon fulfilment of the conditions: $n, m_2-m_1, m_1\gg 1$. This corresponds to the values $\rho\gg n^{-2}$, $(1-\rho)\gg n^{-1}$. In the limiting case $\rho\to 1$ (a strongly excited oscillator, $n(1-\rho)\lesssim 1$) one can reduce formula (11)

$$w_{mn}=\frac{1}{2^n n!}\sqrt{\frac{2(1-\rho)}{\pi m}}H_n^2\left(\sqrt{\frac{m(1-\rho)}{2}}\right)\exp\left\{\frac{-m(1-\rho)}{2}\right\},$$

i.e., w_{mn} is essentially proportional to the square of the wave function of an oscillator in the n-quantum state. In this connection the maximum value of the transition probability is reached for a value of m close to $4n/(1-\rho)$.

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