EVOLUTION OF A COSMOLOGICAL MODEL WITH ROTATION

T. V. RUZMAĬKINA and A. A. RUZMAĬKIN

Institute of Applied Mathematics, U.S.S.R. Academy of Sciences

Submitted December 10, 1968

Zh. Eksp. Teor. Fiz. 56, 1742–1747 (May, 1969)

A nonstationary anisotropic cosmological model with rotation is considered. It is strongly anisotropic in the early stages and gradually evolves into the isotropic Friedman model. The angular dependence of the anisotropy of the residual radiation in later stages of the evolution of the model is determined.

THE part of the Universe observable today is characterized by a high degree of isotropy and homogeneity on the average. It is therefore best described by the homogeneous Friedman model or cosmological models which approach the former in their evolution. The latter are incomparably richer in content than the Friedman model, and are widely discussed at present. However, it is usually assumed that matter expands without rotation. This is because of the mathematical difficulties which one otherwise must cope with when one attempts to obtain the corresponding solution of the Einstein equations (the known solution of Gödel is stationary, cf. also[1]). Only asymptotic ($t \to 0$, $\epsilon \to \infty$) solutions with rotation can be found, which belong to the wide class of solutions of Lifshits and Khalatnikov.[2] In the present paper we investigate the limiting stage of evolution of a homogeneout, anisotropic cosmological model taking account of the rotation of matter. Rotation in anisotropic models has also been considered in the recent preprint of Hawking.[3]

1. We use units where $c = 8\pi G = 1$. Latin indices run from zero to three, Greek indices run from one to three. Differentiation with respect to time is denoted by a dot, covariant differentiation is denoted by a semicolon. The metric of the model

$$ds^2 = dt^2 - a_{11}(t)e^{2x^3}(dx^1)^2 - 2a_{13}(t)e^{x^3}dx^1dx^3 - a_{22}(t)e^{2x^3}(dx^2)^2 - a_{33}(t)(dx^3)^2 \tag{1}$$

admits a three-parameter group of motions G_3, which acts in the space t = const and belongs to the type V in the classification of Bianchi.[4] We further require $g_{12} = g_{23} = 0$ in order to make one of the components of the velocity of the matter (u_2) equal to zero [cf. (3d)]. If we also set $g_{13} = 0$, then we obtain a model (considered by Grishchuk, Doroshkevich, and Novikov[5]) in which the matter does not rotate but moves along the x^3 axis.

At each time instant the metric form (1) corresponds to a space with constant negative curvature with the three-dimensional Ricci tensor P_α^β:

$$P_1{}^1 = P_2{}^2 = P_3{}^3 = -2a_{11}/a, \quad P_\alpha{}^\beta = 0, \quad \alpha \neq \beta,$$
$$a = a_{11}a_{33} - a_{13}^2. \tag{2}$$

Let us write down the Einstein equations for the metric (1) and the energy-momentum tensor $T_i^k = (\epsilon + p)u_i u^k - p\delta_i^k$:

$$\dot{\varkappa}_\alpha{}^\alpha + {}^1\!/_2\varkappa_\alpha{}^\beta\varkappa_\beta{}^\alpha = \varepsilon - p - 2(\varepsilon + p)u_0^2, \tag{3a}$$
$$2\varkappa_3{}^3 - \varkappa_2{}^2 - \varkappa_1{}^1 = 2(\varepsilon + p)u_0u_3, \tag{3b}$$
$$3\varkappa_1{}^3 = 2(\varepsilon + p)u_0u_1, \tag{3c}$$
$$u_2 = 0, \tag{3d}$$
$$-4\frac{a_{11}}{a}\delta_\alpha{}^\beta + \frac{1}{\sqrt{-g}}(\sqrt{-g}\,\varkappa_\alpha{}^\beta)^\cdot = (\varepsilon - p)\delta_\alpha{}^\beta - 2(\varepsilon + p)u_\alpha u^\beta, \tag{3e}$$

where $\kappa_\alpha^\beta = g^{\beta\gamma}g_{\alpha\gamma}$ has the form

$$\varkappa_1{}^1 = \frac{\dot{a}_{11}a_{33} - \dot{a}_{13}a_{13}}{a}, \quad \varkappa_2{}^2 = \frac{\dot{a}_{22}}{a_{22}}, \quad \varkappa_3{}^3 = \frac{a_{11}\dot{a}_{33} - a_{13}\dot{a}_{13}}{a},$$
$$\varkappa_3{}^1 = \frac{\dot{a}_{13}a_{33} - a_{13}\dot{a}_{33}}{a}e^{-x^3}, \quad \varkappa_1{}^3 = \frac{a_{11}\dot{a}_{13} - \dot{a}_{11}a_{13}}{a}e^{x^3}, \quad \varkappa_\alpha{}^\alpha = \frac{\dot{a}}{a} + \frac{\dot{a}_{22}}{a_{22}}. \tag{4}$$

The spatial dependence of the quantities characterizing the matter is easily determined from (3), by substituting (2) and (4). It turns out that ϵ, u_0, and u_3 do not depend on the spatial coordinates, and $u_1 = \tilde{u}_1(t)e^{x^3}$.

For an analysis of the character of the motion of the matter we use the hydrodynamic equations $T_{i;k}^k = 0$ contained in (3) together with the equation of state $p = k\epsilon$ and the entropy density σ, and the identity $u^i u_i = 1$:

$$(\sqrt{-g}\sigma u_0)^\cdot + 2\sqrt{-g}\sigma u^3 = 0, \tag{5a}$$
$$\frac{k}{1+k}\frac{\dot{\varepsilon}}{\varepsilon} + \frac{\dot{u}_1}{u_1} + \frac{u^3}{u_0} = 0, \tag{5b}$$
$$\frac{k}{1+k}\frac{\dot{\varepsilon}}{\varepsilon} + \frac{\dot{u}_3}{u_3} - \frac{u^1u_1}{u_0u_3} = 0 \tag{5c}$$
$$u_0^2 = 1 - u^1u_1 - u^3u_3. \tag{5d}$$

From (5a) and (5b) we easily find the integral

$$\sqrt{-g}\sigma u_0\varepsilon^{-2k/(1+k)}u_1{}^{-2} = \text{const}. \tag{6}$$

The rotation of the matter is usually characterized by the angular velocity (cf., for example, [4])[1)]

$$\omega^i = (-g)^{-1/2}e^{iklm}u_k u_{l;m}, \tag{7}$$

which in our case has only one nonvanishing component $\omega^2 = (-g)^{-1/2}(u_0 u_1 + u_1\dot{u}_3 - \dot{u}_1 u_3)$. Using (5b) and (5c), it can be written in the form $\omega^2 = (-g)^{-1/2}u_1/u_0$.

The magnitude of the angular velocity is

$$\omega = \sqrt{\omega^i\omega_i} = \sqrt{g_{22}}\,\omega^2 = \frac{1}{\sqrt{a}}\frac{\tilde{u}_1}{u_0} \tag{8}$$

and is independent of the spatial coordinates.

2. Near $t = 0$ the general solution of (3) has an asymptotic form of the Kasner type:

$$a_{11} = a_0 t^{2p_1}, \quad a_{22} = b_0 t^{2p_2}, \quad a_{33} = c_0 t^{2p_3}, \quad a_{13} = d_0 t^{2p_3},$$
$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1, \quad p_3 > p_1 \tag{9}$$

---

1)In the nonrelativistic limit this definition goes over into the usual one, ω = curl v.

and hence belongs to the class of singular solutions found by Lifshitz and Khalatnikov.[2]

For the quantities which characterize matter with the equation of state p = 0 (k = 0, $\sigma \sim \epsilon$) we find from (5) for $t \to 0$

$$u_1 = u_1{}^0 e^{x^3}, \quad u_3 = u_3{}^0, \quad u_0 = \frac{|u_3{}^0|}{\sqrt{c_0}} t^{-p_3}, \quad \varepsilon = \varepsilon_0{}^{-1+p_3}. \tag{10}$$

The constants a_0 and b_0 are connected with the scale on the x^1 and x^2 axes. The remaining six constants c_0, d_0, u_1^0, u_3^0, ϵ_0, and one of the p_a satisfy two relations, which follow from (3):

$$3p_3 - 1 = \frac{\varepsilon_0 |u_3{}^0| u_3{}^0}{\sqrt{c_0}}, \quad 3\frac{d_0}{c_0}(p_3 - p_1) = \frac{\varepsilon_0 u_1{}^0 |u_3{}^0|}{\sqrt{c_0}}. \tag{11}$$

Therefore the general solution of (3) depends on four physically arbitrary constants, for example, u_1^0, u_3^0, ϵ_0, and p_3. The proper time of the moving matter is connected with t by the relation $\tau \sim t^{1+p_3}$, i.e., the three-dimensional velocity approaches the velocity of light according to the law $\sqrt{1-v^2} \sim t^{p_3}$. The energy density in the corresponding reference system tends to infinity, $\epsilon \sim \tau^{-(1-p_3)/(1+p_3)}$. The angular velocity has the following behavior near the singularity [cf. (8)]:

$$\omega \approx \frac{u_1{}^0}{|u_3{}^0|\sqrt{a_0}} t^{-p_1}. \tag{12}$$

If the equation of state is changed from p = 0 to $p = \epsilon/3$ ($k = 1/3$, $\sigma \sim \epsilon^{3/4}$), we have[2)]

$$\tilde{u}_1 \sim u_3 \sim t^{(1-p_3)/2}, \quad \varepsilon \sim t^{-2+2p_3}, \quad \tau \sim t^{(3p_3+1)/2}, \tag{13}$$

$$\omega \sim t^{-p_1}, \tag{14}$$

i.e., we obtain qualitatively the same picture when $p_3 < 1/3$, $p_1 < p_2$ are not satisfied simultaneously. In the opposite case, an additional (degenerate) solution appears, where $\epsilon \sim t^{-4/3}$, and the coordinate system tends to a co-moving one. If u_1 is zero, an analogous solution occurs also for p = 0, as has been shown in [5].

We note two important physical consequences of these solutions: a) as a rule, there is no thermodynamic equilibrium in the early stages of evolution (estimates can be found in [5]), and b) the rotation does not affect the character of the singularity ($t \to 0$, $\epsilon \to \infty$).

3. Let us now turn to the other extreme situation: $t \to \infty$. In the case of dustlike matter (p = 0) one can show that the metric (1) becomes isotropic according to the law (cf. also [5] and [3])

$$g_{11} = -t^2\left(1 + \frac{c_1}{t^2}\right)e^{2x^3}, \quad g_{22} = -t^2\left(1 + \frac{c_2}{t^2}\right)e^{2x^3},$$
$$g_{33} = -t^2, \quad g_{13} = c_3 e^{x^3}, \tag{15}$$

where c_1, c_2, and c_3 are arbitrary constants. In this approximation

$$u_1 = u_1{}^0 e^{x^3}, \quad u_3 = u_3{}^0, \quad \varepsilon = \varepsilon_0 t^{-3}, \quad u_0 = 1, \tag{16}$$

where $u_1\epsilon_0/3 = c_3$.

We call attention to the fact that the coordinate system (15) is not co-moving. The corrections to (15) arising from the presence of matter are of the order $t^{-1}\ln t$ and are the same for all $g_{\alpha\beta}$.

For $u_1 = 0$, $c_3 = 0$ we arrive at the model considered in [5]. If also $u_3 = 0$ and $c_2 = -c_1$, one obtains the known cosmological model of Heckmann and Schücking.[4] It follows from (15) and (16) that the matter distribution becomes homogeneous for $t \to \infty$ and the coordinate system becomes co-moving. The angular velocity falls off according to

$$\omega \approx \frac{u_1{}^0}{t^2}\left(1 - \frac{c_1}{2t^2}\right). \tag{17}$$

This result depends on the equation of state. When $p = \epsilon/3$, the metric also becomes isotropic for large t ($g_{\alpha\beta} \to b^2\delta_{\alpha\beta}$) and the time dependence of the density and the velocity of the matter can be expressed by

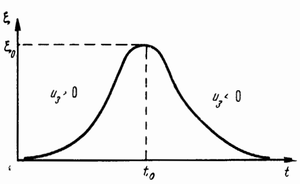
$$\varepsilon b^4 = \beta\xi, \; u_0{}^2 = 1 + \xi^{-1}, \; -u^3u_3 = \xi^{-1} - \alpha^2\sqrt{1+\xi}, \tag{18}$$

$$-u^1u_1 = \alpha^2\sqrt{1+\xi}, \quad (\omega b)^2 = \alpha^2\xi(1+\xi)^{-1/2}, \tag{19}$$

where α and β are constants of integration [cf. (5) and (6)] and the parameter ξ is connected with the time t by

$$4\frac{u_3}{|u_3|}\int\frac{dt}{b} = \int\frac{(2+3\xi)\,d\xi}{\xi\sqrt{1+\xi}\,[1-\alpha^2\xi\sqrt{1+\xi}]^{1/2}}. \tag{20}$$

The qualitative character of the resulting function $\xi = \xi(t)$ is shown in the figure.



For $\alpha = 0$ the rotation vanishes and we arrive at the situation considered in [5]. In this case there are two independent regimes for the motion of matter, $u_3 > 0$ and $u_3 < 0$; the modulus becomes isotropic only in the case $u_3 > 0$. The introduction of rotation changes the picture completely. If the matter first moved with $u_3 > 0$, then the velocity u_3 vanishes at some instant t_0 corresponding to the root of the equation $1 - \alpha^2\xi\sqrt{1+\xi} = 0$ [cf. (18)], and then (for $t > t_0$) becomes negative, i.e., the transition to an anisotropic regime sets in.

If we regard the u_α as small perturbations of the isotropic solution $b = b_0 t^{1/2}$, then we can obtain the solution $\epsilon = \epsilon_0 t^{-2}$, $u_0 \approx 1$, $u_1 = u_1^0 t^{1/2}$, $u_3 = -2(u_1^0)^2 t/b_0^2$, which is valid up to the time $t^* = b_0^4/4(u_1^0)^2$, i.e., one obtains the result of Lifshitz,[5] which he derived in first-order perturbation theory from the Friedman model. In this approximation $v \approx \sqrt{-u^\alpha u_\alpha} = u_1^0/b_0$ = const.

Therefore, although the metric (1) becomes isotropic for $t \to \infty$ when $p = \epsilon/3$, the coordinate system does not become co-moving, the three-dimensional velocity tends to the velocity of light ($\sqrt{1-v^2} \sim t^{-1}$), and large gradients of the density will occur in the co-moving coordinate system, as noted in [5]. The limits of validity of the solutions with the equation of state $p = \epsilon/3$ are given in [5].

Let us estimate the viscosity in the practically important case of small u_α. In first order in u_1, Eq. (3) yields, with account of viscous terms,[3)]

2) Of course, condition (11) is also changed in a corresponding manner.

3) The second viscosity is not considered.

$$u_1 = \frac{3\varkappa_1^3}{{}^8/_3\varepsilon - \eta\varkappa_1^1} \approx u_1^0 t^{1/2}\left(1 + \frac{3\eta}{16\varepsilon t}\right). \quad (21)$$

If we assume that the viscosity is due to Compton scattering of γ quanta by electrons, then $\eta \sim \epsilon m/c\sigma\rho$, where m is the electron mass, $\sigma \sim 10^{-24}$ cm^2 is the bremsstrahlung cross section, and ρ is the density of baryons. It is easy to see that the η in (21) increases with time. However, it becomes comparable with unity at $t = 10^{13}$ sec (the moment of recombination in the hot model) only for the very low value of the present average density of baryons $\bar{\rho}_0 \sim 10^{-36}$ g/cm^3. Since actually $\bar{\rho}_0 \geq 10^{-31}$ g/cm^3, the viscosity is evidently not important.

4. Let us now consider the question of the anisotropy of the residual radiation in the given model, following the method developed in [5]. This problem has also been investigated in [3]. We assume that at some time t_1 the intergalactic medium becomes transparent for radiation so that the photons propagate freely from then on. The anisotropy of the expansion of matter at the moment of escape of the photons leads to the present anisotropy of the residual radiation. If we assume that the radiation was in thermodynamic equilibrium at the time t_1, it will no longer be in thermal equilibrium at the moment of reception t_0, since it becomes anisotropic. However, the propagating quanta do not interact with each other, and therefore, by the Liouville theorem, the distribution function becomes a constant. This implies

$$T_0/T_1 = \omega_0/\omega_1, \quad (22)$$

i.e., in order to determine the anisotropy of the residual radiation $\Delta T/T = (T_0 - \bar{T}_0)/T_0$ ($\bar{T}_0$ is the isotropic temperature) it is sufficient to consider the zero geodesic:

$$\frac{dk_i}{d\lambda} - \frac{1}{2} k^\alpha k^\beta \frac{\partial g_{\alpha\beta}}{\partial x^i} = 0, \quad k^i k_i = 0, \quad (23)$$

with the metric (17). Writing k^α in the form

$$k^1 = \frac{\omega}{t} e^{-x^3} \cos\varphi \sin\theta, \quad k^2 = \frac{\omega}{t} e^{-x^3} \sin\varphi \sin\theta, \quad k^3 = \frac{\omega}{t} \cos\theta,$$

where θ is the angle between the negative x^3 direction and the direction of the light ray at the point of reception, and φ is the azimuthal angle; solving (23) in the linear approximation in the constants c_1, c_2, c_3; and, furthermore, taking account of the motion of the matter at the moment of the emission of light, we obtain for $\Delta T/T$ the following dependence:

$$\frac{\Delta T}{T} = \frac{2}{t_0^2}(c_1 \cos^2\varphi + c_2 \sin^2\varphi) f(\theta) + \frac{4}{t_0^2} c_3 \Psi(\theta) \cos\varphi + \chi(u_\alpha(t_1), \theta). \quad (24)$$

$$f(\theta) = 2\ln\left[\cos\theta + \frac{z^2}{2}(1 - \cos\theta)\right]\mathrm{ctg}^2\frac{\theta}{2} - \frac{(1+\cos\theta)^2}{1 - \cos\theta + 2z^{-2}\cos\theta} \quad (25)$$

$$\Psi(\theta) = 2\,\mathrm{arc\,tg}\frac{(z-1)\,\mathrm{ctg}(\theta/2)}{z + \mathrm{ctg}^2(\theta/2)}\,\mathrm{ctg}^2\frac{\theta}{2} - \left(z + \frac{1+\cos\theta}{2z^{-1}\cos\theta + z(1-\cos\theta)} - \frac{3+\cos\theta}{2}\right)\mathrm{ctg}\frac{\theta}{2}, \quad (26)$$

$$\chi(\theta) = \frac{2}{z}\frac{u_1}{|u_1|}\frac{\sqrt{-u^1 u_1}\sin\theta}{1 - \cos\theta + 2z^{-2}\cos\theta} - \frac{u_3}{|u_3|}\sqrt{-u^3 u_3} \times \frac{1 - \cos\theta - 2/z^2}{1 - \cos\theta + 2z^{-2}\cos\theta}, \quad (27)$$

where $z = t_0/t_1$.

The function $f(\theta)$, which increases in the region $\theta_0 = 2/z$ and is small for the other values of θ, takes account of the anisotropy of the deformation and agrees with that obtained in [5]. The appearance of the function $\Psi(\theta)$ is connected with the nonorthogonality of the metric (1); it is also maximal in the neighborhood of zero and small in the remaining region. If we include only the first two terms in (24), the character of the anisotropy of the residual radiation differs little from that obtained in [5]. The temperature of the background is almost constant over the whole sky except for a small spot around $\theta = 0$. The alternation of surplus and lack of intensity in the spot depends on the relation between c_1, c_2, and c_3.

The function χ describes the Doppler effect caused by the motion of the matter at the moment of the emission of light (we neglect the velocity of the observer at the moment of the reception of the light). Outside the spot the anisotropy of the residual radiation will be determined mainly by χ. As is seen from (27), the sign of χ depends on the signs of u_1 and u_3 at the time t_1 of the escape of the radiation from the matter. If the velocity of the matter at that moment is sufficiently large, then the contribution of χ can become predominant. Formulas (25) to (27) are obtained for the limiting stage of the expansion (Milne model), when the matter is no longer important. Analogous, but more complex formulas are obtained when matter is taken into account, however, the qualitative picture of the anisotropy of the background does not change. The experimental possibilities are discussed in [5]; the difficulties noted there are made worse by the additional parameter $u_1 \neq 0$, which must also be determined by observation.

We are grateful to Academician Ya. B. Zel'dovich and the authors of [5], in particular to A. G. Doroshkevich, for proposing this problem, for valuable advice, and comments.

[1] S. G. Maitra, J. Math. Phys. 7, 1025 (1966).
[2] E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk 80, 391 (1963) [Sov. Phys. Uspekhi 6, 495 (1964)].
[3] St. Hawking, preprint, Cambridge, 1968.
[4] O. Heckmann and E. Schucking, in Gravitation, L. Witten, ed., N. Y., 1962, p. 438.
[5] L. P. Grishchuk, A. G. Doroshkevich, and I. D. Novikov, Zh. Eksp. Teor. Fiz. 55, 2281 (1968) [Sov. Phys. JETP 28, 1214 (1969)].

Translated by R. Lipperheide
198