THE ONE-DIMENSIONAL PROBLEM OF PROPAGATION OF WAVES IN A MEDIUM WITH RANDOM INHOMOGENEITIES

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Submitted May 24, 1968; resubmitted February 20, 1969

Zh. Eksp. Teor. Fiz. 56, 1856–1871 (June, 1969)

The one-dimensional problem of propagation of an harmonic wave through a random-inhomogeneous layer is considered under the assumption that the inhomogeneities are independent and the reflectance phases of each of the inhomogeneities are distributed uniformly. Expressions for the mean field and mean square field (the latter in the absence of losses) are derived and analyzed. It is shown that the mean field in the layer varies according to an exponential law. The linear distribution along the thickness of the layer obtained previously is derived for the mean square field in the case of weak scattering; with increase of scattering properties of the inhomogeneities the distribution changes, remaining antisymmetric with respect to the center of the layer and in the limiting case approaching a step-like shape. Thus at that half of the layer which is directed toward the incident wave the intensity is twice that of the incident wave and in the other half is zero. It is shown that the obtained result can be extended to the general case of a medium which weakly scatters over distances of the order of the wave length and of the order of the correlation range of the inhomogeneities.

INTRODUCTION

THE problem of the propagation of a wave in a randomly-inhomogeneous medium entails no difficulty if it is possible to neglect secondary scattering, but if such scattering must be taken into account the situation becomes much more complicated. The known attempts for such an account are limited, as a rule, to obtaining approximate solutions after various limitations are imposed on the possibility of accumulation of the multiple-scattering effect. In view of this, particular interest attaches to a derivation of an exact solution even in the simplest one-dimensional case.

We solve below the one-dimensional problem of multiple scattering of a wave in a randomly-inhomogeneous layer (for example, scattering by a randomly-inhomogeneous section of a long line) with averaging over an ensemble; we shall find the average field and, assuming no loss in the medium, also the average intensity (the calculation of the latter constitutes the main difficulty). Time averaging likewise reduces to averaging over an ensemble in the case of inhomogeneities that move (or vary in general) randomly and sufficiently slowly. By sufficiently slowly is meant here that the change of the medium must be small during the time of propagation of waves having a maximum scattering multiplicity that is still significant.

Let us explain this using as an example discrete moving inhomogeneities. Upon reflection from a given inhomogeneity, the wave acquires a phase factor that depends on the position of the inhomogeneity; it is over these phase factors that the averaging takes place. In averaging over the ensemble, it is significant that a stationary problem is considered for each realization, i.e., a wave reflected a second time, say, from our inhomogeneity acquires the same phase factor as in the first reflection. It is seen therefore that the more significant the scattering multiplicity, the slower must the inhomogeneities move in order for this scheme to be valid. Scatterings of arbitrarily large multiplicity will be correctly taken into account only in the case of infinitely slow motion of the inhomogeneities. Another extreme case is one in which the inhomogeneity has time to traverse a distance on the order of a wavelength during the time interval between the first and second reflection; the phase factors for the two reflections will then be different and statistically independent. If this holds for all the reflections, then the multiply scattered waves will be additive in energy. This case corresponds to the solution obtained by Ambartsumyan[1] for the one-dimensional problem in which a wave passes through a turbulent layer. It will be shown below that the field intensity, when averaged over the ensemble, differs greatly from that given by Ambartsumyan, provided the scattering is not very weak.

We shall carry out the calculation for the following model, to which, as will be shown at the end of the article, the case of a one-dimensional randomly-inhomogeneous medium that scatters weakly over one wavelength also reduces. Let the inhomogeneous layer represent a chain of discrete inhomogeneities, each of which can assume independently and with equal probability all positions on a section equal to half the wavelength (or its multiples). We assume that the sections do not overlap; we shall calculate the mean values of the quantities of interest to us in the intervals between them (and also in the semi-infinite intervals before the first inhomogeneity and after the last one). It is easy to see that in this case the phase of the coefficient of reflection from each individual inhomogeneity has again a uniform distribution. The intensity reflection coefficient of such a system of inhomogeneities was determined by Gertsenshteĭn and Vasil'ev[2] for an infinite number of infinitely weak inhomogeneities; we shall employ a different method, which enables us not only to obtain the result of [2], but also to find and analyze the field inside the layer.

We denote by k the wave number in the homogeneous layer. Assume that there are n inhomogeneities; we shall characterize them by the transmission and reflection coefficients. We consider two problems: one when a wave exp (ikx) is incident on our inhomogeneities, and

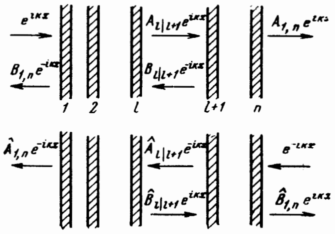

FIG. 1.

another when a wave $\exp(-ikx)$ is incident, and we introduce the notation defined in Fig. 1. For the case when only the i-th and k-th inhomogeneities are present, we shall use the symbols $A_{i,k}$ and $B_{i,k}$. If we consider only one (m-th) inhomogeneity, then we shall use A_m and B_m in place of $A_{m,m}$ and $B_{m,m}$. These two quantities depend on the properties and positions of the m-th inhomogeneity. Thus, if this is an inhomogeneity concentrated at the point x_m, and the square of the refractive index for this inhomogeneity is $1 + \mu\delta(x - x_m)$, then

$$A_m = \frac{1}{1 - i\varepsilon}, \quad B_m = \frac{i\varepsilon}{1 - i\varepsilon} e^{2ikx_m}, \tag{1}$$

where $\epsilon = k\mu/2$. We note that A_m does not depend on the position of the inhomogeneity, while the dependence of B_m on x_m is given by the factor $\exp(2ikx_m)$; these two properties are possessed by A_m and B_m for inhomogeneities of any type, if x_m is taken to mean some definite point of the inhomogeneity. Analogously, $\hat{A}_m$ does not depend on x_m, and the dependence of $\hat{B}_m$ on x_m is given by the factor $\exp(-2ikx_m)$.

From the principle of field superposition and from the uniqueness of the definition of the field in any layer with respect to the waves incident on it from the left and from the right, we get

$$A_{l|l+1} = A_{1,l} + B_{l|l+1}\hat{B}_{1,l},$$
$$B_{l|l+1} = A_{l|l+1}B_{l+1,n},$$
$$A_{1,n} = A_{l|l+1}A_{l+1,n},$$
$$B_{1,n} = B_{1,l} + \hat{A}_{1,l}B_{l|l+1}.$$

This yields

$$A_{l|l+1} = A_{1,l}(1 - \hat{B}_{1,l}B_{l+1,n})^{-1}. \tag{2}$$

We note that $|\hat{B}_1, {}_lB_{l+1,n}| \leq 1$, for according to the energy conservation law we have

$$|A_{i,k}|^2 + |B_{i,k}|^2 \leqslant 1, \quad |\hat{A}_{i,k}|^2 + |\hat{B}_{i,k}|^2 \leqslant 1 \tag{3}$$

(The equality in these relations occurs when there are no losses.) We obtain further

$$A_{1,n} = A_{1,l}A_{l+1,n}(1 - \hat{B}_{1,l}B_{l+1,n})^{-1}, \tag{4}$$

$$B_{l|l+1} = A_{1,l}B_{l+1,n}(1 - \hat{B}_{1,l}B_{l+1,n})^{-1}, \tag{5}$$

$$B_{1,n} = B_{1,l} + A_{1,l}\hat{A}_{1,l}B_{l+1,n}(1 - \hat{B}_{1,l}B_{l+1,n})^{-1}. \tag{6}$$

Similar relations can be written also for the equations $\hat{A}$ and $\hat{B}$.

We exclude from consideration the cases when one of the inhomogeneities reflects the wave completely, i.e., we shall assume that all the A_m and $\hat{A}_m$ differ from zero. Then, using formulas of the type (4), we can easily show that all the $A_{i,k}$ and $\hat{A}$ differ from zero. When (3) is taken into account, this indicates that the quantity $(1 - \hat{B}_1, {}_lB_{l+1,n})^{-1}$ is expanded in a series of powers of $\hat{B}_1, {}_lB_{l+1,n}$; this series converges uniformly over the positions of the inhomogeneities. Substitution of such an expansion in (2), (4)-(6) corresponds to breaking up the field into a sum of waves that are multiply reflected from the segments $(1, l)$ and $(l + 1, n)$. Successive application of formulas of this type makes it possible to represent the field in the form of a series in powers of A_i, B_i, $\hat{A}_i$ and B_i, i.e., in the form of waves that are multiply reflected and transmitted through the various inhomogeneities.

AVERAGE FIELD

From the form of the dependence of B_m on x_m it follows that the average of $(B_m)^N$ over x_m vanishes for integer N. It is easy to prove the following more general relation:

$$\langle (B_{i,k})^N \rangle_{x_i, x_{i+1}, \ldots, x_k} = 0 \tag{7}$$

(the angle brackets denote averaging, the indices are the parameters over which the averaging is carried out; when averaging over the positions of all the inhomogeneities on which the expression under consideration depends, we shall henceforth omit the indices). This relation is proved by induction with respect to the number of inhomogeneities, by taking powers of expressions of the type (6) and averaging; in these expressions, the quantities $(1 - \hat{B}_1, {}_lB_{l+1,n})^{-1}$ are expanded beforehand in powers of $\hat{B}_1, {}_lB_{l+1,n}$.

Using (7), we now calculate the average field. Expanding the right side of (5) and averaging, we get $\langle B_{l|l+1} \rangle_{x_{l+1}, \ldots, x_n} = 0$, i.e., $\langle B_{l|l+1} \rangle = 0$; from (2) we obtain analogously $\langle A_{l|l+1} \rangle = \langle A_{1,l} \rangle$. Finally, putting (4) $l = n + 1$, expanding and averaging, we obtain $\langle A_{1,n} \rangle_x = A_{1,n-1}A_n$, hence $\langle A_{1,n} \rangle = A_1A_2 \ldots A_n$. We thus have

$$\langle B_{1,n} \rangle = 0, \quad \langle B_{l|l+1} \rangle = 0,$$
$$\langle A_{1,n} \rangle = A_1A_2 \ldots A_n, \quad \langle A_{l|l+1} \rangle = A_1A_2 \ldots A_l. \tag{8}$$

The meaning of this result is that no reflection from the inhomogeneities occurs for the average field: the average wave "notes" only those inhomogeneities, through which it has already passed, and does not "feel" the inhomogeneities situated in front of it.

It is interesting to trace the application of the perturbation method to this same problem; we do this for the particular case of identical concentrated inhomogeneities. Thus, let p satisfy the equation

$$\Delta p + k^2(1 + \mu(x))p = 0,$$

where

$$\mu(x) = \sum_{m=1}^{n} \mu\delta(x - x_m),$$

and let a wave $\exp ikx$ be incident on an inhomogeneous layer. We have

$$p = e^{ikx} + \frac{ik}{2}\int e^{ik|x-\xi|}\mu(\xi)p(\xi)\,d\xi.$$

Introducing $u = p\exp(-ikx)$ and denoting $u(x_m)$ by u_m, we obtain

$$u = 1 + i\varepsilon\sum_{m=1}^{n} u_m \cdot (x, x_m);$$

$$\varepsilon = \frac{k\mu}{2}, \quad (x, x_m) = \begin{cases} 1 \text{ for } x > x_m \\ e^{2ik(x_m - x)} \text{ for } x < x_m \end{cases}. \tag{9}$$

It is known that an equation of this kind can be solved by iteration under the condition $|\epsilon| < 1/n$. Assuming that this condition is satisfied, we obtain the following iteration series:

$$u = 1 + i\varepsilon \sum_{m=1}^{n} (x, x_m) - \varepsilon^2 \sum_{m_1=1}^{n} \sum_{m_2=1}^{n} (x, x_{m_1})(x_{m_1}, x_{m_2}) + \ldots \qquad (10)$$

The series terms containing ϵ raised to the power r are taken to mean r-fold scattered waves.[1] For a clear understanding of these terms, it is convenient to examine the scheme of Fig. 2 (which considers triply scattered waves). The broken lines a and b represent two different waves (the observation point is assumed to be behind the layer). The number of waves is equal to the number of broken lines that can be drawn. It is clear, that for broken lines with descending or horizontal links, all the factors (x_i, x_k) and (x, x_k) are equal to unity, while for broken lines having "rises" each such "rise" produces a double phase advance on the path between the corresponding points. We shall show that when averaging over all the positions of the inhomogeneities, any wave with "rises" (i.e., experiencing at least one backward reflection) drops out. In fact, the exponent of the phase factor for such a wave will contain terms of the type $2ikx_l$, corresponding to final rise points. If there are several rises, then some of these terms may cancel each other, but it can be readily seen that some of the x_l must remain. When averaging over these x_l the wave vanishes.

The same considerations apply also when the observation point is inside the layer (Fig. 3). In this case the waves dropping out are those of type b, which do not drop beyond the observation point but have rises, as well as all the waves that rise beyond the observation point (for example, c or d), i.e., the average field will be the same as if the inhomogeneities from the $l+1$-st to the n-th were to be absent. Let us consider again the point of observation behind the layer. It is easy to show that the number of broken lines without rises is equal to C^r_{n+r-1} from which we get, averaging (10),

$$\langle u \rangle = 1 + i\varepsilon C_n^1 + (i\varepsilon)^2 C_{n+1}^2 + \ldots = (1 - i\varepsilon)^{-n}$$

(The condition that ϵ be small becomes immaterial in this case, for the result obtained for small ϵ can be subsequently continued analytically) which coincides with the expression that follows from the general formula (8) and from (1). It is of interest to note that for the "worst" case, when the inhomogeneities are located at distances that are multiples of the wavelength, the series (10) has a convergence radius $|\epsilon| = 1/n$, i.e.,

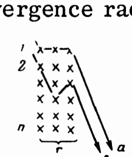

FIG. 2.

[1]We note that such a representation is hardly arbitrary. It is meaningful physically to represent the field in the form of a series of multiply scattered waves, as indicated at the end of the introduction, while expression (10) is obtained from this series by expanding in powers of ϵ the reflection and transmission coefficients (1). From this point of view, a perturbation of r-th order is made up of terms pertaining to all the waves that experience not more than r reflections.

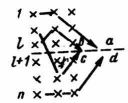

FIG. 3

the larger the number of inhomogeneities, the weaker they should be in order to be able to use the series. On the other hand, the averaged series has a convergence radius $|\epsilon| = 1$ independent of n.

For a large number of weak inhomogeneities, it is convenient to go over to the limiting case of a "continuous" medium,[2] assuming $\epsilon \to 0$ and $n \to \infty$, so that $N = \epsilon n$ is fixed. We then obtain $\langle u \rangle = \exp iN$. We note further that averaging over the positions of the inhomogeneities leads to (8) also when the quantities A_m are in turn independent random quantities with specified distribution; to obtain the final answer, it is then necessary to average over A_m. For example, for concentrated inhomogeneities, when averaging over the positions yields for the average field

$$\frac{1}{1 - i\varepsilon_1} \frac{1}{1 - i\varepsilon_2} \cdots \frac{1}{1 - i\varepsilon_n},$$

we assume that each ϵ_m assumes independently and with equal probability the values $\pm\epsilon$ (i.e., the square of the refractive index is on the average equal to unity). Averaging over all the ϵ_m, we obtain $\langle u \rangle = (1 + \epsilon^2)^{-n}$. Here, too, we can go over to a continuous medium, but a nontrivial result is obtained only if in the limiting transition the fixed quantity is now $N = n\epsilon^2$; in this case $\langle u \rangle = \exp(-N)$.

MEAN SQUARE OF TRANSMITTED WAVE

In calculating the mean square, we shall consider only the case when there are no losses, and then (3) contains only equality signs. Using this, we can assume

$$A_{1,n} = e^{i\varphi_{1,n}} \Big/ \operatorname{ch} \frac{\beta_{1,n}}{2}, \quad B_{1,n} = e^{i\psi_{1,n}} \operatorname{th} \frac{\beta_{1,n}}{2},$$

where β, φ, ψ are real, $\beta \geq 0$. Recognizing that, in accordance with the reciprocity principle, $\hat{A}_{1,n} = A_{1,n}$, we get $\hat{B}_{1,n} = \exp(i\hat{\psi}_{1,n}) \tanh(\beta_{1,n}/2)$. The quantity $\cosh \beta_{1,n}$ is equal to the ratio of the energy density ahead of the layer to the energy density behind the layer. For individual inhomogeneities it is convenient to introduce also the quantity s_m in accordance with the formula $s_m = (\cosh \beta_m - 1)/2$; s_m does not depend on the position of the inhomogeneity and is the ratio of the reflected energy to the transmitted energy (for a concentrated inhomogeneity $s = \epsilon^2$). We shall call this quantity the backward-scattering coefficient of the inhomogeneity. We note that φ_m is likewise independent of the position of the inhomogeneity, and ψ_m depends on x_m in terms of $2kx_m$; analogously, $\hat{\psi}_m$ depends on x_m in terms of $-2kx_m$.

The application of the method of [1] to the case of identical inhomogeneities characterized by coefficient s

[2]In the sense that appreciable scattering is produced only by sections containing many inhomogeneities; a continuous medium in the conventional sense is considered at the end of this paper.

yields for the square of the field passing through the layer the expression

$$1/(1+ns). \tag{11}$$

We shall show that $\langle |A_{1,n}|^2 \rangle$ is smaller than this expression only if $n > 2$. From (4) we have

$$|A_{1,n}|^2 = \frac{|A_{1,n-1}|^2 |A_n|^2}{1+|B_n|^2|\hat{B}_{1,n-1}|^2 - 2|B_n||\hat{B}_{1,n-1}|\cos(\hat{\psi}_{1,n-1}+\psi_n)}.$$

Averaging over x_n (which enters only in ψ_n), and simplifying, we obtain

$$\langle |A_{1,n}|^2 \rangle_{x_n} = \frac{|A_{1,n-1}|^2 |A_n|^2}{1-|B_n|^2|\hat{B}_{1,n-1}|^2} = \frac{|A_{1,n-1}|^2}{1+s|A_{1,n-1}|^2}.$$

When $n = 2$, this expression, with allowance for the fact that $|A_1|^2 = 1/(1+s)$ leads to (11); when $n > 2$, it is necessary to average over the remaining inhomogeneities. Transforming, we write the obtained expression in the following form:

$$\langle |A_{1,n}|^2 \rangle_{x_n} = \frac{\langle |A_{1,n-1}|^2 \rangle}{1+s\langle |A_{1,n-1}|^2 \rangle} + \frac{1}{1+s\langle |A_{1,n-1}|^2 \rangle} \times \left\{ \frac{|A_{1,n-1}|^2 - \langle |A_{1,n-1}|^2 \rangle}{1+s|A_{1,n-1}|^2} \right\}.$$

We shall show that the curly brackets yield a negative quantity upon averaging. Indeed, without the denominator, the mean value would be equal to 0 as a result of the cancellation of the positive and negative values of the numerator. The presence of the denominator decreases the absolute magnitude of both the positive and negative values of the numerator, and it can be readily seen that the positive values decrease more strongly than the negative ones, i.e., the average of the quantity in the curly bracket is smaller than zero. Thus,

$$\langle |A_{1,n}|^2 \rangle < \frac{\langle |A_{1,n-1}|^2 \rangle}{1+s\langle |A_{1,n-1}|^2 \rangle}.$$

Using the inequality for the inverse quantities and making it stronger, we obtain

$$\frac{1}{\langle |A_{1,n}|^2 \rangle} > 1+ns, \quad \text{i.e.} \ \langle |A_{1,n}|^2 \rangle < \frac{1}{1+ns}.$$

In the case of concentrated inhomogeneities it is easy to obtain simple expressions for $\langle |A_{1,n}|^2 \rangle$ for small n by the following method. Putting in (9 $x = x_1$ and x^2, etc., we obtain a system of equations for u_m, and by solving the system we get u_m and then also $u(x)$. Everything reduces then to the calculation of an n-fold integral of $|u|^2$. Such a calculation leads to (11) for $n = 1$ and 2. We further obtain

$$\langle |A_{1,3}|^2 \rangle = \frac{1}{(1+s)\sqrt{1+4s}}, \quad \langle |A_{1,4}|^2 \rangle = \frac{1}{(1+2s)^2}\frac{2}{\pi}K\left(1-\frac{1}{(1+2s)^2}\right),$$

where $K(k)$ is a complete elliptic integral of the first kind. This method cannot be used to find a simple expression for $n = 5$.

Let us consider also the expression for the mean-square of the transmitted field in the form of a perturbation series. This expression can be obtained by multiplying (10) by the complex conjugate and averaging. It is easy to show that the result is a series in even powers of ϵ, i.e., in integer powers of s. We note that, unlike the calculation of the average field, the reflected-wave phase factors, which depend on the positions of the inhomogeneities, may cancel out in the product uu^*, and corresponding terms will no longer drop out in the averaging. For such a cancellation it is not necessary at all that the factors correspond to one and the same wave: for example, for the two waves shown schematically in Fig. 4, the phase factors that depend on the position of the inhomogeneities are the same. The larger the multiplicity of reflection, the greater the assortment of different waves with equal phase factors. In view of this, it is possible to obtain a common expression only for the first terms of the perturbation-theory series:

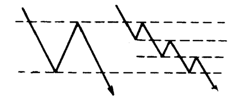

FIG. 4.

$$\langle |A_{1,n}|^2 \rangle = 1 - ns + n^2s^2 - ({}^4\!/_3 n^3 - n^2 + {}^2\!/_3 n)s^3 + ({}^7\!/_3 n^4 - 5n^3 + {}^{17}\!/_3 n^2 - 2n)s^4 + \ldots. \tag{12}$$

Comparing this with (11), we see that the method of [1] yields the same result, accurate to second-order terms in the small-perturbation method. Thus, at small values of ns, it is permissible to use (11). We proceed now to obtain an exact result for arbitrary ns; we shall now already consider the general case of arbitrary discrete inhomogeneities.

Expressing the quantities contained in (4) in terms of β, φ, and ψ, equating the reciprocals of the squares of the moduli of both parts of the resultant equation, and simplifying, we obtain[3)]

$$\operatorname{ch}\beta_{1,n} = \operatorname{ch}\beta_{1,l}\operatorname{ch}\beta_{l+1,n} - \operatorname{sh}\beta_{1,l}\operatorname{sh}\beta_{l+1,n}\cos(\hat{\psi}_{1,l}+\psi_{l+1,n}). \tag{13}$$

We note now, that for any real β the following relation holds true[4)]

$$\frac{2}{1+\operatorname{ch}\beta} = \pi\int_{-\infty}^{+\infty} t\frac{\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}P_\nu(\operatorname{ch}\beta)\,dt, \tag{14}$$

where $\nu = -{}^1\!/_2 + it$. Here $P_\nu(\cosh\beta)$ is a spherical function:

$$P_\nu(\operatorname{ch}\beta) = F(-\nu, 1+\nu; 1; -s), \tag{15}$$

where $s = (\cosh\beta - 1)/2$. Recognizing that $|A_{1,n}|^2 = 2/(1+\cosh\beta_{1,n})$ we represent $|A_{1,n}|^2$ in the form of an integral (14) and average the obtained equation over the positions of the first inhomogeneity. To calculate $\langle P_\nu(\cosh\beta_{1,n})\rangle_{x_1}$ we first obtain, using (13) and the addition theorem

$$+2\sum_{m=1}^{\infty}(-1)^m\frac{\Gamma(\nu+1-m)}{\Gamma(\nu+1+m)}P_\nu^m(\operatorname{ch}\beta_1)P_\nu^m(\operatorname{ch}\beta_{2,n})\cos m(\hat{\psi}_1+\psi_{2,n}). \tag{16}$$

This expression depends on x_1 only via $\hat{\psi}_1$, which contains the term $-2kx_1$. Averaging, we obtain

$$\langle P_\nu(\operatorname{ch}\beta_{1,n})\rangle_{x_1} = P_\nu(\operatorname{ch}\beta_1)P_\nu(\operatorname{ch}\beta_{2,n}).$$

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3)This formula is the relation between the sides and the angles of an acute-angle triangle on a Lobachevskiĭ plane; the existence of a correspondence between the combination of inhomogeneities in a long line and the motions in a Lobachevskiĭ plane was pointed out in [2].

4)The validity of (14) can be verified by substituting in place of P_ν the integral representation

$$\frac{\sqrt{2}}{\pi}\operatorname{cth}\pi t\int_\beta^\infty \frac{\sin tx}{\sqrt{\operatorname{ch}x - \operatorname{ch}\beta}}\,dx$$

and by changing the order of integration.

Averaging subsequently over x^2, etc., we find

$$\langle|A_{1,n}|^2\rangle = \pi\int_{-\infty}^{+\infty} t\frac{\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\prod_{m=1}^{n} P_\nu(\operatorname{ch}\beta_m)\,dt. \tag{17}$$

In particular, for identical inhomogeneities, we have

$$\langle|A_{1,n}|^2\rangle = \pi\int_{-\infty}^{+\infty} t\frac{\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\{P_\nu(\operatorname{ch}\beta)\}^n\,dt, \tag{18}$$

where $\cosh\beta = 2s + 1$ and s is the backward-scattering coefficient of these inhomogeneities. The integrand in (18) has a standard form, making it possible to obtain the asymptotic value of the integral at large values of n. Recognizing that the largest of the maxima of P_ν along the integration path occurs at $t = 0$, we get

$$\langle|A_{1,n}|^2\rangle \sim \pi^{5/2}\sqrt{\frac{2}{a^3}}\frac{b^{3/2+n}}{n^{3/2}};$$

$$a = -\frac{\partial^2 P_\nu}{\partial t^2}\Big|_{t=0} = \frac{\sqrt{2}}{\pi}\int_0^\beta \frac{x^2\,dx}{\sqrt{\operatorname{ch}\beta - \operatorname{ch}x}}$$

$$b = P_{-1/2}(\operatorname{ch}\beta) = \frac{2}{\pi}\frac{K(\operatorname{th}(\beta/2))}{\operatorname{ch}(\beta/2)}$$

In the case of interest to applications, namely that of weak inhomogeneities, we have $a \approx \beta^2/2 \approx 2s$, $b \approx 1 - s/4$, i.e., in this case we get

$$\langle|A_{1,n}|^2\rangle \sim \frac{\pi^{5/2}}{2}\frac{e^{-ns/4}}{(ns)^{3/2}}. \tag{19}$$

In the case of a large number of weak inhomogeneities, we can go over to a "continuous" medium, putting $s \to 0$ and $n \to \infty$, such that $N = sn$ is fixed. Then (18) becomes

$$\langle|A|^2\rangle = \pi\int_{-\infty}^{+\infty} t\frac{\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\exp\left\{-\left(t^2+\frac{1}{4}\right)N\right\}dt. \tag{20}$$

At large values of N we can find an asymptotic expression for (20); we then obtain (19), in which ns should be replaced by N.

Noting that if the inhomogeneities have, besides random positions, also random scattering coefficients, then β_m in (17) must be regarded as random quantities, over which an additional averaging is necessary.

We have seen, using concentrated inhomogeneities as an example, that in the case when ϵ_m assume equally probable values $\pm\epsilon$, the average field greatly differs from the average field for the case when all the ϵ_m are equal to ϵ; it is interesting to note that the mean square of the field is the same in both cases.

We note further that expression (18) makes it possible to find the convergence radius of the perturbation-theory series (12). To this end it suffices to find in the complex s plane the singularity of the analytic function (18) closest to the origin. Recognizing that when $t \to +\infty$ and $0 \le \operatorname{Im}\beta \le \pi$ we have

$$P_\nu \sim \sqrt{\frac{2}{\pi t\operatorname{sh}\beta}}\cos\left(\beta t - \frac{\pi}{4}\right) \quad \text{if} \quad \operatorname{Re}\beta \geqslant 0$$

and

$$P_\nu \sim \sqrt{\frac{2}{\pi t\operatorname{sh}\beta}}\,i\cos\left(\beta t + \frac{\pi}{4}\right) \quad \text{if} \quad \operatorname{Re}\beta \leqslant 0$$

(it is understood that the chosen branch of the root is arithmetic at real positive β), then it is easy to show that such a singular point is $s = \frac{1}{2}(\cos(\pi/n) - 1)$.

Thus, the convergence radius of the perturbation-theory series for the mean squared field is given by the relation $s = \sin^2(\pi/2n)$ or $|\epsilon| = \sin(\pi/2n)$, i.e., it differs little from the minimum convergence radius of the initial iteration series (on the other hand, as we have seen, for the average field the averaging of the series greatly increases the convergence radius at large n). From (18) and (15) it is easy to see that the coefficient of s^m in the perturbation-theory series is a polynomial in n, whose highest order term is

$$\frac{n^m}{4^m m!}\sum_{k=0}^{m} C_m^k(2k+1)|E_{2k}|,$$

where E_k are Euler numbers. We see therefore that so long as $m \ll n$, the terms of the series decrease like powers of ns; however, if the condition $n^2s > \pi^2/4$ is satisfied regardless of the smallness of ns, then the series diverges and its terms will ultimately start to increase. A similar singularity in the behavior of the perturbation-theory series was observed by Kay and Silverman[3] for a somewhat different model of an inhomogeneous medium. We note that at small values of ns the first terms of the series can be used for an approximate calculation also in the case of a diverging series.

Let us consider this question in greater detail, using as an example a solid medium. In this case, it is necessary to retain in the coefficients of the series (12) only the terms of higher order in n, replacing ns by N; the entire series can be obtained from (20) by expanding the exponential and by integrating term by term (it can be readily seen that when $N \neq 0$ this series diverges). It turns out that if we use for the calculations a finite segment of the series, then the absolute value of the correction does not exceed the first discarded term, and is of the same sign as this term, i.e., the perturbation-theory series is asymptotic as $N \to 0$. For concrete calculations it is more convenient to use a somewhat different expression:

$$\langle|A|^2\rangle \sim e^{-N/4}\sum_{m=0}^{\infty}\frac{2m+1}{m!}E_{2m}\left(\frac{N}{4}\right)^m;$$

In this case, too, the absolute value of the correction does not exceed the first discarded term and is of the same sign. At finite values of n, it is impossible to obtain an expression for the general perturbation-theory series, but it is possible to present the following very convenient symbolic relation:

$$\langle|A_{1,n}|^2\rangle = (4B+3)\{F(2B+2, -2B-1; 1; -s)\}^n,$$

where B is the Bernoulli symbol.

The method used above to calculate $\langle|A_{1,n}|^2\rangle$, which consists in first expressing the averaged quantity linearly in terms of $P_\nu(\cosh\beta_{1,n})$, can be applied also to other functions of $\cosh\beta_{1,n}$, for an expression in terms of P_ν can be obtained in the general case on the basis of the Moller–Fock transformation.[4] We shall use this to calculate the distribution function of the field amplitude $|A|$ passing through the layer. On the basis of the indicated transformation, we obtain for the step function $h(x)$

$$h(\operatorname{ch}\beta - \operatorname{ch}\beta_{1,n}) = \frac{\operatorname{sh}\beta}{2}\int_{-\infty}^{+\infty} t\operatorname{th}(\pi t)P_{\nu}^{-1}(\operatorname{ch}\beta)P_\nu(\operatorname{ch}\beta_{1,n})\,dt.$$

Averaging this relation, we obtain for the integral distribution function σ in the case of n identical inhomogeneities with backward-scattering coefficient s

$$\sigma = \frac{\operatorname{sh}\beta}{2}\int_{-\infty}^{+\infty} t\,\operatorname{th}(\pi t)P_\nu^{-1}(\operatorname{ch}\beta)\{P_\nu(1+2s)\}^n dt.$$

Here $\cosh\beta = 2|A|^{-2} - 1$.

For the differential distribution function we obtain

$$\frac{d\sigma}{d\operatorname{ch}\beta} = \frac{1}{2}\int_{-\infty}^{+\infty} t\,\operatorname{th}(\pi t)P_\nu(\operatorname{ch}\beta)\{P_\nu(1+2s)\}^n\,dt$$

$$\frac{i}{\pi}\int_{-\infty}^{+\infty} Q_\nu(\operatorname{ch}\beta)\{P_\nu(1+2s)\}^n t dt.$$

This yields for a continuous medium

$$\frac{d\sigma}{d\operatorname{ch}\beta} = \frac{i}{\pi}\int_{-\infty}^{+\infty} Q_\nu(\operatorname{ch}\beta)\exp\left\{-\left(t^2+\frac{1}{4}\right)N\right\}t\,dt.$$

The last expression coincides (apart from an inessential difference in the integration contour), with that obtained in [2].

MEAN SQUARED FIELD INSIDE THE LAYER

We have seen that the amplitude of the average field is constant between inhomogeneities; we shall now show that this holds for the mean-squared field. We first note that, in analogy with the proof of (7), we can prove the validity of a more general relation: $\langle (B_{i,k})^L (B_{i,k}^*)^M\rangle = 0$ for integer $L \neq M$. Expanding now (2) and the conjugate of (5) in series, multiplying these series, and recognizing that

$$|A_{1,l}|^2 = |\grave{A}_{1,l}|^2 = 1 - \hat{B}_{1,l}\hat{B}_{1,l}^*,$$

we can easily show, using the indicated relation, that $\langle A_{l|l+1} B_{l|l+1}^*\rangle = 0$.[5)] Thus, to calculate the mean-squared field it suffices to determine the mean value of $C_{l|l+1}$, which equals $|A_{l|l+1}|^2 + |B_{l|l+1}|^2$. We note that in the case of identical inhomogeneities, the Ambartsumyan method[1] yields for the mean-squared field the following expression:

$$\frac{1+2(n-l)s}{1+ns} \tag{21}$$

The calculation of $\langle C_{l|l+1}\rangle$, which for small values of n can be carried out by directly evaluating the n-fold integral of $|u|^2$, yields the same expression when n = 2, but when n = 3 a different distribution, shown in Fig. 5, is already obtained.

It turns out that the resultant distribution represents the main properties over the distribution in the general case, too. We note that it is antisymmetrical with respect to the "center" of the inhomogeneous layer. We note further that at small values of s the result coincides with (21), accurate to s^2; on the other hand, when

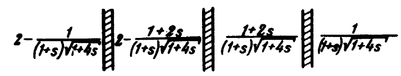

FIG. 5.

s approaches infinity the mean-squared field tends to zero on the right side of the center, and tends to double the intensity of the incident wave on the left side. The fact that the average intensity between the first and second inhomogeneities does not tend to zero following such an "enhancement" of the inhomogeneities is at first glance strange (the same pertains to the case n = 2, when the mean-squared field between the inhomogeneities is equal to unity regardless of their "strength"). The explanation is that for arbitrarily "strong" inhomogeneities there exist arrangements such that the field between the inhomogeneities is much stronger than the field of the incident wave (thus, for identical inhomogeneities, and "transparent" arrangements, in which the wave passes through the layer without being reflected—it is easy to show that such arrangements always exist). We shall make it clear, however, that the mean square of the field is already always smaller than double the intensity of the incident wave.

Proceeding to calculate the mean-squared field, we obtain in the general case from (2), (4), (5), and (14)

$$C_{l|l+1} = \left|\frac{A_{1,n}}{A_{l+1,n}}\right|^2\{1+|B_{l+1,n}|^2\}$$

$$\pi\int_{-\infty}^{+\infty}\frac{t\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}P_\nu(\operatorname{ch}\beta_{1,n})\operatorname{ch}\beta_{l+1,n}\,dt.$$

Averaging over the positions of the first l inhomogeneities is carried out in the same manner as for $\langle|A_{1,n}|^2\rangle$:

$$\langle C_{l|l+1}\rangle_{x_1,\ldots,x_l} = \pi\int_{-\infty}^{+\infty}\frac{t\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\prod_{m=1}^{l}P_\nu(\operatorname{ch}\beta_m)P_\nu(\operatorname{ch}\beta_{l+1,n})\operatorname{ch}\beta_{l+1,n}\,dt. \tag{22}$$

Taking relations similar to (13) and (16) into account, it is easy to obtain

$$\langle\operatorname{ch}\beta_{l+1,n}P_\nu(\operatorname{ch}\beta_{l+1,n})\rangle_{x_{l+1}} = z\bar{z}P_\nu(z)P_\nu(\bar{z}) + \frac{\zeta\bar{\zeta}}{\nu(\nu+1)}P_\nu^1(z)P_\nu^1(\bar{z})$$

$$\frac{\nu+1}{2\nu+1}P_{\nu+1}(z)P_{\nu+1}(\bar{z}) + \frac{\nu}{2\nu+1}P_{\nu-1}(z)P_{\nu-1}(\bar{z});$$

$$z = \operatorname{ch}\beta_{l+1},\quad \zeta = \operatorname{sh}\beta_{l+1},\quad \bar{z} = \operatorname{ch}\beta_{l+2,n},\quad \bar{\zeta} = \operatorname{sh}\beta_{l+2,n}.$$

We see therefore that after averaging (22) over x_{l+1}, further averagings can already be carried out in the same manner as for $\langle|A_{1,n}|^2\rangle$. Calculations yield

$$\langle C_{l|l+1}\rangle = \pi\int_{-\infty}^{+\infty}\frac{t\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\prod_{m=1}^{l}P_\nu(\operatorname{ch}\beta_m)\left\{\frac{\nu+1}{2\nu+1}\prod_{k=l+1}^{n}P_{\nu+1}(\operatorname{ch}\beta_k)\right.$$

$$\left.\frac{\nu}{2\nu+1}\prod_{k=l+1}^{n}P_{\nu-1}(\operatorname{ch}\beta_k)\right\}dt.$$

Expanding the products in the curly brackets, we break this integral up into two and make the substitution $t \to -t$; again combining the integrals we obtain ultimately

$$\langle C_{l|l+1}\rangle = \pi\int_{-\infty}^{+\infty}\frac{t\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\frac{2\nu}{2\nu+1}\prod_{m=1}^{l}P_\nu(\operatorname{ch}\beta_m) \tag{23}$$

$$\times\prod_{k=l+1}^{n}P_{-\nu}(\operatorname{ch}\beta_k)\,dt.$$

5)For concentrated inhomogeneities this can be readily obtained directly, by considering elementary waves of the type shown in Fig. 3. The waves making up $A_{l|l+1}$ have an even number of quantites x_k in the exponent of the phase factor; on the other hand, the waves making up $B_{l|l+1}$ have an odd number. When the waves of the former type are multiplied by the conjugate values of the waves of the latter type, an odd number of x_k remains in the resultant phase factor; averaging over the remaining quantities then yields zero.

Let us consider the case of identical inhomogeneities: $\cosh\beta_i = \cosh\beta = 1 + 2s$. Let $m = n - l$; denoting $\langle C_{l|l+1}\rangle$ by $E_{l|m}$ we get

$$E_{l|m} = \frac{\pi}{i}\int_{-\infty}^{+\infty} \nu \frac{\operatorname{sh}\pi t}{\operatorname{ch}^2 \pi t}(P_\nu)^l (P_{-\nu})^m\, dt.$$

We put $t = -i/2 + x$; the integration contour in the x plane can be shifted by $-i/2$, provided the origin is circled from above; denoting the shifted contour by Γ and making the substitution $x \to -x$, we obtain

$$E_{l|m} = i\pi \int_{\Gamma} \frac{x \operatorname{ch}\pi x}{\operatorname{sh}^2 \pi x}(P_{ix})^l (P_{-ix})^m\, dx$$

$$-i\pi \int_{\Gamma_1} \frac{x\operatorname{ch}\pi x}{\operatorname{sh}^2\pi x}(P_{-ix})^l (P_{ix})^m\, dx,$$

where Γ_1 differs from Γ in the direction of circling around the origin. It follows therefore that $E_{l|m}$ differs from $-E_{m|l}$ by a quantity proportional to the residue of the integrand at the origin. Calculations yield $E_{l|m} + E_{m|l} = 2$. Thus, the plot of E is antisymmetrical with respect to the "center" of the layer (if the "distances" are measured in terms of the number of encountered inhomogeneities).[6] We note that the half-sum of the values of E at points that are symmetrical with respect to the center equals the intensity of the incident wave; in particular, for even n, E at the center of the layer is equal to unity, which agrees with the result of [1].

Let us proceed to consider the case of a continuous medium: let $s \to 0$; $l, m \to \infty$, so that $L = ls$ and $M = ms$ are fixed, we obtain

$$E_{L|M} = e^{M-N/4}\pi \int_{-\infty}^{+\infty} \frac{t\operatorname{sh}\pi t}{\operatorname{ch}^2\pi t}\left(\cos 2Mt + \frac{\sin 2Mt}{2t}\right) e^{-t^2 N}\, dt, \tag{24}$$

where $N = L + M$. The integral in (24) satisfies the diffusion equation in terms of the variables M and N; if we make the substitution $x = N/2 - M$ and $y = N$ (i.e., if we introduce the variables "distances from the center of the "layer" and "layer thickness"), then, putting $E_{L|M} = E(x, y)$, we obtain the diffusion equation for E

$$\frac{\partial E}{\partial y} = \frac{1}{4}\frac{\partial^2 E}{\partial x^2}. \tag{25}$$

The quantity E has a physical meaning only when $-y/2 \le x \le y/2$; however, as seen from (24), E can be regarded also outside this segment, by continuing (24) analytically with respect to M. We use this to obtain an expression that is more convenient for analysis and calculations than (24). Namely: we again calculate E, solving (25) for the specified "initial condition" $E|_{y=0}$. From (24) we can readily get

$$E|_{y=0} = 1 - \operatorname{th} x - \frac{x}{\operatorname{ch}^2 x} = f(x).$$

From this expression we can already gain a qualitative idea of the character of the function E(x, y) (see Fig. 6, which shows also the sections of the E surface by the planes $x = \pm y/2$ bounding the inhomogeneous layer, and three sections by the planes y = const, which give examples of the distribution of E over the layer). Solving (25), we get

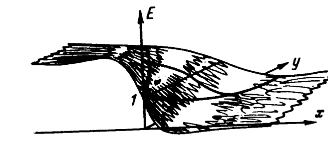

FIG. 6.

$$E = \frac{1}{\sqrt{\pi y}}\int_{-\infty}^{+\infty} e^{-(\xi - x)^2/y} f(\xi)\, d\xi$$

$$1 - \frac{1}{\sqrt{\pi y}}\int_{-\infty}^{+\infty} e^{-(\xi-x)^2/y}\left(\operatorname{th}\xi + \frac{\xi}{\operatorname{ch}^2\xi}\right) d\xi. \tag{26}$$

Expanding here $\tanh\xi + \xi\cosh^2\xi$ in a series and integrating term by term, we obtain a perturbation-theory series that diverges when $x, y \neq 0$:[7]

$$E \sim 1 - 2x(1 - y + {}^3/_2 y^2 - {}^2/_3 x^2 + \ldots). \tag{27}$$

We note that for a continuous medium (21) goes over into

$$1 - 2x/(1+y). \tag{28}$$

In the case of weak inhomogeneities this expression differs from (27) in quantities of second order of smallness. For strong inhomogeneities, on the other hand, both results coincide only in the center of the layer; it can be readily seen from (24) and (26) that a plot of (26) is more gently sloping at the ends of the layer and is steeper in the middle than the straight line (28).

Let us see also what happens to the distribution of E when the inhomogeneities are intensified to infinity, while retaining the geometrical dimensions of the layer. Putting $x, y \to \infty$ with x/y fixed, we see from (26) that E tends to double the intensity of the incident wave in the half of the layer that faces the incident wave, and to zero in the other half.

Returning to expression (23), we note that it is easy to consider the case of smoothly varying inhomogeneities; going over to a continuous medium, we find the same result as for identical inhomogeneities, the only difference being that, for example, the rol of $L = \lim ls$ will be played by $\lim \sum_{i=1}^{l} s_i$. In this sense, all the layers with identical summary backward-scattering coefficients are equivalent: E will be the same at points characterized by the same sum of the backward-scattering coefficients of the inhomogeneities encountered on the wave from the start of the layer.

We note also that it is easy to consider the case when the backward-scattering coefficients of the inhomogeneities are in turn independent random quantities. On going over to a continuous medium, it is necessary to assume here that the mean value and the higher-order moments of the backward-scattering coefficient tend to zero; if the higher-order moments tend to zero sufficiently rapidly compared with the mean value, then

[6] This leads to the following interesting consequence: assume that besides the wave considered above, which is incident on the layer from the left, there is simultaneously incident from the right a unit wave that is not coherent with the other wave; then E in the layer will be constant—its intensity will be equal to two throughout.

[7] By using (25) it is easy to derive an expression for the general term of this series.

it is necessary to simply replace the integral backward-scattering coefficient in the expressions considered above by its mean value.

CERTAIN GENERALIZATIONS

Let us see if it is possible to extend the results to the case of a statistically-inhomogeneous medium in its usual sense. The method employed above is based on the fact that: a) the entire layer can be broken up into statistically independent sections; b) the phase of the coefficient of reflection from each individual section has a uniform distribution and does not depend on the amplitude; c) the reciprocity principle holds and there are no losses (satisfaction of this requirement is not essential for the calculation of the average field). If these conditions are satisfied, then, by averaging in succession over the phases of the reflection coefficients of the individual sections, we obtain, for example for the mean-squared field on the boundaries of these sections, the relation (23), where each β_i is expressed in a well-known manner in terms of the amplitude of the reflection coefficient of the i-th section; if these amplitudes are also random, then an additional averaging of (23) over β_i is necessary.

The main difficulty is the proof of item b), and this is the only reason why we chose the somewhat artificial model considered above. We shall show, however, that for a statistically inhomogeneous medium that is weakly scattering within a wavelength, this condition can be satisfied approximately by correctly choosing the sections. In fact, let us consider a section of space $\lambda/2$ thick (we emphasize that this section is connected with the space and not with the medium). Let a certain realization of the plot of the refractive index be specified in this section. It is natural to assume that all the realizations with plots obtained from the considered plot by cyclic shifting are equally probable. We break up the entire set of realizations into groups, combining into one group the realizations obtained from one another by a cyclic shift. Each group is the analog of one discrete inhomogeneity of the model considered above; the parameter characterizing the magnitude of the cyclic shift is analogous to the quantity x_i, which characterizes the position of the discrete inhomogeneity. The presence of different groups corresponds to the fact that the discrete inhomogeneity can have, besides a random position, also a "random force."

[8]The fact that the layer thickness may not be equal to an integer number half waves is immaterial, since scattering from a half-wave section is weak, i. e., the edge effect is small.

We shall now show that, in the first approximation of the method of small perturbations, the coefficient of reflection from the section has for each group a constant amplitude and a uniformly distributed phase. Indeed, in the approximation under consideration, the coefficients of reflection of the individual sections simply add up; recognizing that when one section is shifted by $\lambda/2$ its reflection coefficient remains unchanged, we find that when the diagram is cyclically shifted within the section by an amount τ, the reflection coefficient changes in the same way as if the entire section were to be shifted as a unit by an amount τ. But in this case the dependence of the reflection coefficient on τ is given by the factor $\exp(2ik\tau)$; recognizing that τ is uniformly distributed over the interval $(0, \lambda/2)$, we obtain the required result.

Thus, breaking up the entire layer into sections of thickness $\lambda/2$, we satisfy condition b) in the first approximation of the method of small perturbations.[8] In order for the coefficients of reflection from the different sections to be statistically independent, it is necessary that the lengths of the sections greatly exceed the correlation radius of the inhomogeneities. The latter can be attained by taking not half-wave sections but sections with thicknesses that are integer multiples of $\lambda/2$ (it is required, however, that the scattering from such sections still be small). Thus, when the indicated conditions are satisfied, it is possible to find the average characteristics of the field at the boundaries of the sections; in view of the weakness of the scattering in each section, it is clear that we obtain the value of the field everywhere.

The author is grateful to M. I. Isakovich, discussions with whom greatly contributed to the performance of this work.

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Translated by J. G. Adashko
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