# SPHERICALLY SYMMETRIC T-MODELS IN THE GENERAL THEORY OF RELATIVITY

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Spherically symmetric models constructed of dustlike matter are considered in a comoving reference frame, and a general solution of the Einstein equations ($\Lambda \neq 0$) is obtained which contains along with the Tolman-Bondi-Lemaitre models an additional class of T-models of a "sphere" with the metric of a synchronously-comoving T-system ($R = r(\tau)$) which represent an inhomogeneous generalization of the anisotropic cosmological model of a "quasiclosed" type with hypercylindrical spatial sections $V_3 = (S_2 \times R_1)$. The T-models of a "sphere" yield a method, which in principle differs from the closed Friedmann model, for realizing the total mass defect maximal in GTR equal to the total rest mass of matter, and are characterized by the fact that the gravitational binding energy for each particle of "dust" exactly compensates its rest mass so that as a result the active mass - the equivalent of the total energy - remains constant in the case of an unrestricted growth of the "sphere" and, in general, does not contain any material contribution. It is of a purely field nature and coincides with the geometrodynamic "massless mass" of the T-regions of the Schwarzschild-deSitter-Kottler fields in which matter is bound gravitationally and is held by the strongest possible vacuum field inside the "event horizon" of the Schwarzschild sphere type. It is shown that the T-models of a "sphere" do not have a classical analogue, and their existence and paradoxical properties are due to the nonlinearity of GTR: a) a mass defect which manifests itself in a characteristic manner through the non-Euclidian nature of the co-moving space $V_3$, b) the presence of T-regions in SSK fields. A detailed discussion is given of the principal properties and the dynamics of the cosmological T-models of a "sphere" ($\Lambda \neq 0$), and they are classified in accordance with Robertson's scheme for a closed Friedmann model in terms of the analogous types $O_1$, $M_1$, $M_2$, $A_1$, $A_2$ of transverse motion of the hypercylinder $V_3$. It is shown that all physically acceptable solutions with $\rho > 0$ must have time singularities of three kinds: collapse of $V_3$ into a line, a point and a sphere, with the infinite types $M_1$, $A_2$ and $M_2$ becoming isotropic in the course of unlimited expansion.

## INTRODUCTION

It is generally accepted that the Tolman solution[1,10] and its cosmological variant for $\Lambda \neq 0$[5,6,11,12] are general and represent all the spherically symmetric models possible in GTR for the distribution and the homologous motion of gravitating dustlike matter without pressure. But the commonly utilized procedure for integrating the Einstein equations is based on the implicit assumption that the condition for a comoving reference frame $G_0^1 = 0$ cannot degenerate into an identity having no content, and therefore the Tolman-Bondi-Lemaitre (TBL) solutions are characterized by an additional requirement on the angular coefficient of the initial metric $(\partial R/\partial\chi)_\tau \neq 0$.

Actually this limitation does not follow from spherical symmetry or from the field equations, and thus, we can leave out of consideration the special type of the so-called T-models of "sphere" for which the comoving reference system is also a synchronous T-reference system according to the terminology of[3,8] with an elementary interval of the form

$$ds^2 = d\tau^2 - e^{\omega(\chi,\tau)}d\chi^2 - r^2(\tau)\{d\vartheta^2 + \sin^2\vartheta d\varphi^2\}, \qquad (1)$$

where $\tau$ is the proper time[1)], and $\chi$ is the radial Lagrangian coordinate of the particles in a spherical shell.

The object of the present work is to obtain a general solution in a unified closed form which would include as a particular case this additional class of exact internal solutions for the metric (1), to carry out a comparative analysis of the basic properties, and to give an invariant characterization and a physical interpretation of the T-models, being guided by their analogy with the T-regions of the Schwarzschild-de Sitter-Kottler fields[3,8] and basing ourselves on a comparison and similarity between relativistic and Newtonian models of a sphere.

The spherically symmetric T-models turn out to be a simple inhomogeneous generalization of the anisotropic cosmological model of a "quasiclosed" type[13-15,3] and for $\Lambda = 0$ they correspond to special configurations, paradoxical in their properties, of a general relativistic dust sphere which have a finite and constant gravitational mass for any unlimited quantity of matter composing them, with this equivalent to the total energy not containing in general any material contribution and having the purely field nature of Wheeler's[16] geometrodynamic "massless mass". In analogy to the closed model of Friedmann[1,3] the T-models of a sphere are topologically "closed upon themselves" and yield a method, which is in principle different from it, of realizing the maximum possible in GTF total mass defect which is exactly equal to the

[1)]Henceforth, with exception of the classical variant, we use for the fundamental velocity of light and the Newtonian gravitational constant $c = G = 1$, while the Einstein constant is $\mathcal{K} = 8\pi$.

total rest mass of matter.

The cosmological T-models of a "sphere" ($\Lambda \neq 0$) are a modification of the T-regions of the Schwarzschild-deSitter-Kottler (SSK) fields and provide an example of general relativistic models and nonstatic fields of an anomalous longitudinal type with an algebraic structure of the Weyl tensor ID, which are not gravitational waves, do not possess Euclidian analogues, and whose existence and unusual properties are due to the nonlinearity of the Einstein equations. Similar special solutions (to this family also belong the "flat" and the "quasiopen" anisotropic models[13-15,3]) are absent in the classical and the linearized gravitational theories, and, consequently, demonstrate a qualitatively new aspect of the interrelationships of the latter and the GTR from the point of view of the correspondence principle.

## THE GENERAL SOLUTION AND THE TOLMAN-BONDI-LEMAITRE MODELS

1. In the reference system comoving with the "dust", which apparently exists in the case of homologous motion (without intersections of particle trajectories) and which in the case of spherical symmetry can always be chosen to be synchronous with a metric of the form

$$ds^2 = d\tau^2 - e^{\omega(\chi,\tau)}d\chi^2 - R^2(\chi,\tau)\{d\vartheta^2 + \sin^2\vartheta d\varphi^2\}, \tag{2}$$

the field equations[1,5] are reduced to a simple system which is equivalent to them (since for $\rho \neq 0$, $R \neq$ const):

$$R' = We^{\omega/2}, \quad R^2 = W^2 - 1 + \frac{2m}{R} + \frac{\Lambda R^2}{3}, \quad \rho = \frac{\mathscr{M}'}{4\pi R^2 e^{\omega/2}},$$
$${}^1\!/_2\omega R - R = (W' + \mathscr{M}'/R)e^{-\omega/2}, \quad m' = W\mathscr{M}', \tag{3}$$

where $\rho$ is the invariant density of the "dust", $\Lambda$ is the cosmological constant, and the prime and the dot denote partial differentiation with respect to $\chi$, $\tau$.

The first integrals of the Einstein equations $M(\chi) \geq 0$, $m(\chi)$ and $W(\chi)$ which are arbitrary functions satisfying only the most general requirements for the existence of a physically admissible solution of (3) define, respectively, the distribution of the total rest mass of the "dust" inside Lagrangian spheres, the effectively gravitating mass—which is equivalent to the total energy of the sphere, and the conserved relativistic specific energy or the active mass of the particles of the layer. Within the latter one can also differentiate between the unit contribution of the rest mass of the "dust" and the kinetic energy of its radial motion and the potential energy of the gravitational and cosmological interactions.

The system (3) is consistent and admits a general solution by means of quadratures with respect to time in closed form:

$$e^{\omega/2} = R\left\{\lambda + W\cdot\int\limits_{R_0}^{R}\left(W^2 - 1 + \frac{2m}{u} + \frac{\Lambda u^2}{3}\right)^{-3/2}du\right.$$
$$\left.+\mathscr{M}'\int\limits_{R_0}^{R}\frac{du}{u}\left(W^2 - 1 + \frac{2m}{u} + \frac{\Lambda u^2}{3}\right)^{-3/2}\right\}, \tag{4}$$
$$\tau - \tau_0 = \int\limits_{R_0}^{R}\left(W^2 - 1 + \frac{2m}{u} + \frac{\Lambda u^2}{3}\right)^{-1/2}du, \quad \rho = \frac{\mathscr{M}'}{4\pi R^2 e^{\omega/2}} \geqslant 0,$$

where the integrals are evaluated in terms of elliptic functions, while in the case $\Lambda = 0$ they are elementary and one can take $R_0(\chi) = 0$.

The solution (4) obtained above implicitly defines the dynamical behavior of the metric and of matter in terms of an irreducible set of not more than two essentially independent integrals of motion which characterize any arbitrary initial distribution of the "dust" in the sphere (a free gravitational field is naturally absent). Indeed, substitution of (4) into the remaining equation $R' = We^{\omega/2}$ gives another relation $\lambda = -\tau_0'/W$ which relates the arbitrary functions $\lambda(\chi)$ and $\tau_0(\chi)$—the moment of the collapse of the layer at the "center" $R(\chi, \tau_0) = 0$ for $\Lambda = 0$, and, moreover, the radial Lagrangian coordinate itself is determined in (2) only with an accuracy up to an arbitrary transformation $\tilde{\chi} = \Phi(\chi)$.

In the absence of material sources ($\rho \equiv 0$) the formulas (4), where one should set $\mathscr{M} \equiv 0$ and $m = \text{const}$, determine the vacuum solution within the class of freely falling systems (2), which, as can be shown[6,8,3] is locally equivalent to the only SSK metrics[17] available for $\Lambda \leq 0$ in vacuum with the invariant parameter of the gravitational mass $m = M$:

$$ds^2 = \left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right)dt^2 - \left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right)^{-1}dR^2$$
$$- R^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2). \tag{5}$$

This limiting case of the models of a sphere (4) corresponds to the motion of a test "reference dust" in SSK fields, with the latter being globally nonstatic in the presence of T-regions where $g_{00} = [1 - (2M/R) - (\Lambda R^3/3)] \leq 0$, and the selected time and radial coordinates (5) appear to interchange their roles, and there exist no spherically symmetric gravitational waves, including shock waves, in accordance with the improved Birkhoff theorem[8,9,18].

For $\Lambda > 0$ the solution (4) in vacuuo will no longer be general since there exists an additional special solution of the initial Einstein equations[5] with $R = \text{const}$ of the form

$$ds^2 = R_0^2\{(d\tilde{\tau}^2 - \text{sh}^2\,\tilde{\tau} d\chi^2) - (d\vartheta^2 + \sin^2\vartheta d\varphi^2)\}. \tag{6}$$

It represents a separable space-time $V_4 = (S_2 \times S_2)$—a direct topological product of twodimensional subspaces of constant positive curvature $\tilde{K}_2 = K_2 = 1/R_0^2 = \Lambda > 0$ each of which possesses maximal mobility. The complete group of automorphisms of this field $V_4$ is $G_6 = \tilde{G}_3 \times G_3$[19] where the transitive group $G_3$ of type IX is the usual group of rotations on a sphere $S_2$, while $\tilde{G}_3$ of type VIII is the group of motions on an indefinite hyperboloid $\tilde{S}_2$ which is open in the time and closed in the spatial direction.

2. If $W^2 = 1 + f > 0$, then the solution (4) can be written in the standard form[1,5,6]:

$$e^{\omega} = \frac{R'^2}{1+f}, \quad \dot{R}^2 = f + \frac{2m}{R} + \frac{\Lambda R^2}{3}, \quad \rho = \frac{m'}{4\pi R^2 R'}, \tag{7}$$

where the radius of curvature of the Lagrangian spheres $R(\chi, \tau)$ is determined by integration of the equation of motion. In the general case it can be expressed in terms of the Weierstrass elliptic functions[11], while for $\Lambda = 0$ it is given by well known formulas in implicit or parametric form[1]. In the TBL models (7) the metric and the distribution of the "dust" essentially depend only on two arbitrary functions $a_0(\chi) = 2m/|f|^{3/2}$ and $\tau_0(\chi)$, since because of the arbitrariness in the choice of the radial Lagrangian coordinate the function $f(\chi)$ can be put in canonical form:

$$f(\chi)=\varepsilon S^2(\chi),\quad R(\chi,\tau)=S(\chi)a(\tau,\chi),\quad e^{\omega/2}=a\left(1+\frac{\partial\ln a}{\partial\ln S}\right),$$

$$S(\chi)=\begin{cases}\sin\chi, & \varepsilon=-1\\ \operatorname{sh}\chi, & \varepsilon=+1\end{cases};\quad \dot{a}^2=\frac{a_0}{a}+\frac{\Lambda a^2}{3}+\varepsilon. \tag{8}$$

The parabolic case $f = 0$ corresponds to $a_0 = \text{const}$, $S(\chi) = \chi$, $m = \frac{1}{2}a_0\chi^3$, $\epsilon = 0$ and, in fact, is characterized by only one physically arbitrary function $\tau_0(\chi)$. The homogeneous isotropic models of Friedmann[1,3,12] the line element of which

$$ds^2=d\tau^2-a^2(\tau)\{d\chi^2+S^2(\chi)[d\vartheta^2+\sin^2\vartheta d\varphi^2]\},$$
$$\dot{a}^2=\frac{a_0}{a}+\frac{\Lambda a^2}{3}-k,\quad \varkappa\rho=\frac{3a_0}{a^3}>0,\quad k=\pm1,0, \tag{9}$$

is contained in (7) as a special case[5,6] correspond to a special choice of initial conditions: a) $a_0 = \text{const}$, $\tau_0 = \text{const}$; b) $f = 0$, $\tau_0 = \text{const}$, so that the connection of the Friedmann models with configurations of a sphere is established by relations of the form

$$R(\chi,\tau)=S(\chi)a(\tau),\quad m(\chi)=\frac{a_0}{2}S^3(\chi),\quad f(\chi)=\varepsilon S^2(\chi). \tag{10}$$

Here the function $S(\chi)$ depending on the sign of $\epsilon = \mp 1, 0$ distinguishes between closed, open and flat models with invariant spatial crosssections $V_3$ of constant positive, negative and zero curvature $K_3 = k/a^2$.

As is well known[3,20] Friedmann models with $\Lambda = 0$ have a complete classical analogue of local properties in Newtonian hydrodynamics of the isotropic expansion of a homogeneous sphere of elliptic, hyperbolic and parabolic type corresponding respectively to $k = \pm 1, 0$. This remarkable analogy extends also to the general case of inhomogeneous models of Tolman[5] which also retain a close connection and exhibit far-reaching similarity of local properties with the Newtonian dynamics of a gravitating dust sphere in Lagrangian formulation:

$$\frac{\partial^2 R}{\partial t^2}=-\frac{G\mathcal{M}}{R^2},\quad \mathcal{M}=4\pi\int_0^R\rho R^2\,dR. \tag{11}$$

The flowing rest mass of the "dust" $\mathcal{M}(\chi)$ contained within "liquid" spheres of radius $R(\chi, t)$ is conserved in homologous motion (in the absence of mixing of layers and of violations of the continuity of the medium) and it is convenient to utilize it as a Lagrangian coordinate.

Within the framework of Newtonian cosmology one can in an adequate manner take into account also the effect on the dynamics of the $\Lambda$-term[21] if one introduces an additional force $\mathbf{F} = \Lambda R/3$, i.e., if one alters the Poisson equation by including the cosmological constant in the sources of the field: $\Delta\varphi = 4\pi G\rho - \Lambda$.

The dynamic equation (7) is identical with the Newtonian energy integral with an altered interpretation of the constants, while the Tolman solution coincides with the general solution of classical hydrodynamics (11) of a homologous centrally-symmetric motion of the "dust" in its own gravitational field:

$$\frac{1}{2}\left(\frac{\partial R}{\partial t}\right)^2-\frac{G\mathcal{M}}{R}=E(\mathcal{M}),\quad \rho=\frac{1}{4\pi R^2}\frac{\partial R}{\partial\mathcal{M}}. \tag{12}$$

Here the law of motion $R(\mathcal{M}, t)$ is expressed in closed form similar to (7), (8), in terms of two essentially arbitrary functions: $E(\mathcal{M})$ = the total energy per unit mass or the equivalent combination of the form $\tilde{a}_0(\mathcal{M}) = G\mathcal{M}/|E|^{3/2}$, and $t_0(\mathcal{M})$ - the instant when the particles of the layer are focused at the center, which are subject only to the requirements which guarantee the homologous nature of the motion of the "dust," and also to additional boundary conditions at the center of the sphere $\mathcal{M} = 0$, where $R(0, t) = 0$, $E(0) = 0$, $t_0(0) = 0$. This set takes into account all possible initial distributions of density and radial velocity of gravitating "dust" in a Newtonian sphere including among them as special cases a) $\tilde{a}_0 = \text{const}$, $t_0 = \text{const}$, b) $E = 0$, $t_0 = \text{const}$ and the classical equivalents of the Friedmann models (9).

If for $f \neq 0$ one gives up the non-Euclidian nature of the physical space $V_3$ of the comoving reference system (2) and if, correspondingly, one identifies the flowing active and proper masses, then the Tolman models (7) admit a quasi-Newtonian interpretation[5] and in their local properties agree completely with the distribution of density and radial velocity of "dust" in a Newtonian arbitrary sphere. Consequently, the classical case gives not only a linear approximation valid only in the limit of weak fields, but represents their exact analogue. This remarkable similarity of relativistic and Newtonian models of a dust sphere is due to the specific properties of spherical symmetry which excludes gravitational radiation and guarantees a considerable similarity of structures of the Einstein and Newtonian gravitational fields: in GTR a modified inverse square law and the principle of superposition with respect to the active mass are preserved.

3. The main differences are due to the nonlinearity of GTR which is expressed first of all by the fact that the role of the active mass of the sphere $m(\chi)$—the effective source of the external field—is played by its total energy[5] which includes in addition to the total rest mass $\mathcal{M}(\chi)$ also the kinetic energy of the radial motion of the "dust" ($m > \mathcal{M}$, if $f > 0$[8,22]) and the gravitational potential binding energy. The latter is obviously not contained in the material energy-momentum tensor, for $\Lambda = 0$ it is negative, as in Newtonian theory, in accordance with the attractive nature of the interaction forces, and leads to a gravitational mass defect $\Delta = m - \mathcal{M} < 0$ for $f < 0$[3,8,22].

In the general case $f \neq 0$ the increment to the flowing active mass $m' = W\mathcal{M}'$ does not coincide with the proper rest mass of the added spherical layer and differs for $W < 1$ by the amount of the negative binding energy, and for $W > 1$ by the positive excess of the residual kinetic energy of infinite separation. These general relativistic effects - the mass defect and the gravitation of kinetic energy—are manifested in a characteristic manner through the non-Euclidian nature of the physical space $V_3$ of the comoving reference system since the distribution of the specific active mass $W = dm/d\mathcal{M}$ also determines the geometry of the spatial cross sections $\tau = \text{const}$[5]. In particular, the sign of the quasi-Newtonian energy $f = 2E$ which differentiates in the case $\Lambda = 0$ the types of motion of the layer of "dust" is opposite to the sign of the local scalar curvature of $V_3$ as, for example, in the Friedmann models[22 21].

Because of the nonlinear contribution of the gravitational potential binding energy the active mass $m(\chi)$ is a nonadditive and possibly even a nonmonotonic function of the proper rest mass $\mathscr{M}(\chi)$ in "semi-closed" models[3,22] with a $W(\chi)$ which can change sign and which has isolated zeros at $\chi = \chi_i^*$. Contrary to Bondi's[5] assertion it is always possible to so choose the arbitrary functions in the TBL solution (7) that $W(\chi_i^*) = 0$, $m'(\chi_i^*) = 0$, $\mathscr{M}'(\chi_i^*) > 0$ and at the same time one must also have $R'(\chi_i^*, \tau) = 0$, $\tau_0'(\chi_i^*) = 0$, while $\exp[\omega(\chi_i^*, \tau)] \neq 0$, so that the metric is regular at these points. In "semi-closed" models one can decrease the gravitational mass of the sphere by adding external layers of "dust" with $W(\chi) < 0$—the "self-screening" effect[8]—and even to make it vanish if one exactly compensates the total rest mass of matter by the total binding energy.

In fields so strong that the mass defect is comparable to the rest mass and the size of the sphere approaches its gravitational radius the nonlinearity of GTR becomes essential and leads to a difference in principle between the relativistic and Newtonian models, and in view of the local validity of the classical theory it has a very specific nature and in the first instance affects the global properties of the solutions of the Einstein equations. In particular, the "super-strong" interaction when the binding energy of the layer exceeds its rest mass, $W(\chi) < 0$, manifests itself geometrically in the non-Euclidean topology of the comoving reference space $V_3$: the latter must necessarily contain in T-regions $R(\chi, \tau) \leq 2m(\chi)$) "orifices"[8]—instantaneous spheres $\chi = \chi_i^*$ with an extremal value of the radius of curvature.

Very instructive in this connection is the example of "semi-closed" Friedmann models[3,8,22] in which the active mass $m(\chi) = \frac{1}{2}a_0 \sin^3\chi$ and the radius of the spheres $R = a(\tau)\sin\chi$ simultaneously pass through a maximum on the "equatorial" sphere $\chi^* = \pi/2$ and tend to zero in the limit $\chi = \pi$.

Although in the region $(\pi/2 < \chi \leq \pi)$ the mass defect of the layer does exceed its rest mass, in the final analysis it leads only to a complete compensation of the material contribution to the limiting critical configuration of the sphere—a closed Friedmann model[22] which corresponds to a topological closure of the space $V_3$: the boundary of the sphere degenerates into the opposite pole of a 3-sphere, while the exterior region with the Schwarzchild field disappears. Consequently, the total gravitational mass defect of a homogeneous sphere in GTR cannot exceed the total rest mass of the "dust", and this differs radically from the result of the Newtonian theory which gives for the total energy of instantaneously static configurations of a homogeneous dust sphere the expression $E = \mathscr{M}c^2 - 3G\mathscr{M}^2/SR$, $\mathscr{M} = 4\pi\rho R^3/3$ which does not have a lower bound, and which can become negative. But already from the quasi-Newtonian approximation which takes into account the equivalence of the active mass and the total energy of the sphere including the gravitational proper binding energy $M = \mathscr{M} - 3GM^2/SR$ it follows that the negative mass defect leads to a cumulative weakening of the interaction and cannot give rise to an inversion of the sign of the total energy—the mass of the sphere made of normal matter with $\rho > 0$, although "self-screening" is possible and $M \to 0$ for $\mathscr{M} \neq 0$ in the limit of complete gravitational binding of matter.

Because of the mass defect in GTR it is possible to construct restricted TBL models—"spheres" with an infinite total rest mass of the "dust" which nevertheless manifest themselves in vacuo as an ordinary sphere with a finite gravitational mass and radius[8]. Their spatial cross sections $V_3$ do not have a classical center of symmetry $R = 0$, but are open in the "radial" direction $(-\infty < \chi < \infty)$, they possess an infinite proper volume and contain within themselves an infinite amount of matter with a density which is finite everywhere. Moreover, in "semi-closed" models containing a denumerable infinity of "orifices" and having the topology of spatial cross sections $V_3$ of the type of a "corrugated" hypercylinder without a center of symmetry $R = 0$ it is possible to guarantee finiteness of the active mass $0 < m(\chi) \leq M$ for any arbitrary proper rest mass of the "dust" $\mathscr{M}(\chi)$ within the whole internal region occupied by matter.

## COSMOLOGICAL T-MODELS OF A "SPHERE"

1. The degenerate case $W \equiv 0$, $R = r(\tau)$ in (3), (4) corresponds to a special class of solutions with the metric of a synchronously-comoving T-system (1):

$$\dot{r}^2 = \frac{r_0}{r} + \frac{\Lambda r^2}{3} - 1, \quad e^{\omega/2} = \dot{r}\left\{\lambda + \mathscr{M}' \int^r \frac{du}{u}\left(\frac{r_0}{u} + \frac{\Lambda u^2}{3} - 1\right)^{-3/2}\right\}$$

$$\tau = \int^r \left(\frac{r_0}{u} + \frac{\Lambda u^2}{3} - 1\right)^{-1/2} du, \quad \rho = \frac{\mathscr{M}'}{4\pi r^2 e^{\omega/2}} \geqslant 0, \tag{13}$$

which is not contained in the TBL solutions (7) and represents a family of cosmological T-models of the "sphere" which have a constant and finite active mass $M = r_0/2$ for any arbitrary total rest mass $\mathscr{M}(\chi)$ of the "dust" of which they are composed. Here the characteristic gravitational radius of the T-models of a "sphere" $r_0 = 2M$ is a constant of integration, $\lambda(\chi)$ is an arbitrary function which, however, (if it differs from zero) can be converted to unity by a permissible transformation of the radial Lagrangian coordinate. The possible values of the parameters $M$ and $\Lambda$, and also the limits for the variation of the angular metric coefficient $r(\tau)$ are determined by the condition $(2M/r + \Lambda r^2/3 - 1) \geq 0$.

It is evident that in the solution (13) only those regions are physically acceptable where the signs of $\exp(\omega/2)$ and $\mathscr{M}'(\chi)$ coincide (in the opposite case the necessary requirement $\rho \geq 0$ is violated), with continuous transition from one sign to the other being excluded due to the inevitable degeneracy if the metric $\exp\omega = 0$. The choice of the independent branch $\exp(\omega/2) > 0$ corresponds to the interpretation of the last equation in (13) as a law of conservation of rest mass or of the number of particles and is dictated by the natural assumption that the proper volume of a spherical layer of "dust" is positive.

Although the T-models for $\Lambda = 0$ can be regarded as limiting configurations of the general relativistic sphere[16] in which the binding energy of each particle of "dust" is exactly equal to and compensates its rest mass, this special class of solutions (13) cannot be obtained by directly setting $f = -1$ in formulas (7), (8). Since the Tolman solution exhausts all the Newton-like models of a sphere, a classical analogue cannot be made to correspond to the T-models, so that the class of spherically-symmetric homologous motions of gravitating "dust" in GTR is in a certain sense wider than in Newtonian dynamics.

The T-models of a "sphere" become possible in GTR only due to the nonlinear effect of the dependence of the active mass on the gravitational and the cosmological binding energies, and their existence is connected in a definite manner with the existence of homogeneous T-regions of the SSK fields which are in principle non-static (5). The latter do not have any Euclidian analogues and represent a maximally strong "attractive" or repulsive" for $\Lambda > 0$ field of an anomalous longitudinal type without material sources which cannot be identified with gravitational waves.

The most remarkable property of the T-models of a "sphere"—the constancy of the active mass $m = M$ and its independence of the distribution of the proper rest mass $\mathscr{M}(\chi)$ and even of the presence of matter—is a consequence of the gravitational mass defect and has a simple explanation. The negative potential binding energy of the particles of "dust" acts on an equal footing with the kinetic energy and the rest mass as an effective source of the field and in view of the condition $W = 0$ for each layer exactly cancels their contribution. As a result the active mass remains unchanged in the course of unrestricted growth of the T-models of a "sphere" and must have a "priming" nature, since in general it contains no material contribution. This arbitrary parameter $M \gtrless 0$ (which also assumes negative values $M \leq 0$ for $\Lambda > 0$) can be interpreted as a gravitational "massless mass" of the vacuum T-regions regions of the SSK fields with a metric of the type[8]

$$ds^2 = \left(\frac{2M}{T} + \frac{\Lambda T^2}{3} - 1\right)^{-1} dT^2 - \left(\frac{2M}{T} + \frac{\Lambda T^2}{3} - 1\right) d\chi^2 - T^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{14}$$

into which the T-models of a "sphere" (13) go over in the limit of empty space $\rho = 0$, and with which they exhibit a very close analogy. The T-models of a "sphere" are constructed on the basis of the T-regions of the SSK fields and appear as a generalization of the latter to the case when space is filled by an unrestricted quantity of matter with $W \equiv 0$ which is bound gravitationally and is held by a maximally strong vacuum field within their boundaries.

The cosmological T-models of a "sphere" give a method that is different in principle from the closed Friedmann models $(\Lambda \neq 0)$[6] for realizing the maximally possible in GTR total mass defect which is equal to the total rest mass of matter, with the latter also manifesting itself in a characteristic manner through the in principle non-Euclidian nature of the physical space $V_3$—as a consequence of the universal nature of the relation between the distribution of the active mass of matter $W(\chi)$ and the geometry of the reference space $V_3$ comoving with it.

A distinguishing feature of a synchronously-comoving T-system is that its orthogonal spatial cross sections $\tau$ = const, each of which has the homogeneous metric

$$dl^2 = d\tilde{\chi}^2 + r^2(\tau)\{d\vartheta^2 + \sin^2\vartheta d\varphi^2\} \tag{15}$$

(which admits as a total group of motions the transitive group $G_4 = G_3 \times G_1$ of type VIII[19]), have basic geometric characteristics—components of the Riemann and Ricci tensors which do not vanish identically and a curvature scalar of the form

$$P_{121}{}^2 = P_{131}{}^3 = 0,\ \ P_{232}{}^3 = 1,\ \ P_1{}^1 = 0,\ \ P_2{}^2 = P_3{}^3 = 1/r^2,\ \ P = 2/r^2$$

and, consequently, they represent three-dimensional hypercylinders with the non-Euclidian connectivity $V_3 = (S_2 \times R_1)$—a direct topological product of an ordinary sphere and an open straight line. In view of the constancy of the radius of curvature of the Lagrangian spheres $S_2$ the physical space $V_3$ does not have a classical center of symmetry, is open in both "radial" directions $(-\infty < \chi < \infty)$, has an infinite proper volume and can contain within itself an unlimited quantity of matter.

Owing to the total neutralization of the rest mass by the binding energy the "dust" in the T-models becomes "passive" and is "inscribed" into the initial SSK T-regions without any essential distortion of their local properties, appearing to replace the test "reference liquid" of the synchronous T-system (14). In addition to the "priming" parameters $M$ and $\Lambda$ of the vacuum T-regions matter brings with it only one physically arbitrary integral $\mathscr{M}'(\chi) \geq 0$ which measures the increase in the invariant proper rest mass when a spherical layer of "dust" is added. In virtue of the special assumptions $(W \equiv 0)$ on the nature of the distribution of matter in the T-models of a "sphere" the irreducible set of initial data of Cauchy contains only one essentially arbitrary function which characterizes the inhomogeneous distribution of the density of "dust" and, consequently, this special class of solutions is of zero measure compared to the Tolman class.

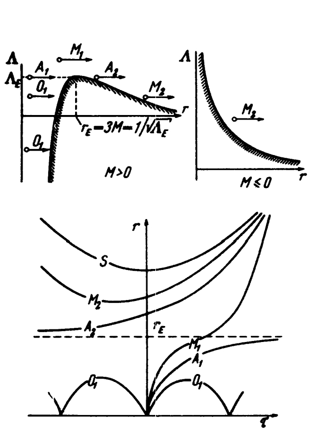
The indicated solution of (13) in the form of special cases a) $\mathscr{M}'/\lambda$ = const and b) $\lambda \equiv 0$ when this single physically arbitrary function of the type $\mathscr{M}'(\chi)$ reduces to a constant (we know, by the way, that for $\lambda \equiv 0$ we can set $\mathscr{M}' = 1$) includes the spatially homogeneous metric of anisotropic models with $\rho = \rho(\tau)$ of a "semi-closed" type[13,15,3], among them also those for $\Lambda \neq 0$. Although the latter possess a higher symmetry[14], since in addition to the ordinary rotation group $G_3$ of type IX which is transitive on the spheres $S_2$ they admit an additional one-parameter subgroup $G_1$ with a commuting operator for shifts along the Killing field $\xi^1 = \delta_1^1$ in the radial direction, nevertheless they give a sufficiently complete representation also of the properties of the inhomogeneous T-models.

The quasiisotropic homogeneous T-models with $\lambda \equiv 0$ in analogy with completely isotropic Friedmann models are possible only for regions occupied by matter since a more general anisotropic solution $(\lambda \neq 0)$ exists also in vacuo for $\mathscr{M}' = 0$ where it represents automatically homogeneous T-regions of the SSK fields with a nonstatic metric which can be reduced to the canonical form (14) by the transformation $T = r(\tau)$. In a certain sense the T-models can be treated as a peculiar superposition of homogeneous vacuum and quasiisotropic solutions, wherein either one of the two arbitrary coefficients $\lambda(\chi) \neq 0$ or $\mathcal{M}'(\chi) > 0$ can be, without restricting the generality, taken equal to unity; in particular, it is more convenient to take $\lambda = \pm 1$.

2. The family of geodesic parallel spatial cross sections $\tau$ = const which describe the temporal evolution of the T-models of a sphere'' represents as a whole a nonstatic hypercylinder $V_3$ containing matter. The tensor of the velocities of its longitudinal $(H_{\parallel} = \omega/2)$ and transverse $(H_{\perp} = \dot{r}/r)$ deformations is invariantly characterized by the presence of a general expansion and of anisotropy (a local rotation is incompatible with spherical symmetry).

The radius of curvature $r(\tau)$ of the hypercylinder $V_3$ satisfies an equation of the Friedmann type (9) with $k = +1$, so that the principal results of the analysis of the dynamics of ''closed'' isotropic models with $\Lambda \neq 0$[23,12] are also applicable to the T-models of a ''sphere.'' The transverse motions of the hypercylinder are similar to the law for the expansion of a 3-sphere in the corresponding closed Friedmann models[21], and they can also be classified in accordance with Robertsons[24] scheme (extended by taking into account the additional possibility $r_0 \leq 0$ for $\Lambda > 0$) in terms of analogous types $O_1$, $M_1$, $M_2$, $A_1$ and $A_2$ of behavior of its peripheral dimensions (cf., diagram), with the equivalent of the static Einstein model being absent.

The specific properties of the T-models are contained in the behavior of the radial components of $\exp(\omega/2) > 0$ which determines the mutual proper distances between the particles of different Lagrangian spheres, and are manifest in the dynamics of inhomogeneous longitudinal deformations along the generators of the hypercylinder $V_3$. Although ''passive'' matter with $W = 0$ gives no contribution to the constant gravitating mass under the action of which the motion of each layer of ''dust'' occurs, nevertheless it affects the dynamics of the longitudinal deformations of $V_3$.




Classification of the cosmological T-models of a "sphere" according to the type of temporal behavior of the radius of curvature $r = r(\tau)$ of the hypercylinder $V_3 = (S_2 \times R_1)$. The curves $\dot{r}^2 = 2M/r + \Lambda r^2/3 - 1 = 0$ the turning points for the transverse expansion of the hypercylinder $V_3$ solved with respect to $\Lambda(r) = 3r^{-2}(1-2M/r)$, divide the half-plane $(\Lambda, r \geqslant 0)$ into an allowed ($\dot{r}^2 < 0$ – shaded) regions. To each model for a given $\Lambda$ corresponds a segment of the straight line $\Lambda$ = const in the allowed region of values of $r(\tau)$. The points of intersection of this straight line with the line $\dot{r}^2 = 0$ give the roots of the characteristic equation $r_j^* = r_j^*(M, \Lambda)$, which also determine the boundaries between the $T$ and the $R$-regions of the SSK fields. If $M > 0$, $\Lambda > \Lambda_E$, there are no intersections, there exists only one T-region $(0 < r < \infty)$, corresponding to the type $M_1$ – a monotonic unbounded expansion from the singularity $r = 0$ with an asymptotic approach to the de Sitter type S.

The latter are determined only by the local characteristics of the distribution of matter—the rest mass $\mathcal{M}'(\chi)$ of a given layer, and do not depend on the remaining matter in contrast to the TBL models where, as in a Newtonian sphere, only the external spherical layers of ''dust'' do not affect the dynamics of the properties of the internal region. In the T-models the inhomogeneity of the density of the ''dust'' and of the velocity field of the longitudinal deformations of $V_3$ are so interconnected that a homogeneous distribution of matter $\rho = \rho(\tau)$ is not only a necessary but a sufficient criterion for a complete metrical homogeneity of the field $V_4$ and this is in accordance with the Birkhoff theorem concerning the impossibility of it having a free radiation part.

In the general case the solution of (13) can be expressed explicitly in parametric form in terms of the Weierstrass elliptic functions with the invariants $g_2 = 1/12$ and $g_3 = 1/216 - \Lambda r_0^2/48$:

$$r = \frac{r_0}{4}\{\wp(\eta) - \wp(\eta_0)\}^{-1}, \quad \tau = \frac{1}{\dot{\wp}(\eta_0)}\left[2\eta\zeta(\eta_0) + \log\frac{\sigma(\eta-\eta_0)}{\sigma(\eta+\eta_0)}\right],$$

$$e^{\omega/2} = \frac{\dot{\wp}(\eta)}{\wp(\eta) - \wp(\eta_0)}\left\{\lambda + \mathcal{M}'\int^{\eta}\frac{[\wp(\eta) + g_2]^2}{4\wp^3(\eta) - g_2\wp(\eta) - g_3}d\eta\right\}, \quad (16)$$

where $\wp(\eta_0) = -g_2$, $\dot{\wp} = d\wp/d\eta$. The integral in (16) can also be evaluated in terms of elliptic functions by means of factorization if one finds the roots of the characteristic equations $4z^3 - g_2 z - g_3 = 0$, but the resulting expression is too awkward and is not reproduced here. The requirements of $r(\eta) \geq 0$ and of the reality of the time $\tau$ in (16) determine the region of physically admissible values of the parameter $\eta$ in the complex plane.

For certain particular values of the parameters $r_0 = 2M$ and $\Lambda$

$$\frac{2M}{r} + \frac{\Lambda r^2}{3} - 1 = 0$$

degenerates into a linear one $(\Lambda = 0, M > 0)$ or into a quadratic one $(M = 0, \Lambda = 3\alpha^2 > 0)$, or has a multiple root $(9M^2\Lambda = 1)$ the solution of (13) can be written in terms of elementary functions. In particular, the last case of a double root $r_E = 3M > 0$, $\Lambda_E = 1/r_E2 = 3\alpha_E^2$,

$$\begin{aligned} r(\tau) &= r_E[1 + x(\tau)], \\ \alpha_E\tau &= \ln[x + 2 + \sqrt{x^2 + 4x + 3}] \\ &\quad - \frac{1}{\sqrt{3}}\ln\frac{1}{|x|}[3 + 2x + \sqrt{3(x^2 + 4x + 3)}], \\ e^{\omega/2} &= \frac{x}{\sqrt{3}}\sqrt{\frac{x+3}{x+1}}\left\{\lambda + \frac{\mathcal{M}'}{\sqrt{3}}\left[\frac{2x^2 + x - 3}{2x^2}\sqrt{\frac{x+1}{x+3}}\right.\right. \\ &\quad \left.\left. - \frac{1}{2\sqrt{3}}\ln\frac{1}{|x|}(3 + 2x + \sqrt{3(x^2 + 4x + 3)})\right]\right\}, \end{aligned} \quad (17)$$

corresponds to the asymptotic types of Lemaitre-Eddington[21,24] $A_1$ in the region $(-1 \leq x < 0, 0 \leq r < r_E)$ and $A_2$ for $(0 < x < \infty, r_E < r < \infty)$ with an unbounded time scale $(-\infty < \tau < \infty)$ for the transverse expansion of the hypercylinder $V_3$.

3. In the course of a detailed investigation of the dynamics and of the time singularities of the T-models of a ''sphere'' we shall restrict ourselves in the main to the simple and the most important case of the T-collapse of ''dust'' with $\Lambda = 0$, $M > 0$, for which the

solution of (13) is expressed in a convenient parametric form:

$$r=\frac{r_0}{2}(1-\cos\eta),\quad \tau=\frac{r_0}{2}(\eta-\sin\eta),$$
$$e^{\omega/2}=\varepsilon\operatorname{ctg}\frac{\eta}{2}+\mathscr{M}'\left(1-\frac{\eta}{2}\operatorname{ctg}\frac{\eta}{2}\right). \tag{18}$$

The cycloidal dependence of the radius of curvature $r(\tau)$ and the temporal behavior of the peripheral dimensions of the hypercylinder, and together with them of the proper volume elements of $V_3$ and of the density of the "dust" in general terms remind one of the closed Friedmann model[1,3]; the phase of general expansion starts from the singular state, and is then replaced, also simultaneously throughout the whole space, by unlimited compression. At the singular points $\eta = 2\pi n$, where n is an integer, the metric becomes degenerate—$r(\tau) = 0$, and the density of matter $\rho \to \infty$, so that these singularities prevent continuation of the solutions (a formal continuation of the metric into the region $r(\tau) < 0$ differ only by a reversal of the sign of $r_0 < 0$). The proper time for the existence of T-models of such an oscillating type $O_1$ with $-\infty < \Lambda < \Lambda_E$ is restricted in both directions, and the period $\Delta\tau = \pi r_0$ for $\Lambda = 0$ and tends to infinity as one approaches the type $A_1(\Lambda \to \Lambda_E)$.

The dynamics of longitudinal deformations of the hypercylinder $V_3$ can be regarded as the result of a superposition of pulsations of initial T-regions of the Schwarzchild field and of aperiodic monotonic motions of the "dust" in a quasiisotropic T-model, and in this case one can no longer restrict oneself in (18) to a single cycle $(0 < \eta < 2\pi)$, as in the closed Friedmann model, but one has to consider all intervals of permissible values of the angular parameter in the region $(-\infty < \eta < \infty)$ which satisfy the requirement $\exp(\omega/2) > 0$.

The nature of the variation of the longitudinal dimensions of the layer does not necessarily coincide with the transverse compression of expansion of $V_3$, and as a result of the inhomogeneity of the velocity of the deformations along the "axis" of the hypercylinder all different combinations of their common behavior are simultaneously possible, and this results in leading to a fairly broad class of permissible motions of the "dust" in the T-models of a "sphere."

The replacement of phases of compression of the longitudinal dimensions of the hypercylinder $V_3$ by those of expansion in the general case occurs nonsimultaneously, and is not even necessary. For example, for the T-collapse of the "dust" (18) with $\epsilon = +1$ in the interval $(0 < \eta < 2\pi)$ the regular minimum for the longitudinal distances between Lagrangian spheres on $V_3$ exists only for $\mathscr{M}'(\chi) > \frac{1}{2}\pi$, and for the layer with $\mathscr{M}' = 2/\pi$ there exists an instant of general instananeous rest of the "dust" $\eta = \tau_1$. A monotonic unrestricted compression along the generators of $V_3$ necessarily leads to the appearance of additional intermediate singularities $\exp[\omega(\chi, \tau^*)] = 0$, which, generally speaking, are reached nonsimultaneously by particles of the different "liquid" spheres and obviously correspond to the contraction of individual parts or of the whole hypercylinder $V_3$ into a spherical $\delta$ layer $S_2$. Along these caustics the density of the "dust" becomes infinite, and in going through a singularity changes sign[13] (and does this an infinite number of times in the interval $-\infty < \eta < \infty$).

Near the singularities $r = 0$ the term containing $\Lambda$ becomes nonessential, and, moreover, the behavior of the metric of the anisotropic T-models $(\lambda \neq 0)$ is also independent of the presence of matter. It turns out to be exactly similar to the nonsimultaneous anisotropic collapse of the Tolman models (7), (8) with $\tau_0(\chi) = \text{const}$[1,7]: at each point of $V_3$ the peripheral distances decrease as $\tau^{2/3}$, the radial lengths increase indefinitely $\sim\tau^{1/3}$, and as a result of this the hypercylinder $V_3$ contracts into a line, its proper volume elements tend to zero $\sim\tau$, and the density tends to infinity according to the inverse law. The nature of the special homogeneous solution with $\lambda = 0$ in the neighborhood of one of the singular points $r = 0$ ($\eta = 0$ in (18)) does not differ from the case of the quasiisotropic simultaneous collapse of the Tolman models (7), (8) with $\tau_0 = \text{const}$[1,7] (which become homogenized near the singularity $R(\chi, \tau_0) = 0$): all the linear dimensions tend to zero according to the same law $\sim\tau^{2/3}$, and an instantaneous contraction of the whole distribution of matter within $V_3$ into a point occurs. The density of the "dust" becomes infinite in accordance with the same law $\kappa\rho = \frac{4}{3}\tau^2$ as in the isotropic Friedmann models since the anisotropy of the tensor of the velocities of the deformations disappears, and the anisotropy of the curvature of $V_3$ does not affect the dynamics of the collapse.

4. The existence of time singularities $\rho \to \infty$ is a general property of the cosmological T-models for $\Lambda \leq 0$, in accordance with the Landau-Raychaudhuri[25] theorem concerning the inevitable development of a fictitious singularity in the synchronous system and its necessary transformation into a physical one for a synchronously comoving system due to the focussing of the geodesic world lines for particles on a caustic which always exists for a normal time-like congruence of geodesics in Einstein gravitational fields $V_4$ if $\Lambda \leq 0$. But in the case $\Lambda > 0$, i.e., cosmological repulsion, this theorem is no longer applicable and, in particular, closed Friedmann models (9) having no singularities are possible: the Einstein static model E and dynamic types $A_2$, $M_2$. Therefore, the problem of singularities in the corresponding T-models of a "sphere" of similar types $A_2$ and $M_2$, when the geometric singularity of the vacuum SSK metrics (5), (14) at the "center" $R = 0$ has a spatial character $(M < 0)$, or is situated beyond the boundaries of the initial T-region ($M_2$ and $A_2$ with $0 < \Lambda < \Lambda_E$), requires a separate investigation.

Very instructive in this connection is the simple example of T-models due to de Sitter-Lanczos with $M = 0$, $\Lambda = 3\alpha^2 > 0$:

$$r=\frac{1}{\alpha}\operatorname{ch}\alpha\tau$$
$$e^{\omega/2}=\varepsilon\operatorname{sh}\alpha\tau+\mathscr{M}'[(\tfrac{1}{2}\pi-\operatorname{arc\,tg}\operatorname{sh}\alpha\tau)\operatorname{sh}\alpha\tau-1], \tag{19}$$

which also belong to the monotonic regular type $M_2$ and are constructed on the basis of T-regions of an everywhere regular space-time $V_4$ of constant positive curvature with the metric (5), (14) with $M = 0$. Although the metric (19) with $\epsilon = 0$, $\mathscr{M}' > 0$ and $\epsilon = -1$,

$\mathscr{M}' > 1/\pi$ is regular over the whole time interval $(-\infty < \tau < \infty)$, this group of solutions is physically unacceptable, since for them $\exp(\omega/2) < 0$ and $\rho < 0$, while the remaining T-models have a singularity of the "disc" type $\exp[\omega(\chi, \tau^*)] = 0$ for $r(\tau^*) > 1/\alpha$. If $\epsilon = +1$, then all the $\mathscr{M}' > 0$ are admissible, and the phase of the transverse compression of the hypercylinder $(-\infty < \tau < 0)$ is not realized, while its general unrestricted expansion begins with the "disc" singularity and asymptotically becomes isotropic as $\tau \to \infty$: $\exp(\omega/2) \alpha r(\tau) \alpha \exp(\alpha\tau)$. For the physically admissible solutions with $\epsilon = -1$ and $\mathscr{M}' < 1/\pi$ the phase of the transverse expansion of $V_3$ is not realized, and general compression begins with $r = \infty$ and continues for an infinite time up to the singularity $\tau = \tau$ $\tau = \tau^*(\chi)$. One could expect that all the physically acceptable T-models of a "sphere" with $\rho > 0$ of regular types $A_2$ and $M_2$ must in analogy with the deSitter-Lanczos T-models have a "disc" singularity $\exp[\omega(\chi, \tau)] = 0$, although for $M \neq 0$ this question remains open since it requires an investigation of a sufficiently complicated time dependence of the radial component of $\exp(\omega/2)$ in (16) and the determination of its zeros. We note that the infinite types $M_1$, $A_2$, and $M_2$ of the cosmological T-models of a "sphere" approach asymptotically for $r(\tau) \to \infty$ the deSitter S type and, therefore, they also become isotropic in the course of unbounded expansion.

One can easily show that also in vacuo the singularities $r = 0$ of the solution of (13) with $\mathscr{M}' \equiv 0$ (which is equivalent to the nonstatic parts of the SSK metric (14) in T-regions) are true ones for $M \neq 0$, since the canonical invariants of Petrov $-\frac{1}{2}\alpha_1 = \alpha_2 = \alpha_3 = r_0/2r^3$ and the Kretschmann scalar $J = C_{ijkl}C^{ijkl} = 12r_0^2/r^6$ constructed from the Weyl tensor of conformal curvature become infinite. If $M > 0$ these geometric singularities at the "centre" $T = 0$ have a timelike character, are always situated in T-regions (14) on spacelike hypersurfaces and correspond to an anisotropic collapse of invariant cross sections of $V_3 = (S_2 \times R_1)$ into a line, so that they cannot be identified with a localized point mass. But for $M < 0$ they have a spacelike nature, are situated in R-regions (5) and become unaccessible for geodesics of any synchronous system (2).

In contrast to the pseudosingularities of the SSK metrics (5), (14) at the boundaries of the T-regions of the type of the Schwarzchild sphere $R = T = 2M$ for $\Lambda = 0$ (where $r_0/r + \Lambda r^2/3 - 1 = 0$, $\exp(\omega/2)\alpha(\tau - \tau_0) \to 0$ and the hypercylindrical spatial cross sections of $V_3$ degenerate into a sphere $S_2$) the singularity at the "centre" $R = T = 0$ is nonremovable with the exception of the cases of the Minkowski-deSitter spacetime with $M = 0$. We note that the metric of the factorizable spacetime (6) gives one more example of an everywhere regular vacuum Einstein field with $\Lambda \neq 0$.

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