

DYNAMICS OF ELECTROACOUSTIC WAVES IN FLUIDS AND PLASMA

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Wave processes are studied with account taken of electrostriction effects in the high-frequency electromagnetic field. We consider the development of a parametric instability by a plane electromagnetic wave due to the interaction of the radiation with perturbations of density in a fluid with a positive dielectric constant. We also consider the transfer of electromagnetic radiation by acoustic perturbations in a plasma with a negative dielectric constant.

1. INTRODUCTION. BASIC EQUATIONS

THE term electroacoustic wave was first introduced by Volkov<sup>[1]</sup>, who showed that small modulations of a traveling electromagnetic wave in a plasma are unstable owing to the interaction of the electromagnetic wave with density oscillations. It was subsequently found that this effect is a particular case of a general class of nonlinear unstable electromagnetic waves of large amplitude, the instability being due to the dependence of the refractive index on the wave intensity. Phenomena of this kind lead to self-focusing and self-modulation of light.<sup>[2-4]</sup> The two most widely known mechanisms associated with this phenomena are the well-known optical Kerr effect and the electrostriction effect; these two effects lead to qualitatively different nonlinear phenomena.<sup>[5,6]</sup> The instability first studied by Volkov is due to electrostriction effects.

The present work is devoted to a further investigation of nonlinear electroacoustic waves. We start from the general equations for nonstationary electroacoustic processes, these equations consisting of the hydrodynamic equations and Maxwell's equations for media in which the dielectric constant  $\epsilon(\omega, \rho)$  is a slowly varying function of time, owing to which the density in the medium is a functional of the nonstationary intensity of the wave field  $\rho = \rho\{|\mathbf{E}|^2\}$ . Certain important features of these equations are discussed below in the present section. In this work we shall not take account of dissipative effects nor spatial dispersion.

In Sec. 2 we consider the development of the parametric instability of an electromagnetic wave in media in which  $\epsilon > 0$ . We investigate the basic kinds of nonlinear stationary electroacoustic waves that can propagate in such media, in particular, solitary waves. It is shown that the latter can be divided into two different classes (in our terminology, the optical class and the acoustic class). A qualitative analysis carried out in Sec. 2 shows the relation between nonlinear acoustic waves and parametric instabilities associated with electrostriction.

In Sec. 3 we study nonlinear electroacoustic processes in media with negative dielectric constant in situations in which rather intense electromagnetic waves with varying amplitude are incident on such media. The results of this section lead to the conclusion that it is possible to propagate nonlinear rarefac-

tion waves (density) filled by the high-frequency electromagnetic field that is stored within them.

We start with the Maxwell equations

$$\text{rot } \vec{\mathcal{H}} = \frac{1}{c} \frac{\partial \vec{\mathcal{D}}}{\partial t}, \quad \text{rot } \vec{\mathcal{E}} = -\frac{1}{c} \frac{\partial \vec{\mathcal{H}}}{\partial t}, \tag{1.1}$$

where the induction  $\vec{\mathcal{D}}(t)$  is related to the field  $\vec{\mathcal{E}}$  by the expression

$$\vec{\mathcal{D}}(t) = \vec{\mathcal{E}}(t) + \int_0^\infty f(\tau; t) \vec{\mathcal{E}}(t - \tau) d\tau; \tag{1.2}$$

where the function  $f(\tau, t)$  defines the dielectric constant for the nonstationary medium, as introduced by Pitaevskii.<sup>[7]</sup>

$$\epsilon(\omega; t) = 1 + \int_0^\infty f(\tau; t) e^{-i\omega\tau} d\tau + \text{c.c.} \tag{1.3}$$

If the properties of the medium vary slowly in time but the field is monochromatic, i.e.,  $(\partial f/\partial t \ll \partial f/\partial \tau)$

$$\vec{\mathcal{E}} = 1/2[\mathbf{E}(t)e^{-i\omega t} + \text{c.c.}], \quad \vec{\mathcal{H}} = 1/2[\mathbf{H}(t)e^{-i\omega t} + \text{c.c.}], \tag{1.4}$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are slowly varying functions (as compared with the phase factor), as an approximation we can write

$$\begin{aligned} \frac{\partial \vec{\mathcal{D}}}{\partial t} &= \frac{1}{2} \left[ \frac{\partial \epsilon}{\partial t} \mathbf{E} - i\omega \epsilon \mathbf{E} + \frac{i}{2} \frac{\partial^2(\omega \epsilon)}{\partial \omega^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\partial(\omega \epsilon)}{\partial \omega} \frac{\partial \mathbf{E}}{\partial t} \right] e^{-i\omega t} + \text{c.c.}, \\ \frac{\partial \vec{\mathcal{H}}}{\partial t} &= \frac{1}{2} \left[ -i\omega \mathbf{H} + \frac{\partial \mathbf{H}}{\partial t} \right] e^{-i\omega t} + \text{c.c.} \end{aligned} \tag{1.5}$$

In obtaining Eq. (1.5) we have neglected terms that contain products of the time derivatives.

The dielectric properties in the medium vary (in the effects considered below) together with the density  $\rho = \rho(t, \mathbf{r})$ . As shown in<sup>[7]</sup>, we can write

$$\epsilon = \epsilon(\omega; \rho) + \frac{i}{2} \frac{\partial^2 \epsilon}{\partial \omega \partial \rho} \frac{\partial \rho}{\partial t}; \tag{1.6}$$

where  $\epsilon(\omega, \rho)$  is the dielectric constant of the stationary medium, in which we take  $\rho = \rho(t)$ . Substituting Eqs. (1.4)–(1.6) in Maxwell's equations (1.1) we obtain the following basic equations for the field amplitude in the approximation used here:

$$\begin{aligned} c \text{rot } \mathbf{H} &= -i\omega \epsilon \mathbf{E} + \frac{1}{2} \omega \frac{\partial^2 \epsilon}{\partial \omega \partial \rho} \frac{\partial \rho}{\partial t} \mathbf{E} + \frac{\partial \epsilon}{\partial \rho} \frac{\partial \rho}{\partial t} \mathbf{E} \\ &+ \frac{\partial(\omega \epsilon)}{\partial \omega} \frac{\partial \mathbf{E}}{\partial t} + \frac{i}{2} \frac{\partial^2(\omega \epsilon)}{\partial \omega^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \end{aligned} \tag{1.7}$$

$$c \text{rot } \mathbf{E} = i\omega \mathbf{H} - \partial \mathbf{H} / \partial t. \tag{1.8}$$

These equations are supplemented by the hydrodynamic equations, which govern the behavior of  $\rho$ :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{f}_e, \quad (1.9)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (1.10)$$

where  $\mathbf{f}_e$  is the force density of the electromagnetic field which, in accordance with the results of<sup>[7]</sup>, can be written in the form<sup>1)</sup>

$$\mathbf{f}_e = \frac{1}{16\pi} \left\{ \nabla \left[ |\mathbf{E}|^2 \rho \left( \frac{\partial \epsilon}{\partial \rho} \right)_s \right] - |\mathbf{E}|^2 \nabla \epsilon \right\}. \quad (1.11)$$

In this expression and in all other formulas [in particular in Eq. (1.6)] we must regard all derivatives with respect to  $\rho$  as being at constant entropy  $S$  since we neglect dissipative processes in the present analysis.

Taking the scalar product of Eqs. (1.7) and (1.8) with  $\mathbf{E}^*$  and  $\mathbf{H}^*$  respectively, we then subtract one from the other and add the complex conjugate of equations, thus obtaining

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \frac{\partial(\omega \epsilon)}{\partial \omega} |\mathbf{E}|^2 + |\mathbf{H}|^2 - \frac{\partial^2(\omega \epsilon)}{\partial \omega^2} \operatorname{Im} \left( \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E}^* \right) \right] \\ &= -\frac{\partial \epsilon}{\partial \rho} \frac{\partial \rho}{\partial t} |\mathbf{E}|^2 - c \operatorname{div}([\mathbf{E} \cdot \mathbf{H}] + [\mathbf{E} \mathbf{H}^*]). \end{aligned} \quad (1.12)^*$$

The first two terms in the left side of Eq. (1.12) correspond to the energy density of the electromagnetic field in a dispersive medium.<sup>[9]</sup> The last term describes the variation of energy density associated with the variation of wave frequency. The first term on the right has the meaning of work per unit time for a change in the energy density of the electromagnetic field in the medium, this work being done by the hydrodynamic forces. This expression has been obtained by Pitaevskii<sup>[7]</sup> from somewhat different considerations.

In accordance with Eqs. (1.9) and (1.10), the increment in hydrodynamic energy is given by

$$\frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \rho U \right) = \mathbf{v} \mathbf{f}_e - \operatorname{div} \left[ \mathbf{v} \left( \frac{\rho v^2}{2} + p + \rho U \right) \right], \quad (1.13)$$

where  $U$  is the internal energy of the medium per unit mass while  $\mathbf{f}_e$  is the force density (1.11) acting on the medium as exerted by the field. Adding Eqs. (1.12) and (1.13), after some simple transformations we obtain the following equation, which expresses the conservation of the total energy of the medium and the field:

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \frac{1}{16\pi} \left[ \frac{\partial(\omega \epsilon)}{\partial \omega} |\mathbf{E}|^2 + |\mathbf{H}|^2 - \frac{\partial^2(\omega \epsilon)}{\partial \omega^2} \operatorname{Im} \left( \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E}^* \right) \right] \right. \\ & \left. + \frac{\rho v^2}{2} + \rho U \right\} = -\operatorname{div} \left\{ \frac{c}{16\pi} [[\mathbf{E} \cdot \mathbf{H}] + [\mathbf{E} \mathbf{H}^*]] \right. \\ & \left. + \mathbf{v} \left( \frac{\rho v^2}{2} + p + \rho U + \frac{|\mathbf{E}|^2}{16\pi} \rho \frac{\partial \epsilon}{\partial \rho} \right) \right\}. \end{aligned} \quad (1.14)$$

We now wish to consider in greater detail the equations for the wave field. Eliminating  $\mathbf{H}$  from Eqs. (1.7) and (1.8) and limiting ourselves to the one-dimensional case we have

$$\begin{aligned} & \left( \omega^2 \epsilon + c^2 \frac{\partial^2}{\partial x^2} \right) E + i \frac{\partial(\omega^2 \epsilon)}{\partial \omega} \frac{\partial E}{\partial t} - \frac{1}{2} \frac{\partial^2(\omega^2 \epsilon)}{\partial \omega^2} \frac{\partial^2 E}{\partial t^2} \\ & + i \omega \left( \frac{1}{2} \omega \frac{\partial^2 \epsilon}{\partial \omega \partial t} + 2 \frac{\partial \epsilon}{\partial t} \right) E = 0. \end{aligned} \quad (1.15)$$

Everywhere in what follows we shall assume that the perturbation of the density  $\rho$  away from the equilibrium value  $\rho_0$  is small i.e.,

$$(\rho - \rho_0) / \rho_0 \ll 1. \quad (1.16)$$

Thus, we can write

$$\begin{aligned} \epsilon(\omega; \rho) &= \epsilon_0 + \frac{\partial \epsilon}{\partial \rho_0} (\rho - \rho_0); \\ \epsilon_0 &= \epsilon(\omega, \rho_0); \quad \frac{\partial \epsilon}{\partial t} = \frac{\partial \epsilon}{\partial \rho_0} \frac{\partial \rho}{\partial t}. \end{aligned} \quad (1.17)$$

Since the rate of change of the density is of the order of the rate of change of the complex amplitude  $\partial \mathbf{E} / \partial t$ , we have

$$\partial(\rho - \rho_0) / \partial t \ll \omega(\rho - \rho_0). \quad (1.18)$$

Substituting Eq. (1.17) in Eq. (1.15) we have

$$\begin{aligned} & \left[ \omega^2 \epsilon_0 + \omega^2 \frac{\partial \epsilon}{\partial \rho_0} (\rho - \rho_0) + c^2 \frac{\partial^2}{\partial x^2} \right] E \\ & + i \frac{\partial(\omega^2 \epsilon_0)}{\partial \omega} \frac{\partial E}{\partial t} - \frac{1}{2} \frac{\partial^2(\omega^2 \epsilon_0)}{\partial \omega^2} \frac{\partial^2 E}{\partial t^2} = 0, \end{aligned} \quad (1.19)$$

where we have neglected terms of higher order than  $\omega^2(\rho - \rho_0)E/\rho_0$ .

Equation (1.19) together with the hydrodynamic equations serves as the original set for describing electroacoustic waves in the fluid. We shall limit our analysis to transverse waves and consider in detail two cases:  $\epsilon(\omega, \rho_0) > 0$  (the medium can support the propagation of traveling electromagnetic waves) and  $\epsilon(\omega, \rho_0) < 0$  (in the linear approximation the medium cannot support the propagation of traveling waves).

## 2. ELECTROACOUSTIC WAVES WITH $\epsilon_0 > 0$

When  $\epsilon_0 > 0$  it is convenient to rewrite the expression for the field amplitude  $E$ , replacing Eq. (1.4) by

$$\mathcal{E} = \frac{1}{2} [E(t, x) e^{i(kx - \omega t)} + \text{c.c.}]; \quad (2.1)$$

where  $k$  is the "unperturbed" wave number, defined by the relation

$$c^2 k^2 = \epsilon_0 \omega^2. \quad (2.2)$$

In Eq. (1.19) we are to understand the substitution  $E \rightarrow E e^{i\mathbf{k} \cdot \mathbf{x}}$ ; in view of Eq. (2.2) we then have

$$\begin{aligned} & i \left( \frac{\partial E}{\partial t} + u \frac{\partial E}{\partial x} \right) + \frac{u}{2k} \frac{\partial^2 E}{\partial x^2} \\ & - \frac{1 - ku'}{2ku} \frac{\partial^2 E}{\partial t^2} + \frac{uk}{2\epsilon_0} \left( \rho_0 \frac{\partial \epsilon}{\partial \rho_0} \right) \mathbf{v} E = 0. \end{aligned} \quad (2.3)$$

Here and below,

$$\mathbf{v} = (\rho - \rho_0) / \rho_0, \quad (2.4)$$

while  $u$  is the group velocity of a wave characterized by frequency  $\omega$  in the unperturbed medium:<sup>2)</sup>

<sup>2)</sup> It will be useful to have the following formulas that derive from (2.2):

$$\frac{d(\omega^2 \epsilon_0)}{d\omega} = \frac{2kc^2}{u} \quad \frac{d^2(\omega^2 \epsilon_0)}{d\omega^2} = \frac{2c^2(1 - ku')}{u^2};$$

We also recall that for gases  $\rho_0(\partial \epsilon_0 / \partial \rho_0) = \epsilon_0 - 1$ .

\* $[\mathbf{E} \cdot \mathbf{H}] \equiv \mathbf{E}^* \cdot \mathbf{H}$ .

<sup>1)</sup> In the case of a plasma where,  $\epsilon = 1 - \omega_p^2 / \omega^2$ , the expression in (1.11) coincides with the corresponding force defined in [8].

$$u = \left( \frac{\partial \omega}{\partial k} \right)_{\rho=\rho_0}; \quad u' = \left( \frac{\partial u}{\partial \omega} \right)_{\rho=\rho_0} \quad (2.5)$$

Equation (2.3) together with the hydrodynamic equations constitute the initial system for nonlinear electro-acoustic waves in the region  $\epsilon_0 > 0$ .

In a number of cases it is convenient to write the equation for the field in hydrodynamic form. For this purpose we write

$$E = ae^{i\varphi}, \quad (2.6)$$

where  $a$  is a positive quantity; taking real and imaginary parts in (2.3) we have

$$-(\varphi_t + u\varphi_x) + \frac{u}{2k} \left[ \frac{a_{xx}}{a} - \varphi_x^2 \right] - \frac{1 - ku'}{2ku} \left[ \frac{a_{tt}}{a} - \varphi_t^2 \right] + \frac{uk}{2\epsilon_0} \left( \rho_0 \frac{\partial \epsilon}{\partial \rho_0} \right)_v = 0; \quad (2.7)$$

$$(a^2)_t + u(a^2)_x + \frac{u}{k} (a^2\varphi_x)_x - \frac{1 - ku'}{ku} (a^2\varphi_t)_t = 0. \quad (2.8)$$

We now wish to consider the stability of the nonlinear plane wave described by Eqs. (1.9), (1.10), (2.7) and (2.8). Assuming that all quantities representing deviations from the initial plane wave are proportional to  $\exp[i(\kappa x - \Omega t)]$ , we obtain the following dispersion equations:

$$\left[ (\Omega - \kappa u)^2 - \frac{u^2 \kappa^4}{4k^2} \right] (\Omega^2 - c_s^2 \kappa^2) = \frac{u^2 \kappa^4}{4k^2 \tau^2}, \quad (2.9)$$

where

$$\tau^2 = \frac{8\pi c^2}{E_0^2 \omega^2 \rho_0 (\partial \epsilon / \partial \rho_0)^2}, \quad c_s^2 = \left( \frac{\partial p}{\partial \rho} \right)_s. \quad (2.10)$$

The quantity  $\tau$  has the dimensions of time and, as will be evident below, defines the characteristic time for the nonlinear parametric processes;  $c_s$  is the velocity of sound in the medium. In deriving Eq. (2.9) we have neglected terms of order  $\Omega/kc$ , which are unimportant in the stability investigation.

We first consider Eq. (2.9) for small values of  $\kappa$ . In this case it is easy to see that the roots of Eq. (2.9) are given by

$$\Omega_{1,2} = \pm c_s \kappa \left\{ 1 + \frac{u^2}{c_s^2 \tau^2 [4k^2 (u \mp c_s)^2 - \kappa^2 u^2]} \right\}^{1/2}, \quad (2.11)$$

$$\Omega_{3,4} = \kappa u \mp \frac{u \kappa^2}{2k} \left[ 1 + \frac{1}{(u^2 - c_s^2) \kappa^2 \tau^2} \right]^{1/2}. \quad (2.12)$$

These formulas have been obtained under the assumption that

$$c_s^2 \tau^2 k^2 (1 - c_s/u)^2 \gg 1, \quad (2.13)$$

which is equivalent to placing an upper bound on the amplitude of the initial electromagnetic wave  $E_0$ .

Equations (2.11) and (2.12) are valid for sufficiently large values of  $\kappa$  which, however, must satisfy the condition

$$|\kappa/k| \ll 1, \quad (2.14)$$

which derive from the condition of validity for the initial equations (2.7) and (2.8) (in deriving these equations we have assumed that  $\varphi_x \ll k$ ). In the region in which they apply, Eqs. (2.11) and (2.12) describe perturbations that propagate with velocities that are respectively close to the acoustic velocity ( $\pm c_s$ ) and the electromagnetic group velocity ( $\pm u = \pm \partial \omega / \partial k$ ). For

this reason, we call these branches the acoustic branch and the optic branch.

It follows from Eq. (2.11) that when  $u \gg c_s$  the dispersion equation for the perturbation of the wave that obtained by Volkov<sup>[1]</sup> for the plasma case:<sup>3)</sup>

$$\Omega = \pm c_s \kappa [1 + 1/c_s^2 \tau^2 (4k^2 - \kappa^2)]^{1/2}. \quad (2.15)$$

However, the instability region found by Volkov in<sup>[1]</sup> on the basis of Eq. (2.15) lies outside the region of applicability of his approximation, which is subject to the condition in (2.14), as is the present analysis. The more exact expression given above (2.11) is not suitable for this purpose since the region in which the roots are complex lies outside its region of validity. A correct investigation of the instability region must be carried out using the original fourth-order equation (2.9).

We consider two cases separately:  $u > c_s$ ,  $u < c_s$ ,

1.  $u > c_s$ . It is shown in the Appendix that the instability region in this case is given by the inequalities

$$0 \leq \left( 1 - \frac{c_s}{u} \right) - \alpha \left( 1 - \frac{c_s}{u} \right)^{-1/2} \leq \frac{\kappa}{2k} \leq \left( 1 - \frac{c_s}{u} \right) + \alpha \left( 1 - \frac{c_s}{u} \right)^{-1/2}, \quad (2.16a)$$

$$-\left( 1 - \frac{c_s}{u} \right) - \alpha \left( 1 - \frac{c_s}{u} \right)^{-1/2} \leq \frac{\kappa}{2k} \leq -\left( 1 - \frac{c_s}{u} \right) + \alpha \left( 1 - \frac{c_s}{u} \right)^{-1/2}, \quad (2.16b)$$

where

$$\alpha^2 = 1/4uc_s k^2 \tau^2. \quad (2.17)$$

The expressions for the roots that assume complex values in the regions described by (2.16a) and (2.16b) are of the form

$$\Omega_{1,3} = \kappa c_s + \frac{\kappa(u - c_s) - \kappa^2 u / 2k}{2} \mp \frac{1}{2} \left\{ \left[ \kappa(u - c_s) - \frac{\kappa^2 u}{2k} \right]^2 - \frac{\alpha^2 \kappa^2 u^2}{1 - c_s/u} \right\}^{1/2}, \quad (2.18a)$$

$$\Omega_{1,4} = \kappa c_s + \frac{\kappa(u - c_s) + \kappa^2 u / 2k}{2} \mp \frac{1}{2} \left\{ \left[ \kappa(u - c_s) + \frac{\kappa^2 u}{2k} \right]^2 - \frac{\alpha^2 \kappa^2 u^2}{1 - c_s/u} \right\}^{1/2}. \quad (2.18b)$$

When  $\alpha = 0$  Eq. (2.18a) taken with the appropriate signs becomes the expression for  $\Omega_1$  and  $\Omega_3$  given by Eqs. (2.11) and (2.12) where we also take  $\alpha = 0$ . Correspondingly, when  $\alpha = 0$  Eq. (2.18b) is converted into the expressions for  $\Omega_1$  and  $\Omega_4$  given by Eqs. (2.11) and (2.12). Thus, we see that an instability appears in the present case when the acoustic and optical branches approach each other closely enough in some region ( $1 - c_s/u \approx |\kappa| 2k > 0$ ). The maximum growth rate is reached when  $\kappa = \kappa_0^{(\pm)}$ , where

$$\kappa_0^{(\pm)} = \pm 2k(1 - c_s/u). \quad (2.19)$$

The magnitude of the maximum growth rate is then given by

<sup>3)</sup>For a plasma (where  $\epsilon = 1 - \omega_0^2/\omega^2$ ) the quantity  $\tau$  defined by Eq. (2.10) corresponds with the appropriate nonlinear time  $\tau = 2(2\pi\rho_0)^{1/2} c\omega/E_0\omega_0^2$  introduced in [1]. We also note that the oscillation branch given by our equation (2.12) does not appear in [1] since the approximation used there corresponds to  $u \rightarrow \infty$ .

$$\text{Im } \Omega(\kappa_0^{\pm}) = \frac{1}{2\tau} \left( \frac{u}{c_s} - 1 \right)^{1/2} \quad (2.20)$$

2.  $u < c_s$ . In this case, as follows from the analysis given in the Appendix, the instability can only appear when

$$0 \leq |\kappa| \leq (c_s^2 - u^2)^{-1/2\tau^{-1}}, \quad (2.21)$$

while the corresponding frequencies are given by Eqs. (2.12), which define the optical branch. The maximum growth rate in this case obtains when

$$\kappa_0^{-1} = \pm \sqrt{2\tau} (c_s^2 - u^2)^{1/2} \quad (2.22)$$

being given by

$$\text{Im } \Omega(\kappa_0) = \frac{u}{4k\tau^2(c_s^2 - u^2)}. \quad (2.23)$$

The instability given by Eqs. (2.12) and (2.21)–(2.23) is identical with the instability of a plane wave in a medium in which the nonlinear effects are due to the Kerr effect (cf. for example the reviews<sup>[10-12]</sup> and also<sup>[13]</sup>). Obviously in the paper by Volkov<sup>[1]</sup> an instability of this kind does not appear (cf. remark in footnote<sup>3)</sup>).

All of the analysis given above only describes the initial state in the instability process. An investigation of the further development of the instability at longer times lies beyond the framework of the linear approximation for the perturbation of the initial wave. Qualitative considerations similar to those given in<sup>[14,15]</sup> lead to the conclusion that as a result of the instability there will develop soliton envelopes<sup>[10-12]</sup> which are associated with stationary solutions of Eqs. (2.7), (2.8) and the hydrodynamic equations.

Let us now consider in greater detail the structure of these solitons. With reasonable limitations, which will be indicated below [cf. Eq. (2.39)], the hydrodynamic can be linearized, yielding the following results in the one-dimensional case:

$$\nu_{tt} = c_s^2 \nu_{xx} = c_s^2 (a^2)_{xx} / E_c^2. \quad (2.24)$$

Here,  $\nu$  is the relative density variation [cf. Eq. (2.4)] and  $E_c^2$  denotes the quantity

$$E_c^2 = \frac{16\pi c_s^2}{|\partial \epsilon / \partial \rho_0|} = 2E_0^2 \tau^2 \omega^2 \rho_0 \left| \frac{\partial \epsilon}{\partial \rho_0} \right| \frac{c_s^2}{c^2} \quad (2.25)$$

( $E_c$  characterizes the field level at which the interaction between electromagnetic and acoustic waves becomes important).

We now consider the stationary solutions of Eqs. (2.7), (2.8), and (2.24), which are described by expressions of the form

$$a = a(x - wt), \quad \nu = \nu(x - wt), \quad \varphi = Ct + \varphi_1(x - wt), \quad (2.26)$$

where  $C$  is a fixed frequency shift due to the nonlinear processes. Substituting Eq. (2.26) in Eqs. (2.7) and (2.8) we have

$$\frac{\partial \varphi_1}{\partial x} = -\frac{k}{u} \frac{(a^2 - a_0^2)(u - w)}{a^2}. \quad (2.27)$$

The remaining expressions are found to be different for the two cases:  $u > c_s$  and  $u < c_s$ .

1)  $u > c_s$ . In this case

$$a^2 = a_0^2 - (a_0^2 - a_{min}^2) \text{sech}^2[(x - wt)/\delta], \quad (2.28)$$

$$\delta = 2\tau \left[ \frac{w^2 - c_s^2}{1 - a_{min}^2/a_0^2} \right]^{1/2}, \quad (2.29)$$

$$C = 0, \quad \nu = \nu_0 + \frac{c_s^2}{E_c^2} \frac{a^2 - a_0^2}{w^2 - c_s^2}, \quad (2.30)$$

where  $\nu_0$  is the relative density variation in a plane wave characterized by amplitude  $a_0 = |E_0|$ . Thus, in the case being considered the field amplitude and the density within the soliton are always smaller than the corresponding values in the plane wave.<sup>4)</sup> In this case the velocity of propagation of this rarefaction wave is determined from the Equation

$$(w^2 - c_s^2)(w - u)^2 = \frac{a_{min}^2}{a_0^2} \frac{u^2}{4k^2\tau^2}. \quad (2.31)$$

(It follows in particular, that  $w^2 > c_s^2$ .) Equation (2.31) for the nonlinear stationary wave is analogous to the dispersion equation (2.9) and can be studied in the same way (cf. Appendix). It is found that if  $u$  and  $c_s$  satisfy Eq. (2.31) then all four roots of the equation are real. Correspondingly, in this case we have two kinds of solitons, the acoustic soliton whose velocity is approximately

$$w_a^2 \approx c_s^2 + \frac{a_{min}^2}{a_0^2} \frac{u^2}{4k^2\tau^2(u - c_s)^2} \quad (2.32)$$

and the optical soliton whose velocity is approximately

$$w_o \approx u \pm \frac{a_{min}}{a_0} \frac{u}{2k\tau(u^2 - c_s^2)^{1/2}}. \quad (2.33)$$

Substituting these expressions in Eq. (2.29) we obtain the following expressions for the reciprocal length of the acoustic and optical solitons:

$$\delta_a^{-1} \approx \left( \frac{a_0^2}{a_{min}^2} - 1 \right)^{1/2} k \left( 1 - \frac{c_s}{u} \right), \quad (2.34)$$

$$\delta_o^{-1} \approx \frac{1}{2\tau u} \left( 1 - \frac{a_{min}^2}{a_0^2} \right) \left( 1 - \frac{c_s}{u} \right)^{-1}. \quad (2.35)$$

2).  $u < c_s$ . In this case

$$\varphi = -\frac{A^2}{4E_c^2} \frac{c_s^2}{c_s^2 - u^2} \left( \nu_0 \frac{\partial \epsilon}{\partial \rho_0} \right) \frac{uk}{\epsilon_0} t, \quad (2.36)$$

$$a = A \text{sech} \left( \frac{x - wt}{\delta} \right), \quad \delta = \frac{2a_0\tau}{A} (c_s^2 - u^2)^{1/2}, \quad (2.37)$$

$$\nu = -\frac{c_s^2}{c_s^2 - u^2} \frac{a^2}{E_c^2}. \quad (2.38)$$

In a soliton described by these relations, as the field amplitude increases the density of the medium is reduced; the amplitude of the wave is the maximum at the center of the soliton while the density is, correspondingly, a minimum. At infinity  $a \rightarrow 0$ ,  $\nu \rightarrow 0$ . The velocity of the soliton is equal to the group velocity of the wave  $u$  so that it can also be regarded as optical (however, it is fundamentally different from the optical soliton for  $u > c_s$ , which is considered below).

The soliton described by Eqs. (2.36)–(2.38) is qualitatively completely analogous to the solution of the envelope in a medium where nonlinear effects are due to the Kerr effect and where the wave, under these conditions is unstable against self-modulation. In this

<sup>4)</sup>The term "soliton" is used here in place of the more exact term solitary perturbation of a plane wave (since  $a \rightarrow a_0$  when  $x \rightarrow \pm\infty$ ).

connection we recall that the instability that occurs when  $u < c_s$  [for the condition given in Eq. (2.21)] is also qualitatively analogous to the self-modulation instability due to the Kerr effect.

If we now introduce the results obtained in<sup>[14,15]</sup> we find that the development of the instability described by Eq. (2.21)–(2.23) leads to the decay of the wave and to separate wave packets which are qualitatively like the soliton described by Eqs. (2.36)–(2.38). The amplitude of these waves is of the order of the amplitude of the original wave and their dimensions, as is evident from Eq. (2.37), is of the order of the length of the perturbation responsible for the maximum growth rate.

When  $u > c_s$  the length of the acoustic solitons determined by Eq. (2.34) is of the same order of magnitude (when  $a_0 \sim a_{\min}$ ) as the length corresponding to the maximum growth rate (2.19). A qualitative analysis is similar to that carried out in<sup>[14,15]</sup> leads to the conclusion that the result of the development of the instability for  $u > c_s$  is the formation of acoustic solitons.

We now consider briefly the case of a medium in which  $c_s \rightarrow 0$ . A simple example of such a medium is a "cold" plasma. In this case the dispersion equation is the equation given by Volkov (2.15) where  $c_s = 0$ . This equation was first obtained by Gorbunov.<sup>[16]</sup> It follows from this equation that there is no instability region in the case at hand.

The approximation used by Gorbunov is equivalent to neglecting terms with  $\varphi_{xx}$  and  $a_{xx}$  in Eqs. (2.7) and (2.8). Correspondingly, in his approximation it is not possible to obtain solitons. It follows from the results given above that in this case only the optical solitons exist [(Eqs. (2.28)–(2.30), (2.33) and (2.35)] where we must take  $c_s = 0$ .

In concluding this section we now discuss the condition under which the linearized hydrodynamic equations apply. The expressions for the density of the solitons (2.30) and (2.38) yield limitations on the parameters of the medium and the amplitude in the original wave:

$$\frac{u^2 \rho_0}{|u^2 - c_s^2| \epsilon_0} \left| \frac{\partial \epsilon_0}{\partial \rho_0} \right| \gg 1, \quad \frac{a_0^2}{16\pi} \ll \frac{|u^2 - c_s^2|}{|\partial \epsilon_0 / \partial \rho_0|}. \quad (2.39)$$

If we now make use of the condition

$$|1 - c_s / u| \ll 1, \quad (2.40)$$

which follows from the inequality  $k^{-1} \ll \delta$  and Eqs. (2.34) and (2.37), the second relation in (2.39) is satisfied when  $a_0^2 \ll E_c^2$ .

### 3. ELECTROACOUSTIC WAVES WITH $\epsilon_0 < 0$

In the linear approximation an electromagnetic field can penetrate such a medium only to a distance of the order of the skin depth  $\mu^{-1}$  where

$$\mu^2 = -\omega^2 \epsilon_0 / c^2. \quad (3.1)$$

The structure of the nonlinear stationary skin depth (i.e., in which the amplitude of the incident wave does not change with time) has been studied in<sup>[16,17,18]</sup> for the plasma case, in which it has been shown, in particular, that the depth of penetration of the field is not less

than  $1/\mu$ .<sup>5)</sup> This result can be easily generalized for an arbitrary medium characterized by  $\epsilon < 0$ .

We consider here nonstationary processes that occur when the amplitude of the field incident on the medium with negative  $\epsilon_0$  varies in time. In this case the changes in field amplitude and density in the medium lead to the possibility of propagation of nonlinear electroacoustic pulses which, penetrating into the depth of the medium, carry the electromagnetic field with them. The characteristic dimensions of such pulses, as will be evident below, is of order  $\mu^{-1}$  while their velocity does not exceed the velocity  $c_s$ .

We introduce the notation that will be used in the present section. The frequency at which  $\epsilon_0$  changes sign is denoted by  $\omega_0$  [ $\epsilon_0(\omega_0) = 0$ ]. If the incident wave is close to  $\omega_0$  we can write

$$\epsilon_0(\omega) \approx \beta(\omega^2 - \omega_0^2) / \omega^2 \quad \text{if } (\omega_0 - \omega) / \omega \ll 1, \quad (3.2)$$

where  $\beta$  is a factor of order unity. We will assume that spatial dispersion can be neglected. Furthermore, we make the assumption that  $T$ , the characteristic time for the variation of the complex amplitude of the field  $E$ , is large compared with  $(\omega_0 - \omega)^{-1}$ . Introducing (3.1) and (3.2) we can write this condition in the form

$$T \gg \omega / (\mu c)^2. \quad (3.3)$$

As we have done everywhere above, we assume that the amplitude in the field is small ( $|E|^2 / E_c^2 \ll 1$ ). Under these conditions the spatial scales of the variation of amplitude density and other slowly varying quantities, as we have already noted, will be of order  $L \sim \mu^{-1}$ . It then follows from Eq. (3.3) that in the basic equations for the field (1.19) the time derivatives are small compared with the term  $c^2 \partial^2 E / \partial x^2$  and can be neglected.

Now, substituting  $E = ae^{i\varphi}$  in Eq. (1.19) we obtain the basic equations for the field in the form

$$\partial(a^2 \varphi_x) / \partial x = 0, \quad (3.4)$$

$$a_{xx} - \frac{\mu^2}{v^2} (\gamma^2 + \nu) a - a \varphi_x^2 = 0, \quad (3.5)$$

where we have used the notation

$$\gamma^2 = -\epsilon_0 / |\rho_0 \partial \epsilon / \partial \rho_0|, \quad \nu = (\rho - \rho_0) / \rho_0. \quad (3.6)$$

From Eq. (3.4) we have  $a^2 \varphi_x = P(t)$  where  $P(t)$  is an arbitrary function of time. In the present formulation of the problem the field  $E(x, t)$  must vanish when  $x \rightarrow +\infty$  so that  $P(t) = 0$ . Thus, we obtain the first basic equation in the form

$$a_{xx} - \mu^2 \gamma^2 (\gamma^2 + \nu) a = 0, \quad (3.7)$$

where  $a^2(x, t) = |E|^2$ . The second basic equation is the linearized hydrodynamic equation (2.24). The condition that must be satisfied for linearization, as will be shown below (cf. footnote<sup>6)</sup> below) is of the form

<sup>5)</sup>Here we have in mind the skin depth for a transverse wave. The nonlinear theory for the stationary skin depth for the longitudinal wave has been investigated in [19].

<sup>6)</sup>The validity of linearizing the hydrodynamic equations derives primarily from Eq. (3.9) if we require that  $\nu \ll 1$ . In particular, under these conditions we obtain (3.8).

$$\gamma^2 = (\omega_0^2 - \omega^2) / \omega_0^2 \ll 1, \tag{3.8}$$

that is to say, the frequency of the incident wave must be close to the threshold frequency  $\omega_0$ .

We consider first the stationary waves that satisfy Eqs. (3.7) and (2.24). Making the substitutions

$$v = v(x - wt), \quad a = a(x - wt),$$

in these equations we have

$$v = -\frac{a^2 - a_0^2}{E_c^2(1 - M^2)} - \gamma^2, \tag{3.9}$$

$$\frac{2a_0^2}{\kappa^2} a_{xx} - a(a_0^2 - a^2) = 0, \tag{3.10}$$

where  $a_0$  is a constant of integration and

$$M = w / c_s, \quad \kappa^2 = 2\mu^2 a_0^2 / \gamma^2 E_c^2 (1 - M^2). \tag{3.11}$$

The quantity  $M$  might be called the electroacoustic Mach number. The bounded solutions of Eqs. (3.10) are of the form

$$a^2 = a_2^2 - (a_2^2 - a_1^2) \operatorname{sn}^2 \left( \frac{\kappa}{2} (x - wt) \sqrt{\frac{a_2^2 - a_1^2}{a_0^2 q^2}}, q^2 \right), \tag{3.12}$$

where

$$q^2 = \frac{(a_2^2 - a_1^2)(a_0^4 - a_2^2 a_1^2)}{(a_2^2 - a_0^2)(2a_0^4 - a_1^2 a_0^2 - a_2^2 a_1^2)}, \tag{3.13}$$

while the constants  $a_2$  and  $a_1$  must satisfy one of the following equations:

$$a_2^2 + a_1^2 = 2a_0^2 \quad \text{or} \quad a_1 = 0, \quad a_2^2 \geq 2a_0^2. \tag{3.14}$$

It follows from Eq. (3.12) that

$$\max a^2(x, t) = a_2^2, \quad \min a^2(x, t) = a_1^2. \tag{3.15}$$

The wavelengths and the frequencies that follows from Eq. (3.12) are determined by the relations

$$\lambda = \frac{4a_0 K(q^2)}{a_2 \kappa}, \quad \Omega = \frac{2\pi c_s M}{\lambda}, \tag{3.16}$$

where  $K(q^2)$  is an elliptic integral of the first kind. When  $a_1 \rightarrow 0$  and  $a_2^2 \rightarrow 2a_0^2$  we have

$$q \rightarrow 1, \quad 1 - \operatorname{sn}^2(z, q) \rightarrow \operatorname{sech}^2 z, \quad \lambda \sim -\ln|1 - a_2^2 / 2a_0^2| \rightarrow \infty, \tag{3.17}$$

That is to say, in this case the periodic wave approaches a sequence of solitons.

We now consider the structure of these solitons in greater detail. It follows from the definition of  $v$  that  $v \rightarrow 0$  as a function of distance from the soliton. We then have from Eq. (3.9)<sup>6)</sup>

$$a_0^2 = \gamma^2 E_c^2 (1 - M^2), \quad \kappa^2 = 2\mu^2. \tag{3.18}$$

Introducing new notation for the soliton amplitude we have<sup>7)</sup>

$$a(x, t) = a_m \operatorname{sech}[\mu(x - c_s M t)], \tag{3.19}$$

$$v(x, t) = -2\gamma^2 \operatorname{sech}^2[\mu(x - c_s M t)], \tag{3.20}$$

$$a_m^2 = 2\gamma^2 E_c^2 (1 - M^2). \tag{3.21}$$

Thus, all of the solitons are of the same length and have amplitudes that are uniquely related to the electroacoustic Mach number by (3.21).

We wish to consider further the limiting case of stationary waves that are obtained when  $a_1 = a_2 = a_0$ :

$$a(x, t) = a_0, \quad v(x, t) = -\gamma^2. \tag{3.22}$$

The solution in (3.22) describes a wave of constant amplitude. It is easy to be convinced, however, that this wave is unstable. If we substitute the following expressions in Eq. (3.7) and (2.24)

$$a(x, t) = a_0 + \delta a \exp[i(\kappa x - \Omega t)], \tag{3.23}$$

$$v(x, t) = -\gamma^2 + \delta v \exp[i(\kappa x - \Omega t)],$$

where  $\delta a$  and  $\delta v$  are the amplitudes of the small perturbations, we obtain the following dispersion equation:

$$\Omega^2 = c_s^2 (\kappa^2 - \kappa_0^2), \quad \kappa_0^2 = 2\mu^2 a_0^2 / \gamma^2 E_c^2. \tag{3.24}$$

It then follows that perturbations characterized by  $\kappa^2 < \kappa_0^2$  grow exponentially. Precisely the same considerations obtain for the stability of the periodic stationary wave such as those given in (3.12), at least in the limiting case  $a_2 - a_1 \ll a_0$ . On the other hand, it can be shown<sup>[20]</sup> that solitons (at least of sufficiently low amplitude when the Mach number is close to unity) are stable.

We now wish to consider certain characteristic features of the processes that arise when a modulated wave of large amplitude is incident on a medium characterized by  $\epsilon_0 < 0$ . We remark first of all that the basic equations (3.7) and (2.24) satisfy solutions that describe two kinds of waves: 1) electroacoustic waves in which the variation in relative density  $v$  is connected with  $a$  in such a way that if  $v \neq 0$  in the vicinity of a given point then  $a \neq 0$ . A simple example of these waves is given by the solitons (3.19)–(3.21); 2) pure acoustic waves in which  $v \neq 0$ ,  $a = 0$  [it will be evident that these solutions satisfy the basic equations (3.7) and (2.24)]. The velocity of propagation of the latter is  $c_s$ . The velocity of the first waves, the electroacoustic waves, is smaller than the acoustic velocity (the simplest example is given by the stationary electroacoustic waves considered above).

We are now in a position to obtain the general form of the asymptotic behavior of the electroacoustic waves that propagate in an unperturbed medium. If we assume  $v$  and  $a$  are small at the leading edge of such a wave we can neglect the last term in (3.7); in this case the asymptote of the solution for  $x \rightarrow +\infty$  is of the form

$$a(x, t) = A(t) e^{-\mu x}, \tag{3.25}$$

where  $A(t)$  is an arbitrary function of time. The asymptote of the solitons in (3.19) is a particular case of (3.25).

Excitation in the skin layer by a modulated wave of large amplitude leads, first of all, to the emission of acoustic waves which propagate with velocity  $c_s$ . Beyond these there move with a smaller velocity the electroacoustic waves given by the asymptotic expression (3.25). A detailed investigation of the nature and evolution of these waves will be carried out in a separate work. Here we note that a preliminary investigation of this question gives some reason for believing that the final result of this evolution is the formation of electroacoustic solitons (3.19)–(3.21).

In conclusion, we consider some of the consequences

<sup>7)</sup>If we take  $M = 0$  and  $x \rightarrow x + \text{const}$  in Eqs. (3.19)–(3.21) we obtain formulas that describe the structure of the nonlinear skin layer for a fixed amplitude of the incident wave such as those obtained by Silin. [18]

of the general relations given above for a plasma. For example, let us consider a nonisothermal plasma characterized by  $T_e > T_i$  (for reasons of simplicity we write  $T_i = 0$ ) in which case  $c_s^2 = T_e/m$ ; and the dimensions of the soliton are determined, in accordance with Eq. (3.19), by the relation

$$\delta = \mu^{-1} = c / (\omega_0^2 - \omega^2)^{1/2}. \quad (3.26)$$

In this case the amplitude of the soliton (3.21) is of the form

$$a_m = \sqrt{2} [(\omega_0^2 - \omega^2)(1 - M^2)]^{1/2} E_c / \omega_0, \quad (3.27)$$

$$E_c^2 = 16\pi\rho_0 c_s^2 \omega^2 / \omega_0^2.$$

When  $M = 0$  this quantity is  $2E_p$  where  $E_p$  is the limiting value of the field amplitude of the incident wave on the plasma for which the plasma can still exert a skin effect; this relation has been obtained by Silin [(cf. <sup>[18]</sup>, Eq. (3.9))].

The lifetime of electroacoustic waves in the plasma is limited by dissipation effects. The latter are due primarily to ion-ion collisions and Landau damping for the ion acoustic waves, and secondarily to electron-ion collisions which cause damping of the high-frequency field. The Landau damping for the ion-acoustic waves with characteristic length  $\mu^{-1}$  is given by

$$\gamma_\alpha = \mu c_s \sqrt{\frac{\pi m_e}{8 m_i}}. \quad (3.28)$$

This quantity is approximately 100 times smaller than the characteristic frequency for electroacoustic waves, which is of order  $\mu c_s$ . Hence, if the plasma is of low-density, so that the damping due to ion-ion collisions is smaller than the Landau damping, the damping of the ion-acoustic wave in the plasma can be regarded as small.

The order of magnitude of the damping of the high-frequency electromagnetic field due to electron-ion collisions is <sup>[21]</sup>

$$\gamma_\omega \approx \frac{1}{2} \pi \frac{e^4 n_0}{T_e^{3/2} m_e^{1/2}} \ln \left( 0.37 \frac{T_e}{e^2 n_0^{2/3}} \right), \quad (3.29)$$

where  $n_0$  is the unperturbed electron density. If this quantity is small compared with  $\mu c_s$  i.e.,

$$\frac{\gamma_\omega}{\mu c_s} \sim \frac{e^4 n_0}{T_e^{3/2} m_e^{1/2}} \left( \frac{m_i}{m_e} \right)^{1/2} \ln \left( 0.37 \frac{T_e}{e^2 n_0^{2/3}} \right) \ll 1, \quad (3.30)$$

this damping mechanism will become effective only at distances that are significantly greater than the characteristic dimensions of the electroacoustic waves.

It is possible that the formation of solitons inside a plasma with  $\omega < \omega_0$  has been observed in experiments carried out by Levin and Khodataev<sup>[22]</sup> but a more detailed quantitative comparison of the results obtained here with the experimental data would require taking account of the bounded dimensions of the system used in these experiments. A calculation of this kind is beyond the framework of the present paper.

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#### APPENDIX

We note, first of all, that Eqs. (2.11) and (2.12) hold when

$$\kappa^2 / k^2 \ll |c_s - u|^2 / u^2. \quad (A.1)$$

In this case Eq. (2.11) does not lead to complex roots while Eq. (2.12) will have complex roots when

$$u < c_s, \quad \kappa^2 / k^2 < 1 / |c_s^2 - u^2| \tau^2 k^2. \quad (A.2)$$

This is the only unstable region for the condition in (A.1).

We note further that if  $|c_s - u| \gtrsim u$  then (A.1) always holds in the region of applicability of the present analysis by virtue of (2.14). Now, let  $|c_s - u| \ll u$  i.e.,

$$c_s / u \sim 1. \quad (A.3)$$

The condition in (A.1) is not satisfied when

$$|c_s - u|^2 / u^2 \lesssim \kappa^2 / k^2, \quad (A.4)$$

in which the right side of this inequality is bounded by (2.14).

We now write Eq. (2.9) in the form

$$z(z + 2c_s / u)(z - a)(z - b) = (2c_s / u)a^2, \quad (A.5)$$

where

$$z = \frac{\Omega / \kappa - c_s}{u}, \quad a = 1 - \frac{c_s}{u} + \frac{\kappa}{2k}, \quad b = 1 - \frac{c_s}{u} - \frac{\kappa}{2k}, \quad (A.6)$$

where  $\alpha$  is a parameter given by Eq. (2.17).

The right side of Eq. (A.5) is very small; by virtue of Eqs. (2.13) and (A.4) the condition that (A.1) be unsatisfied is given by

$$4(c_s / u)a^2 \ll |1 - c_s / u|^2 \lesssim \kappa^2 / k^2. \quad (A.7)$$

If we write  $\alpha = 0$ , the roots of Eq. (A.5) become

$$z_1(0) = 0; \quad z_2(0) = -2c_s / u, \quad z_3(0) = a, \quad z_4(0) = b. \quad (A.8)$$

For values of  $\alpha$  that satisfy Eq. (A.7) the roots  $z_1(\alpha)$  will not be very different from the values in (A.8) in which the complex roots can only appear in the case in which some of the roots fuse in the region given by (A.7). In turn, for this to happen it is necessary that the corresponding values in (A.8) be sufficiently close together. It will be evident that the root  $z_2(\alpha)$  cannot fuse with one of the other roots so that it cannot be complex. Furthermore, since we have

$$\frac{c_s}{u} \sim 1, \quad a \sim \frac{1}{uk\tau}, \quad |c_s - u| \gtrsim \frac{1}{k\tau}, \quad (A.9)$$

it follows from this relation and (A.4) that

$$|b - a| = |\kappa / k| \gtrsim a. \quad (A.10)$$

Thus

$$|z_4(\alpha) - z_3(\alpha)| \gtrsim a. \quad (A.11)$$

The relations in (A.10) and (A.11) eliminate the possibility that  $z_1(\alpha) \sim z_4(\alpha) \sim z_3(\alpha)$  so that we need only consider the case in which  $z_1(\alpha) \sim \alpha$  and

$$\begin{aligned} \text{a) } & |z_1(\alpha)| \sim |z_4(\alpha)| \ll |z_3(\alpha)| \ll |z_2(\alpha)|, \\ \text{b) } & |z_1(\alpha)| \sim |z_3(\alpha)| \ll |z_4(\alpha)| \ll |z_2(\alpha)|. \end{aligned}$$

In case a) only the roots  $z_{1,4}(\alpha)$  can become complex. In order to find these roots we replace Eq. (A.5) by the quadratic equation

$$z(z - b) + a^2 / 2a = 0. \quad (A.12)$$

Assuming that  $1 - c_1 / u \approx \kappa / 2k + O(\alpha)$  in the present case we can, with a desired degree of accuracy, replace the quantity  $a$  by  $2(1 - c_s / u)$ . As a result we

find that complex roots will appear only when  $u > c_S$ . In this case the values of  $\kappa$  must lie in the range given by Eq. (2.16a) while the frequencies are given by (2.18a) (2.18a).

In case b) the complex roots can be determined approximately from the equation  $z(z - a) + \alpha^2/2b = 0$ . Proceeding as above, we again find  $u > c_S$  as well as Eq. (2.18b) and the inequality in (2.16b).

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