# RADIATION SPECTRUM OF AN ELECTRON MOVING IN A CONSTANT ELECTRIC FIELD

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The energy spectrum of the classical radiation of an electron moving in a constant electric field is found and, in particular, the radiation spectrum of a uniformly accelerated charge is determined. The region on the particle's trajectory where radiation with a given wave vector **k** is generated is found, which allows one to indicate specific experimental conditions for observation of the spectrum.

## 1. INTRODUCTION

THE motion of a charge in a constant electric field in the particular one-dimensional case reduces to uniformly accelerated (hyperbolic) motion, characterized by constant acceleration in the co-moving Lorentz system. The electromagnetic field of a charge undergoing such motion was considered by Born[1] and Schott.[2] Using Born's solution, Pauli reached the conclusion that a charge does not radiate during hyperbolic motion[3] (also see von Laue[4]). On the other hand Schott[5] and later Drukey[6] and Bondi and Gold[7] reached the opposite conclusion. The most complete argumentation in favor of radiation is given in the articles by Fulton and Rohrlich.[8,9]

Here we cite and comment upon the basic formulas of the classical theory of radiation,[10] which will be used in what follows.

From the expressions for the electromagnetic field created by a moving charge[1)] (at the moment t′ the charge is located at the point $\mathbf{x}(t')$, has a velocity $\mathbf{v} = \dot{\mathbf{x}}(t')$, and an acceleration $\mathbf{w} = \ddot{\mathbf{x}}(t')$; the field is observed at the point $\mathbf{x} = \mathbf{x}(t') + \mathbf{R}$ at the moment $t = t' + R$),

$$\mathbf{E} = \frac{e(1-v^2)(\mathbf{R}-\mathbf{v}R)}{(R-\mathbf{R}\mathbf{v})^3} + \frac{e[\mathbf{R}[(\mathbf{R}-\mathbf{v}R)\mathbf{w}]]}{(R-\mathbf{R}\mathbf{v})^3}, \quad \mathbf{H} = \frac{[\mathbf{R}\mathbf{E}]}{R}, \tag{1*}$$

it is seen that if the charge is being accelerated then at large distances from it, namely for

$$R \gg (1-v^2)/w, \tag{2}$$

the field becomes a wave field (i.e., it is described by only the second term in (1) for which E = H; **E**, **H**, and **R** form a right-handed set of three orthogonal vectors, and the field falls off like $R^{-1}$) and consequently such a charge radiates.

The energy, radiated by the charge into a solid angle $d\Omega$ at the moment t′ during the time interval dt′, in the wave zone passes through the area element $R^2 d\Omega$ in the direction $\mathbf{n} = \mathbf{R}/R$ at the moment $t = t' + R$ during the time interval $dt = (1 - \mathbf{v}\cdot\mathbf{n})dt'$, and this energy is determined by the relation

$$d\mathscr{E}_\mathbf{n} = \frac{([\mathbf{E}\mathbf{H}]\mathbf{n})}{4\pi} R^2 d\Omega\, dt = \frac{e^2}{4\pi}\frac{[\mathbf{n}[(\mathbf{n}-\mathbf{v})\mathbf{w}]]^2}{(1-\mathbf{v}\mathbf{n})^6} d\Omega\, dt. \tag{3}$$

The first equality determines the total energy passing through the area element. The second equality determines the radiation energy and is written down for condition (2), when only the second (wave) term remains of the field (1). We shall be interested not in the intensity $d\mathscr{E}_\mathbf{n}/dt$ of the radiation registered by an observer, but in the intensity $d\mathscr{E}_\mathbf{n}/dt'$ of radiation from the source, i.e., the energy radiated per unit time by the particle.

It is not difficult to calculate the total (i.e., integrated over all directions **n**) intensity $d\mathscr{E}_\mathbf{n}/dt'$ of radiation from the source:

$$\frac{d\mathscr{E}}{dt'} = \frac{e^2}{4\pi}\int \frac{[\mathbf{n}[(\mathbf{n}-\mathbf{v})\mathbf{w}]]^2}{(1-\mathbf{v}\mathbf{n})^5} d\Omega = \frac{2}{3}e^2\left[\frac{\mathbf{w}^2}{(1-v^2)^2} + \frac{(\mathbf{w}\mathbf{v})^2}{(1-v^2)^3}\right] = \frac{2}{3}e^2 a^2 \tag{4}$$

($a_\mu$ denotes the particle's 4-acceleration[2)]). Thus, the energy radiated per unit time by a particle is positive, relativistically invariant, and proportional to the square of its 4-acceleration. The relativistic invariance of $d\mathscr{E}/dt'$ is a consequence of the symmetry of dipole radiation (see[10], Sec. 73) and allows one to perform the calculation in an arbitrary coordinate system. In the system where v = 0 the second equality in (3) holds for arbitrary R, and this means that one can establish the existence of radiation with regard to the field in any arbitrary neighborhood of the charge, and not just in the wave zone.[9]

For a charge moving in a constant electric field and, in particular, for a uniformly accelerated charge[3)] $a^2$ is a constant quantity, different from zero, and consequently such a charge radiates. However, Pauli[3] noted that the field of a charge undergoing hyperbolic motion does not have a wave zone at any distance from the source at the moment t = 0 when the velocity of the charge is equal to zero, and consequently there is no corresponding radiation. But the moment of observation t = 0 does not satisfy condition (2) for the wave zone

$$t - t' \gg \frac{1-v^2(t')}{w(t')}\Bigg|_{\text{hyperb. mot.}} = \sqrt{a^{-2}+t'^2}$$

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[1)]The Gaussian system of units is used; the velocity of light c = 1; the notation is $p_\mu = (\mathbf{p}, p_0)$, $pk = \mathbf{p}\cdot\mathbf{k} - p_0 k_0$; $u_\mu$ is the 4-velocity, $a_\mu = du_\mu/ds$ is the 4-acceleration and s is the proper time.

*[RE] ≡ R × E.

[2)]We recall that $a_\mu = \{(1-v^2)^{-1}\mathbf{w} + (1-v^2)^{-2}(\mathbf{v}\cdot\mathbf{w})\mathbf{v},\ (1-v^2)^{-2}\mathbf{v}\cdot\mathbf{w}\}$.

[3)]By definition uniformly accelerated motion means $\mathbf{w} \neq 0$, $\dot{\mathbf{w}} = 0$ in the system where the particle is at rest, or in covariant form $a^2 \neq 0$, $\Gamma_\mu = 0$ where $\Gamma_\mu = (da_\mu/ds) - u_\mu a^2$. For charged particles $(2/3)e^2\Gamma_\mu$ is the 4-force due to radiation damping. [10]

for any moment of emission t′; in any case the moment of observation t must be much larger than the reciprocal of the acceleration: $t \gg a^{-1}$.

## 2. RADIATION SPECTRUM OF AN ELECTRON MOVING IN A CONSTANT ELECTRIC FIELD

Let us consider the radiation spectrum of an electron moving in a constant electric field.

As is well known (see the problem in Sec. 66 of[10]) the radiation energy spectrum of an electron is determined by the formula

$$d\mathscr{E}_{\mathbf{k}} = |j_\mu(k)|^2 d^3k / 4\pi^2, \qquad (5)$$

where $j_\mu(k)$ is the Fourier component of the charge's current 4-vector:

$$j_\mu(k) = \frac{e}{m}\int_{-\infty}^{\infty} ds\, \pi_\mu(s) \exp\{-ik_\alpha x_\alpha(s)\}. \qquad (6)$$

Here $x_\alpha(s)$ and $\pi_\alpha(s) = m(dx_\alpha/ds)$ are the charge's 4-coordinate and 4-momentum, depending on the proper time s ($ds = \sqrt{1-v^2}\, dt'$) and obeying the equations of motion

$$\frac{d\pi_\mu}{ds} = \frac{e}{m} F_{\mu\nu}\pi_\nu, \quad x_\mu(s) = \int_0^s \frac{\pi_\mu(s)}{m} ds + x_\mu(0). \qquad (7)$$

For motion in an arbitrary constant field $F_{\mu\nu}$ one can represent the momentum $\pi_\mu$ in the form

$$\pi_\mu(s) = f_1 p_\mu + f_2 F_{\mu\nu} p_\nu + f_3 F_{\mu\nu}^* p_\nu + f_4 F_{\mu\nu} F_{\nu\lambda} p_\lambda, \qquad (8)$$

where $p_\mu$ is a constant 4-vector ($p^2 = -m^2$) determined by the condition $\pi_\mu(0) = p_\mu$ and which is the limiting value for $\pi_\mu$ when the field is turned off, $F^*_{\mu\nu} = (i/2)\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$, and the $f_k$ are four scalar functions of s:

$$f_1 = \frac{\varepsilon^2 \cos\eta s + \eta^2 \operatorname{ch}\varepsilon s}{\varepsilon^2+\eta^2}, \quad f_2 = \frac{e}{m}\frac{\varepsilon \operatorname{sh}\varepsilon s + \eta \sin\eta s}{\varepsilon^2+\eta^2},$$

$$f_3 = \frac{e}{m}\frac{\varepsilon \sin\eta s - \eta \operatorname{sh}\varepsilon s}{\varepsilon^2+\eta^2}, \quad f_4 = \left(\frac{e}{m}\right)^2 \frac{\operatorname{ch}\varepsilon s - \cos\eta s}{\varepsilon^2+\eta^2}. \qquad (9)$$

The quantities $\epsilon$ and $\eta$ appearing here are related to the well known field invariants $F = (1/4)F^2_{\mu\nu}$ and $G = (1/4)F_{\mu\nu}F^*_{\mu\nu}$ by the relations

$$\varepsilon = \frac{e}{m}(\sqrt{F^2+G^2} - F)^{1/2}, \quad \eta = \frac{e}{m}(\sqrt{F^2+G^2}+F)^{1/2}. \qquad (10)$$

For an electric field $\eta = 0$ and $\epsilon = eE/m$, where E is the magnitude of the electric field intensity, and we obtain

$$\pi_\mu(s) = p_\mu + \operatorname{sh}(\varepsilon s)\frac{F_{\mu\nu}p_\nu}{E} + (\operatorname{ch}\varepsilon s - 1)\frac{F_{\mu\nu}F_{\nu\lambda}p_\lambda}{E^2}. \qquad (11)$$

Concerning the phase $f(s) \equiv -ik_\alpha x_\alpha(s)$ appearing in the current $j_\mu(k)$, having represented it in the form of the integral of $-ik_\alpha\pi_\alpha(s)/m$ over s, and having simplified this last expression with the aid of the dimensionless invariant parameters z, $\xi$, and $\nu$ given by

$$z \operatorname{sh}\xi = \frac{F_{\mu\nu}p_\nu k_\mu}{eE^2}, \quad z \operatorname{ch}\xi = -\frac{F_{\mu\nu}F_{\nu\lambda}p_\lambda k_\mu}{eE^3},$$

$$\nu = \frac{1}{eE}\left(kp - \frac{F_{\mu\nu}F_{\nu\lambda}p_\lambda k_\mu}{E^2}\right), \qquad (12)$$

we obtain

$$f(s) = +i[z \operatorname{sh}(\varepsilon s - \xi) - \nu\varepsilon s + z\operatorname{sh}\xi]. \qquad (13)$$

Substituting the momentum (11) and the phase (13) into the expression for the current and evaluating the quadratic combination $|j_\mu(k)|^2$ which determines the spectrum, we obtain

$$j_\mu(k)j_\mu(-k) = e^2\left\{\left(\frac{(F_{\mu\nu}p_\nu)^2}{m^2E^2} - 1\right)|B_0|^2 + \frac{(F_{\mu\nu}p_\nu)^2}{m^2E^2}(|B_1|^2 - |B_2|^2)\right\}, \qquad (14)$$

where

$$B_{0,1,2} = \int_{-\infty}^{\infty} ds\, e^{f(s)}\{1, \operatorname{sh}\varepsilon s, \operatorname{ch}\varepsilon s\}.$$

It is obvious that the change of variable $\epsilon s - \xi = u$ expresses $B_0$ in terms of the function

$$R_\nu(z) = \int_{-\infty}^{\infty} du\, e^{i(z \operatorname{sh} u - \nu u)} = i\pi H^{(1)}_{i\nu}(iz) = 2e^{\nu\pi/2}K_{i\nu}(z), \qquad (15)$$

which to within a factor is a Hankel function of the first kind or a Macdonald function. For real z and $\nu$ this function is real. Since $B_1 = -i\partial B_0/\partial a$ and $B_2 = i\partial B_0/\partial b$ where $a = z\cosh\xi$ and $b = z\sinh\xi$, one can express the functions $B_{1,2}$ in terms of $R_\nu(z)$ and its derivative with respect to z, $R'_\nu(z)$. As a result we obtain the following expression for the spectrum $d\mathscr{E}_{\mathbf{k}}$:

$$d\mathscr{E}_{\mathbf{k}} = \frac{m^2}{4\pi^2E^2}\left\{\left[\left(1-\frac{\nu^2}{z^2}\right)\gamma^2 - 1\right]R_\nu^2(z) + \gamma^2 R_\nu'^2(z)\right\}d^3k, \qquad (16)$$

in which the invariant parameter

$$\gamma = \sqrt{(F_{\mu\nu}p_\nu)^2}/Em = \sqrt{m^2+p_\perp^2}/m \qquad (17)$$

characterizes the motion, and the invariant variables z and $\nu$ are related to the wave vector **k** and the momentum **p** by the equations

$$z = \frac{\sqrt{(F_{\mu\nu}p_\nu)^2(F_{\mu\nu}k_\nu)^2}}{eE^3} = \frac{\sqrt{m^2+p_\perp^2}\,k_\perp}{eE}, \quad \nu = \frac{\mathbf{p}_\perp\mathbf{k}_\perp}{eE} = \frac{p_\perp k_\perp}{eE}\cos\varphi; \qquad (18)$$

the subscript ⊥ denotes the component perpendicular to the field **E**, and $\varphi$ is the angle between $\mathbf{p}_\perp$ and $\mathbf{k}_\perp$. We note that $|\nu|$ is always less than or equal to z.

Formula (16) for the radiation spectrum of an electron in a constant electric field is the analogue of the well known Schott's formula for the radiation spectrum of an electron in a magnetic field. During motion in a constant electric field the square of the particle's 4-acceleration is a constant:

$$a^2 = \left(\frac{1}{m}\frac{d\pi_\mu}{ds}\right)^2 = \frac{e^2}{m^4}(F_{\mu\nu}p_\nu)^2 = \left(\frac{eE}{m}\right)^2\frac{m^2+p_\perp^2}{m^2}, \qquad (19)$$

so that the parameter $\gamma$ is nothing else but the dimensionless acceleration of the particle expressed in units of eE/m: $\gamma = am/eE$. As to the radiation damping force $(2/3)e^2\Gamma_\mu$, it is a function of s and vanishes only in the case $p_\perp = 0$ or $\gamma = 1$. Thus, in the special case $\gamma = 1$ (and consequently $\nu = 0$) formula (16) describes the radiation spectrum of a uniformly accelerated charge:

$$d\mathscr{E}_{\mathbf{k}} = \frac{m^2}{\pi^2E^2}K_0'^2(z)\,d^3k. \qquad (16')$$

A characteristic feature of the spectrum (16) is its independence of $p_{||}$ and $k_{||}$, the components of the vectors **p** and **k** parallel to the field **E**. This is a consequence of the fact that in a homogeneous constant field a change of these components is equivalent to a change of the time origin. Therefore, if we represent $d^3k$ in the form

$$d^3k = dk_{||}k_\perp dk_\perp d\varphi = (eE/m\gamma)^2 dk_{||} z\, dz\, d\varphi \qquad (20)$$

and integrate the spectrum over $k_{||}$, the resulting integral will diverge linearly. And what is more, since as $z \to 0$ the functions

$$R_\nu(z) \approx 2\ln(2/z) - 2C, \quad R_\nu'(z) \approx -2/z, \quad C = 0.577\ldots,$$

then the spectrum possesses a logarithmic singularity $d\mathscr{E}_{\mathbf{k}} = z^{-1}dz$ at the point $k_\perp = 0$ (or $z = 0$).

The divergence of the total radiation energy is not surprising since the intensity of the radiation $d\mathscr{E}/dt'$ is finite, see Eq. (4), and therefore the total radiation energy during the infinite time interval associated with the entire trajectory of the particle is infinite.

We shall show that the radiation associated with a definite value of the wave vector **k** is produced in a definite segment of the particle's trajectory, the so-called "coherence interval" whose position and length are proportional to $k_{\|}/k_\perp$. It is obvious that the coherence interval is that segment of the particle's trajectory on which the functions $R_\nu(z)$ and $R'_\nu(z)$ are formed, being integrals over the proper time s (or over $u = \epsilon s - \xi$). Let us consider the function $R_\nu(z)$ (see Eq. (15)) assuming that the acceleration is not too large, i.e., $\gamma \sim 1$.[4)] In this case the ratio $\nu/z \lesssim 1$, but not too close to unity so that $1 - (\nu/z)^2 \sim 1$. It is not difficult to see that for $z \ll 1$, in the integral over u which determines the function $R_\nu(z)$, values of u lying near $u = 0$ in an interval $\Delta u \sim \ln z^{-1}$ are effective. For $z \gtrsim 1$ one can estimate the effective range of integration over u by the saddle-point method. From such an estimate it follows that on the real axis, values of u lying close to Re $u_0$ ($u_0$ is the saddle point) in an interval $\Delta u \sim |\mathrm{Im}\, u_0| + |f''(u_0)|^{-1/2}$ are effective. For the function $R_\nu(z)$ the saddle point lies on the imaginary axis $u_0 = i\cos^{-1}(\nu/z)$ and $|f''(u_0)|^{-1/2} = (z^2 + \nu^2)^{-1/2}$. From here it is seen that for $z \gtrsim 1$ values of u lying close to $u = 0$ in an interval $\Delta u \sim \cos^{-1}(\nu/z) \sim 1$ will be effective in the integral (15). Consideration of the function $R'_\nu(z)$ leads to the same results.

Thus, the coherence interval is determined by values of the proper time s lying inside the interval $s_0 - \Delta \lesssim s \lesssim s_0 + \Delta$ with center at the point

$$s_0 = \frac{\xi}{\varepsilon} = \frac{1}{\varepsilon}\operatorname{Arsh}\frac{F_{\mu\nu}p_\nu k_\mu}{eE^2 z} = \frac{m}{eE}\left[\operatorname{Arsh}\frac{k_{\|}}{k_\perp} - \operatorname{Arsh}\frac{p_{\|}}{\sqrt{m^2 + p_\perp^2}}\right], \tag{21}$$

and length

$$\Delta \sim \frac{m}{eE}\begin{cases}\ln z^{-1} & \text{for} \quad z \ll 1\\ 1 & \text{for} \quad z \gtrsim 1\end{cases}. \tag{22}$$

Only the position of the interval depends on $k_{\|}$, its length does not depend on $k_{\|}$ and is practically a constant, of the order of m/eE. However, if we change to the time relative to the laboratory frame

$$t = x_0(s) = \frac{m}{eE}\left(-\frac{p_{\|}}{m} + \frac{p_{\|}}{m}\operatorname{ch}\varepsilon s + \frac{p_0}{m}\operatorname{sh}\varepsilon s\right),$$

then the picture becomes complicated due to the nonlinear dependence of t on s, and there is a substantial change of t when s passes through the interval $\Delta \sim \epsilon^{-1}$. As a result both the position

$$t_0 = t(s_0) = \frac{m}{eE}\left(-\frac{p_{\|}}{m} + \gamma\frac{k_{\|}}{k_\perp}\right), \tag{23}$$

and the length of the coherence interval

$$\Delta t = t(s_0 + \Delta) - t(s_0 - \Delta) = 2\frac{m}{eE}\gamma\frac{k_0}{k_\perp}\operatorname{sh}\varepsilon\Delta \tag{24}$$

depend on $k_{\|}$.

Thus, one can say that radiation with a given momentum **k** is emitted at the moment $t_0$ with an uncertainty $\Delta t$. Dividing the energy contained in the interval $\Delta k_{\|}$ of the spectrum by the corresponding "radiation time" $\Delta t_0 = m\gamma\Delta k_{\|}/eEk_\perp$, we obtain the invariant quantity

$$\frac{1}{\Delta t_0}\int_{\Delta k_{\|}} d\mathscr{E}_{\mathbf{k}} = \frac{e^4E^2}{4\pi^2m^2\gamma^4}\left\{\left[\left(1 - \frac{\nu^2}{z^2}\right)\gamma^2 - 1\right]R_\nu^2(z) + \gamma^2R_\nu'^2(z)\right\}z^2\,dz\,d\varphi, \tag{25}$$

which one can call the spectral distribution of the intensity of the radiation, provided the uncertainty $\Delta t$ in the determination of the radiation time $\Delta t_0$ is sufficiently small: $\Delta t \ll \Delta t_0$ for all $k_{\|}$ from the interval $\Delta k_{\|}$. Since, however, $\Delta t \sim \Delta t_0$ the spectrum of the intensity (25) is only determined to within terms of order unity. Thus, if expression (25) is integrated over z and $\varphi$ for the case $p_\perp = 0$ (corresponding to uniformly accelerated motion), then we obtain

$$\frac{d\mathscr{E}}{dt_0} = \frac{2e^4E^2}{\pi m^2}\int_0^\infty K_0'^2(z)z^2\,dz = \frac{9\pi}{32}\cdot\frac{2}{3}e^2\left(\frac{eF}{m}\right)^2,$$

which differs from the intensity (4) by the factor $9\pi/32$.

From the existence of a finite coherence interval it follows that in order to observe the spectrum in a given range of angles and frequencies, it is not necessary to realize motion over all space. For this purpose it is sufficient that the considered motion should be realized over a segment which should contain within it the coherence interval given by (23) and (24), and the segment should be many times larger than the coherence interval (in practice several times larger). Outside of this segment the motion may be arbitrary; it may lead to radiation with the same wave vector **k**, but this radiation will originate at a time differing substantially from $t_0 \pm \Delta t$, and one can always separate out this radiation by turning-on the spectrometer only during the time when the electron is traveling along the coherence interval given by Eqs. (23) and (24).

The discussion given here indicates that the radiation spectrum and the coherence length of a uniformly accelerated charge are not distinguished by anything and do not possess any special features distinguishing them from the general case of radiation in an electric field for $\gamma \neq 1$. We also note that in this general case the work done by the radiation damping force per unit time in the laboratory frame

$$\tfrac{2}{3}e^2\Gamma_0 ds/dt = -\tfrac{2}{3}e^2(a^2 - \varepsilon^2) \tag{26}$$

is constant and is not equal to the intensity of the radiation (with the opposite sign) $-(2/3)e^2a^2$, but approaches this value as $\gamma \equiv a/\epsilon$ increases.

Now let us consider the radiation spectrum $d\mathscr{E}_{\mathbf{k}}$ when the particle's acceleration is large: $\gamma \gg 1$. In this case, as we see below, large values of z and $\nu$ close to each other are important: z, $|\nu| \sim \gamma^3$, $1 - (\nu/z)^2 \sim \gamma^{-2}$. In this region one can replace the functions $R_\nu(z)$ and $R'_\nu(z)$ by their asymptotic values, which are obtained by the saddle-point method, taking second and third derivatives into account:

$$R_\nu(z) \approx 2\sqrt{\pi}\left(\frac{2}{\nu}\right)^{1/3}\Phi(y), \quad R_\nu'(z) \approx 2\sqrt{\pi}\left(\frac{2}{\nu}\right)^{2/3}\Phi'(y); \tag{27}$$

here $y = (\nu/2)^{2/3}(z^2/\nu^2 - 1)$, and $\Phi(y)$ is the Airy function

[4)]The case $\gamma \gg 1$, when $1-(\nu/z)^2 \ll 1$, reduces to the radiation of a charge in a crossed field; we consider it below.

defined by the integral

$$\Phi(y)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}du'\exp\left\{i\left(\frac{u'^3}{3}+yu'\right)\right\}, \quad (28)$$

in which the variable $u'$ is related to the proper time s or to the variable $u = \epsilon s - \xi$ (see Eq. (15)) by the relation $u' = (\nu/2)^{1/3}u + i(\nu/2)^{1/3}(\tan\alpha - \alpha)$ where $\alpha = \tan^{-1}[(z/\nu)^2 - 1]^{1/2}$.

Instead of z and $\varphi$ it is convenient to introduce new variables v and $\tau$:

$$v=\frac{z}{\gamma^3},\quad \tau=\frac{F_{\mu\nu}{}^*p_\nu k_\mu}{m\sqrt{(F_{\mu\nu}k_\nu)^2}}=\sqrt{\gamma^2-1}\sin\varphi, \quad (29)$$

whose effective values will be of the order of unity. Using (27) and (29) in formula (16) and taking into consideration that $\gamma \gg 1$, we obtain the following expression for the spectrum $d\mathscr{E}_{\mathbf{k}}$:

$$d\mathscr{E}_{\mathrm{k}}=\frac{m^2}{\pi\gamma^2E^2}\left(\frac{2}{v}\right)^{2/3}\left\{\tau^2\Phi^2(y)+\left(\frac{2}{v}\right)^{2/3}\Phi'^2(y)\right\}d^3k, \quad (30)$$

in which $y = (v/2)^{2/3}(1 + \tau^2)$, and in accordance with (20) and (29) one can use

$$d^3k=\left(\frac{eE}{m}\right)^2\gamma^3\,dk_{\parallel}\,v\,dv\,d\tau. \quad (20')$$

instead of $d^3k$. Of course, the fact that the spectrum, as before, does not depend on $k_{\parallel}$ here also leads to an infinite total energy but in contrast to expression (16) it possesses an integrable singularity $d\mathscr{E}_{\mathbf{k}} \sim v^{-1/3}dv$ at the point $k_\perp = 0$ (or $v = 0$).

The coherence interval is now determined by the region of formation of the Airy function and its derivative. Considering in (27) the relation between the integration variable $u'$ and the proper time s, it is not difficult to show that the coherence interval is determined by the same position of the center $s_0$ given by Eq. (21), but has a length which is $\gamma$ times smaller

$$\Delta\sim\frac{m}{eE}\left(\frac{2}{v}\right)^{1/3}\begin{Bmatrix}1 & \text{for} & y\ll 1\\ y^{1/2} & \text{for} & y\gtrsim 1\end{Bmatrix}\sim\frac{m}{eE}\frac{1}{\gamma}. \quad (31)$$

In terms of the time measured in the laboratory frame, the coherence interval is characterized by the same position $t_0$ given by Eq. (23), but the length

$$\Delta t\sim\frac{m}{eE}\cdot 2\frac{k_0}{k_\perp}.$$

Since $\gamma \gg 1$ the "radiation time" $\Delta t_0$ of energy in the interval $\Delta k_{\parallel}$ is now well determined: $\Delta t_0 \gg \Delta t$ if $\gamma\Delta k_{\parallel} \gg k_0$, and the quantity

$$dI_{\mathrm{k}}=\frac{1}{\Delta t_0}\int_{\Delta k_{\parallel}}d\mathscr{E}_{\mathrm{k}}=\frac{e^2}{\pi}\left(\frac{eE\gamma}{m}\right)^2\left(\frac{2}{v}\right)^{2/3}\left\{\tau^2\Phi^2(y)+\left(\frac{2}{v}\right)^{2/3}\Phi'^2(y)\right\}v^2\,dv\,d\tau \quad (32)$$

will be the spectral distribution of the radiation intensity. One can represent the integral over $\tau$ of this distribution in the form

$$dI=-\frac{e^2}{\sqrt{\pi}}\left(\frac{eE\gamma}{m}\right)^2\left\{\int_{v^{2/3}}^{\infty}dx\,\Phi(x)+\frac{2}{v^{2/3}}\Phi'(v^{2/3})\right\}v\,dv. \quad (33)$$

Carrying out the integration over v, we obtain the total intensity

$$I=-\frac{2e^2}{\sqrt{\pi}}\left(\frac{eE\gamma}{m}\right)^2\int_0^{\infty}dx\,x\Phi'(x)=\frac{2}{3}e^2\left(\frac{eE\gamma}{m}\right)^2, \quad (34)$$

in exact agreement with formula (4).

We note that for $\gamma \gg 1$ the electric field in the electron's rest system appears to be very close to a crossed field (i.e., very close to a field for which $\mathbf{E} \perp \mathbf{H}$, $E = H$). Indeed, one can show that the limiting formulas (29) and (32)–(34) for the energy spectrum and radiation intensity in an electric field are exact for the radiation in a crossed field provided that, in the invariant variables v, $\tau$, and in the parameter $eE\gamma/m = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^2$, by $F_{\mu\nu}$ one understands the field tensor of the crossed field.

The spectrum (16′) will always be observed when uniformly accelerated motion is realized on a segment which is much longer than the coherence interval. For example, the one-dimensional motion of an electron according to the law $x(t) = v\sqrt{(v/w)^2 + t^2}$ (v is the velocity of the charge for $t = \pm\infty$, w is the acceleration for $t = 0$) for v close to unity is very close to uniformly accelerated motion for times which are small in comparison with the characteristic time associated with a change of the momentum $t \ll t_\pi = v/w\sqrt{1 - v^2}$, and its radiation spectrum

$$d\mathscr{E}_{\mathrm{k}}=\frac{e^2v^4}{\pi^2w^2}\frac{z^2}{z_1^2}K_0'^2(z_1)\,d^3k,\quad z_1=\frac{v}{w}\sqrt{k_\perp^2+(1-v^2)k_{\parallel}^2},\quad z=\frac{v}{w}k_\perp,$$

actually coincides with the spectrum (16′) in the region $k_\perp \gg k_{\parallel}\sqrt{1 - v^2}$, i.e., those k for which the time $\Delta t \sim vk_0/wk_\perp$ required for generation of the radiation is small in comparison with the characteristic time $t_\pi$. Whereas here the total radiated energy is finite: $\mathscr{E} = \pi e^2w^2t_\pi/4$, equal to the total energy calculated by integration of the radiation intensity (4) over the time, and this also agrees with the total work done by the radiation damping force. However, just as in the case of a constant field, one can only determine the spectral distribution of the intensity to within terms of order unity since the time $\Delta t_0$ for radiation of energy into the interval $\Delta k_{\parallel}$ is stated with an uncertainty $\Delta t \sim \Delta t_0$.

During the passage of the electron through the coherence interval the radiation probability will be optimal, $\sim 1/137$ if $z \sim \gamma^3$, i.e., $k_\perp \sim eE\gamma^2/m$. In this region the coherence length $c\Delta t \sim (mc^2/eE\gamma)^2k_0$. For a field $E = 3 \times 10^6$ V/cm and $\gamma \sim 1$ this gives $k_\perp \sim 6\ \mathrm{cm}^{-1}$ and $c\Delta t \sim 2.8 \times 10^{-2}\ k_0$ [cm] ($k_0$ is expressed in $\mathrm{cm}^{-1}$). Measurement of the spectrum for $\gamma \sim 1$ is of interest as a confirmation of the transformation of Schott's energy of acceleration[2,6,8] into radiation.


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