

ON THE THEORY OF GAPLESS SUPERCONDUCTIVITY IN ALLOYS CONTAINING PARAMAGNETIC IMPURITIES

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The theory of superconductors containing paramagnetic impurities^[1] is generalized to the case when the scattering of electrons by an isolated impurity is not weak. An exact calculation of the scattering of electrons by an impurity with spin \mathbf{S} leads to the appearance of local states lying inside the gap. It is shown that the order parameter does not change substantially near an impurity. For finite impurity concentrations, these states are responsible for the experimentally-observed broadening of the region of gapless superconductivity. For the calculations it is assumed that the impurity spin \mathbf{S} is a classical vector.

THE theory of superconductors containing a small number of magnetic impurities was developed by Abrikosov and Gor'kov (AG).^[1] They showed that the critical temperature T_c of such an alloy decreases monotonically with an increase of the concentration n of magnetic impurities and vanishes at a certain value $n = n_{cr}$. On the other hand, it was found that the absorption threshold ω_0 in such a system is not in good agreement with the value of the order parameter Δ at $T = 0$; the decrease of ω_0 with concentration n takes place somewhat more rapidly than for the quantity $\Delta(n)$. According to the calculation^[1] the value at which the threshold ω_0 tends to zero corresponds to a concentration $n'_{cr} = 0.91n_{cr}$. Thus, in the narrow region $n'_{cr} < n < n_{cr}$ an interesting situation occurs in which the system exhibits the Meissner effect ($\Delta \neq 0$), and at the same time its energy spectrum starts from zero ($\omega_0 \equiv 0$), just like in a normal metal. This phenomenon is called gapless superconductivity.¹⁾

Somewhat later the phenomenon of gapless superconductivity in magnetic alloys was confirmed in the tunneling experiments by Reif and Woolf.^[3,4] However, in contrast to the prediction of the AG theory, the experimental value of n'_{cr} turned out to be substantially smaller and approximately equal to $0.5 n_{cr}$. An opinion was expressed that ferromagnetic ordering of the impurity spins at low temperatures^[5] in principle might be responsible for the observed broadening of the region of existence of gapless superconductivity.²⁾

In this article we show that agreement with the data^[3,4] can also be obtained for a paramagnetic phase (the impurity spins are not ordered) if, in contrast to the AG theory, the interaction between an electron and an impurity is not assumed to be weak. The essential point consists in the fact that, as shown in^[6], an exact calculation of the scattering of an electron by an isolated magnetic impurity in the case of a superconductor leads to the appearance of local impurity levels inside the energy gap, corresponding to excited states of

Cooper pairs near the impurity. Below we will see that simultaneous account of the scattering processes on many impurities leads to the appearance of an absorption edge with a minimum energy ω_0 which is smaller than in the AG theory; in accordance with this the smallest concentration n'_{cr} at which the threshold vanishes ($\omega_0 = 0$) also turns out to be smaller and dependent on the relative magnitude of the interaction (i.e., the ratio of the exchange interaction J of an electron with an impurity to the Fermi energy μ).

Let us write the interaction of an electron (at the point \mathbf{r}) with the i -th impurity in the form

$$V_i = U(\mathbf{r} - \mathbf{r}_i) + \sigma \mathbf{S} J(\mathbf{r} - \mathbf{r}_i), \quad (1)$$

where \mathbf{S} is the spin of the impurity, and σ denotes the Pauli matrices ($\sigma_i^2 = 1$). In the AG theory V_i is taken into account in the Born approximation. In this case the final results actually do not depend on whether we regard the spin \mathbf{S} as a classical vector or as an operator. In higher-order approximations the above statement ceases to be valid as a consequence of the well-known Kondo anomaly, which is a reflection of the specific commutation rules of the spin operator.^[7] In the following account we shall neglect the Kondo effect, i.e., we shall regard the spin \mathbf{S} as a classical vector. This is apparently valid provided the impurity spin is sufficiently large ($S \gg 1$). A more detailed analysis shows that in practice this restriction is not too strong.

1. LOCAL LEVELS NEAR AN ISOLATED IMPURITY

Regarding the spin as a classical vector, let us first consider the case of a single impurity located at the origin of coordinates ($\mathbf{r}_i = 0$). For the calculations we shall use Gor'kov's technique in matrix form, as set forth in^[5].

The Green's function $\hat{\mathcal{G}}_\omega(\mathbf{p}, \mathbf{p}')$ for a superconductor in the presence of an impurity is given by the expression

$$\hat{\mathcal{G}}_\omega(\mathbf{p}, \mathbf{p}') = \hat{\mathcal{G}}_\omega^{(0)}(\mathbf{p}) \delta_{\mathbf{p}\mathbf{p}'} + \hat{\mathcal{G}}_\omega^{(0)}(\mathbf{p}) \hat{\mathfrak{I}}_{\mathbf{p}\mathbf{p}'}^{(0)}(\omega) \hat{\mathcal{G}}_\omega^{(0)}(\mathbf{p}'), \quad (2)$$

where $\hat{\mathfrak{I}}_{\mathbf{p}\mathbf{p}'}^{(0)}(\omega)$ is the vertex part, which satisfies the equation

¹⁾ Later it was found that a similar situation also occurs in a number of other cases (see the discussion given in the book [2], Ch. 8)

²⁾ Private communication from L. P. Gor'kov.

$$\hat{\mathfrak{z}}_{pp'}^{(0)}(\omega) = \hat{V}_{pp'} + \int \hat{V}_{pp_1} \hat{\mathfrak{G}}_{\omega}^{(0)}(\mathbf{p}_1) \hat{\mathfrak{z}}_{p_1 p'}^{(0)}(\omega) \frac{d\mathbf{p}_1}{(2\pi)^3}. \quad (3)$$

The matrices $\hat{\mathfrak{G}}_{\omega}^{(0)}$ and \hat{V} appearing here have the form form^[5]

$$\hat{\mathfrak{G}}_{\omega}^{(0)}(\mathbf{p}) = \begin{pmatrix} \mathfrak{G}_{\omega}^0(\mathbf{p}) & -i\sigma_y \mathfrak{F}_{\omega}^0(\mathbf{p}) \\ -i\sigma_y \mathfrak{F}_{\omega}^0(\mathbf{p}) & \mathfrak{G}_{\omega}^0(\mathbf{p}) \end{pmatrix} = -\frac{1}{\omega^2 + \Delta^2 + \xi^2} \begin{pmatrix} i\omega + \xi & i\sigma_y \Delta \\ i\sigma_y \Delta & -i\omega + \xi \end{pmatrix}, \quad (4)$$

$$\hat{V}_{pp'} = \begin{pmatrix} V_{\alpha\beta}(\mathbf{p}, \mathbf{p}') & 0 \\ 0 & V_{\beta\alpha}(\mathbf{p}', \mathbf{p}) \end{pmatrix}, \quad (5)$$

where $\omega \equiv \omega_n = \pi T(2n + 1)$, $\xi = (\mathbf{p}^2/2m) - \mu$. In writing down Eqs. (2) and (3) it was assumed that the parameter Δ is unchanged by the introduction of a single impurity. Justification of this assumption will be given below.

In connection with the integration over \mathbf{p}_1 in Eq. (3), we now exclude the region of momenta far from the Fermi surface. One can do this in general form by introducing the amplitude $\hat{f}_{pp'}$ for the scattering of an electron by an impurity in place of the potential $\hat{V}_{pp'}$, in analogy to the way this is done for a nonmagnetic impurity.^[8] Then

$$\hat{\mathfrak{z}}_{pp'}^{(0)}(\omega) = \frac{2\pi}{m} \hat{f}_{pp'} + p_0 \int \hat{f}_{pp_1} \hat{\mathfrak{G}}_{\omega}^{(0)} \hat{\mathfrak{z}}_{p_1 p'}^{(0)} \frac{d\mathbf{p}_1}{4\pi}. \quad (6)$$

The momenta entering into this equation lie on the Fermi surface: $|\mathbf{p}| = |\mathbf{p}_1| = |\mathbf{p}'| = p_0$. The bar denotes integration over the energy,

$$\hat{\mathfrak{G}}_{\omega}^{(0)} = \int_{-\infty}^{+\infty} \hat{\mathfrak{G}}_{\omega}^{(0)}(\xi) \frac{d\xi}{\pi} = \frac{1}{\sqrt{\omega^2 + \Delta^2}} \begin{pmatrix} -i\omega & -i\sigma_y \Delta \\ -i\sigma_y \Delta & i\omega \end{pmatrix}. \quad (7)$$

The amplitude f has the dimension of a length and satisfies the equation

$$\hat{f}_{pp'} = \frac{m}{2\pi} \hat{V}_{pp'} - \int \frac{\hat{V}_{pp_1} \hat{f}_{p_1 p'}}{\xi(\mathbf{p}_1)} \frac{d\mathbf{p}_1}{(2\pi)^3}. \quad (8)$$

In order to determine the vertex part $\hat{\mathfrak{z}}_{pp'}^{(0)}(\omega)$, we expand the corresponding quantities in Eq. (6) in series of Legendre polynomials:

$$\hat{f}_{pp'} = \sum_{l=0}^{\infty} (2l+1) \hat{f}_l P_l(\mathbf{nn}'), \quad (9)$$

$$\hat{\mathfrak{z}}_{pp'}^{(0)} = \sum_{l=0}^{\infty} (2l+1) \hat{\mathfrak{z}}_l^{(0)} P_l(\mathbf{nn}'),$$

where $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. As a result for the l -th harmonic we obtain

$$\hat{\mathfrak{z}}_l^{(0)}(\omega) = \hat{f}_l + p_0 \hat{f}_l \hat{\mathfrak{G}}_{\omega}^{(0)} \hat{\mathfrak{z}}_l^{(0)}(\omega). \quad (10)$$

Let us choose the direction of \mathbf{S} along the z axis; then one can write the solution of Eqs. (10) and (7) in the form

$$\hat{\mathfrak{z}}_l^{(0)}(\omega) = \begin{pmatrix} \mathfrak{z}_1^+ & 0 & 0 & -\mathfrak{z}_2 \\ 0 & \mathfrak{z}_1^- & \mathfrak{z}_2^* & 0 \\ 0 & -\mathfrak{z}_2^* & (\mathfrak{z}_1^+)^* & 0 \\ \mathfrak{z}_2 & 0 & 0 & (\mathfrak{z}_1^-)^* \end{pmatrix}, \quad (11)$$

where

$$\mathfrak{z}_{1\pm} = \frac{(2\pi/m) f_l \pm \Delta}{R_l} \frac{\sqrt{\omega^2 + \Delta^2} - i p_0 f_l \mp \omega}{\varepsilon_l \sqrt{\omega^2 + \Delta^2} \pm i\omega \sqrt{\Delta^2 - \varepsilon_l^2}}, \quad (12)$$

$$\mathfrak{z}_2 = \frac{(2\pi/m) p_0 f_l^+ f_l^- \Delta^2}{R_l} \frac{1}{\varepsilon_l \sqrt{\omega^2 + \Delta^2} + i\omega \sqrt{\Delta^2 - \varepsilon_l^2}}, \quad (13)$$

$$\varepsilon_l = \Delta(1 + p_0^2 f_l^+ f_l^-) / R_l, \quad R_l = [(1 + p_0^2 f_l^+ f_l^-)^2 + p_0^2 (f_l^+ - f_l^-)^2]^{1/2},$$

for $J > 0$; for $J < 0$ one must make the following sub-

stitutions: $\mathfrak{z}_1^+ \rightleftharpoons \mathfrak{z}_1^-$ and $\mathfrak{z}_2 \rightleftharpoons \mathfrak{z}_2^*$. An asterisk denotes the operation of complex conjugation. The amplitudes f_l^{\pm} describe the scattering of an electron by an impurity in a state with orbital momentum l and spin projection $\pm 1/2$ in a normal metal.

In order to determine the possible bound states in the system associated with the presence of an impurity we must, according to general rules (^[9], Secs. 7, 17), construct the analytic continuation of the Matsubara vertex $\mathfrak{z}^{(0)}(\omega)$ into the upper half-plane of complex values $\epsilon = i\omega$. Setting $i\omega = \epsilon + i\delta$ in Eqs. (12) and (13), one can easily see that the function $\Gamma_l(\epsilon) = \mathfrak{z}_l^{(0)}(i\epsilon - \delta)$ satisfies the formulated condition in the complex ϵ plane with cuts along the real axis from $-\infty$ to $-\Delta$ and from Δ to ∞ . The pole of $\Gamma_l(\epsilon)$ at $\epsilon = \epsilon_l - i\delta$ gives the energy ϵ_l of the bound state with orbital momentum l .³⁾

Expression (14) for ϵ_l was previously obtained in ^[6] by another method; an expression was given there for the wave function of this state. Here we shall not cite the corresponding formulas and only note that the contribution of the pole $i\omega = \epsilon_l$ to the Green's function (2) for $T = 0$ in coordinate space falls off at distances $r \sim \xi_0 [1 - (\epsilon_l/\Delta)^2]^{-1/2}$ as one goes away from the impurity, in contrast to the case of a pure superconductor for which the Green's function $\hat{\mathfrak{G}}_{\omega}^{(0)}(\mathbf{r} - \mathbf{r}')$ falls off at the coherence length ($|\mathbf{r} - \mathbf{r}'| \sim \xi_0 = v/\Delta$). The considerations given below for finite impurity concentrations indicate that, as a consequence of the rapid oscillations of the electron wave functions at the Fermi surface, the interaction of these local levels becomes important for $n \sim n_{cr}$ (compare with the case of two impurities in article^[6]).

Finally we present an expression, which will be useful in what follows, for the energy levels ϵ_l in terms of the phases δ_l^{\pm} for the scattering of an electron by an impurity:

$$\varepsilon_l = \Delta \cos(\delta_l^+ - \delta_l^-), \quad \text{tg } \delta_l^{\pm} = p_0 f_l^{\pm}. \quad (15)$$

2. SPATIAL VARIATIONS OF Δ

Now let us briefly discuss the question of the spatial variation of the parameter $\Delta(\mathbf{r})$ near a paramagnetic impurity. For a nonmagnetic impurity this problem was studied in detail in ^[12], and it was shown that the entire variation of $\Delta(\mathbf{r})$ near an impurity reduces to small oscillating (over atomic distances) corrections. This is in complete agreement with the fact that, in general, the ordinary scattering does not have any effect on the critical temperature T_c of a superconductor (see ^[9], Sec. 39).

The situation is somewhat different in the case of a magnetic impurity. As we now see, here the correction $\delta\Delta(\mathbf{r}) = \Delta(\mathbf{r}) - \Delta$ remains finite after averaging the atomic oscillations; however, it is extremely small in magnitude. Near the critical temperature ($\Delta = \epsilon_l = 0$)

³⁾A bound state (for $l = 0$) having an energy less than the superconductor's energy gap Δ was found in articles ^[10,11] for impurity spin $S = 1$. However, in this case the physical situation is very complicated due to the above-mentioned Kondo anomaly, ^[7] which even in a normal metal may in principle lead to a bound state of an electron on an impurity. In particular, in article ^[11] it is shown that within the framework of the dispersion method used there, the question of the actual existence of such states for a superconductor remains open.

the variation $\delta\Delta(\mathbf{r})$ was previously calculated in [13], for $n \ll n_{cr}$ the relative correction to the AG result for $T_C(n)$ turned out to be insignificantly small, $\sim 10^{-6}$ to 10^{-7} , which justifies the assumption in the AG theory about the constancy of Δ in space. In the presence of isolated states inside the gap, this conclusion remains valid at absolute zero.

Variation of the parameter $\Delta(\mathbf{r})$ near an impurity leads to the appearance of an additional term of the form

$$\delta\hat{\mathcal{G}}_{\omega}(\mathbf{p}, \mathbf{p}') = \hat{\mathcal{G}}_{\omega}^{(0)}(\mathbf{p}) \delta\hat{\Delta}_{\mathbf{p}-\mathbf{p}'} \hat{\mathcal{G}}_{\omega}^{(0)}(\mathbf{p}'), \quad (16)$$

in the right-hand side of Eq. (2), where the matrix $\delta\hat{\Delta}_{\mathbf{q}}$ has the form

$$\delta\hat{\Delta}_{\mathbf{q}} = -i\sigma_y \delta\Delta_{\mathbf{q}} \begin{pmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{pmatrix}. \quad (17)$$

By definition,

$$\Delta_{\mathbf{q}} = -|g|T \sum_{\omega} \mathcal{G}_{\mathbf{14}} \left(\mathbf{p} + \frac{1}{2}\mathbf{q}, \mathbf{p} - \frac{1}{2}\mathbf{q}; \omega \right) \frac{d\mathbf{p}}{(2\pi)^3}, \quad (18)$$

where $\mathcal{G}_{\mathbf{14}}$ is the element, standing in the upper right-hand corner, of the matrix (2) with the additional term (16) taken into consideration, and g is the electron-electron interaction constant corresponding to attraction ($g < 0$). Considering the second term in (2) as a perturbation, to first order in $\mathfrak{E}^{(0)}$ one can write

$$\begin{aligned} & \left[|g|^{-1} - T \sum_{\omega} \int \left(\mathcal{G}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathcal{G}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \right. \right. \\ & \quad \left. \left. - \mathfrak{F}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathfrak{F}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \right) \frac{d\mathbf{p}}{(2\pi)^3} \right] \delta\Delta_{\mathbf{q}} \\ & = T \sum_{\omega} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\left(\mathcal{G}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathcal{G}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \right. \right. \\ & \quad \left. \left. - \mathfrak{F}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathfrak{F}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \right) \mathfrak{E}_2(\mathbf{q}, \omega) \right. \\ & \quad \left. + \mathcal{G}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathfrak{F}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \mathfrak{E}_1^+(\mathbf{q}, \omega) \right. \\ & \quad \left. + \mathfrak{F}_{\omega}^0 \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \mathcal{G}_{\omega}^0 \left(\mathbf{p} - \frac{\mathbf{q}}{2} \right) \mathfrak{E}_1^-(\mathbf{q}, \omega) \right]. \quad (19) \end{aligned}$$

Performing the integration over ξ (for $q \ll p_0$), we obtain

$$L(q) \delta\Delta_{\mathbf{q}} = -\frac{\pi^2}{m p_0} \sum_l (2l+1) \left(1 - \frac{\varepsilon_l^2}{\Delta^2} \right) \varphi_l(q), \quad (20)$$

where

$$\begin{aligned} L(q) & = \pi T \sum_{\omega} \frac{1}{\sqrt{\omega^2 + \Delta^2}} \int_0^{\frac{1}{2}} \frac{\Delta^2 + \frac{1}{4}(vq\mu)^2}{\omega^2 + \Delta^2 + \frac{1}{4}(vq\mu)^2} d\mu, \\ \varphi_l(q) & = T \sum_{\omega} \frac{\omega^2 \Delta}{\omega^2 + \varepsilon_l^2} \int_0^{\frac{1}{2}} \frac{d\mu}{\omega^2 + \Delta^2 + \frac{1}{4}(vq\mu)^2}. \quad (21) \end{aligned}$$

The left-hand side of Eqs. (19) and (20) has the standard form of all similar problems relating to a variation of the order parameter under the influence of a static external field (see, for example, [14]). In coordinate space

$$\delta\Delta(r) = -\frac{1}{2m p_0 r} \sum_l (2l+1) \left(1 - \frac{\varepsilon_l^2}{\Delta^2} \right) \int_0^{\infty} \frac{q \sin qr \varphi_l(q)}{L(q)} dq. \quad (22)$$

According to Eqs. (21) the dimensionless functions $L(q)$ and $\varphi_l(q)$ have a characteristic scale of variation coinciding in order of magnitude with the reciprocal of the coherence length ($q \sim \xi_0^{-1}$). From here and from

Eq. (22) one can at once see that the relative change of the order parameter can be written in the form

$$\frac{\delta\Delta(r)}{\Delta} = -\frac{1}{p_0^2 \xi_0 r} F\left(\frac{r}{\xi_0}\right), \quad (23)$$

where F is a certain dimensionless function varying over distances $r \sim \xi_0$. For $r \approx \xi_0$ the correction $\delta\Delta/\Delta$ is insignificantly small: $\sim 10^{-8}$, a fact which has been utilized from the very beginning. Apparently one can expect a substantial variation only over atomic distances; however, the model itself becomes inapplicable much sooner—at distances of the order of the interaction radius $r \sim v/\omega_D \sim 10^{-6}$ cm.

One can evaluate the function $F(r/\xi_0)$ in two limiting cases: $r \gg \xi_0$ and $r \ll \xi_0$. Here we present the results for the case of absolute zero.

At distances which are large in comparison with the coherence length, small values of $q \ll \xi_0^{-1}$ are important in the integral (22). Using the expansion [14]

$$L(q) = 1 + \frac{1}{2} (qv/\Delta)^2$$

and the value $\varphi_l(0) = \Delta/2(\Delta + \varepsilon_l)$ it is easy to obtain

$$\frac{\delta\Delta(r)}{\Delta} = -\frac{3\pi}{4} \sum_l (2l+1) \left(1 - \frac{\varepsilon_l}{\Delta} \right) \frac{\Delta}{\mu} \frac{1}{p_0 r} \exp\left(-\frac{2\sqrt{3}r\Delta}{v}\right), \quad r \gg \xi_0 = \frac{v}{\Delta}.$$

Similarly, at small distances ($r \ll \xi_0$) large momenta $q \gg \xi_0^{-1}$ for which $L(q) \approx \ln q \xi_0$ and $\varphi_l(q) \approx n(q \xi_0)/q \xi_0$ play the major role. Carrying out the integration in (22), in the logarithmic approximation we obtain

$$\frac{\delta\Delta(r)}{\Delta} = -\frac{1}{2(p_0 r)^2} \sum_l (2l+1) \left(1 - \frac{\varepsilon_l^2}{\Delta^2} \right), \quad p_0^{-1} \ll r \ll \xi_0. \quad (25)$$

3. FINITE CONCENTRATIONS

Now let us discuss the case when the concentration of magnetic impurities is finite but small in the sense that the impurities do not exert any significant influence on the thermodynamic properties of a metal in its normal state. This is equivalent to the condition that the electron mean free path l be large in comparison with the lattice constant $1/p_0$. We shall assume that the impurities are randomly distributed in the crystal and that one can neglect the spin interaction of neighboring impurities. For averaging over the positions of the impurities we shall use the technique developed by Abrikosov and Gor'kov; [1, 9] in this connection we shall treat the scattering by a given impurity exactly.

In contrast to (2) the Green's function of a superconductor averaged over the coordinates and spin directions of the impurities, $\hat{\mathcal{G}}_{\omega}(\mathbf{p})$ depends on only one momentum \mathbf{p} and satisfies the following equation:

$$\hat{\mathcal{G}}_{\omega}(\mathbf{p}) = \hat{\mathcal{G}}_{\omega}^{(0)}(\mathbf{p}) + n \hat{\mathcal{G}}_{\omega}^{(0)}(\mathbf{p}) \hat{\mathfrak{E}}_{pp}(\omega) \hat{\mathcal{G}}_{\omega}(\mathbf{p}) \quad (26)$$

(n denotes the impurity concentration). The exact vertex part $\hat{\mathfrak{E}}_{pp}'$ in turn satisfies an equation containing the exact Green's function $\hat{\mathcal{G}}$:

$$\hat{\mathfrak{E}}_{pp}'(\omega) = \hat{V}_{pp'} + \int \hat{V}_{p_1 p'} \hat{\mathcal{G}}_{\omega}(\mathbf{p}_1) \hat{\mathfrak{E}}_{p_1 p'}(\omega) \frac{d\mathbf{p}_1}{(2\pi)^3}. \quad (27)$$

In writing down these equations the physically-reasonable assumption has been made that interference effects associated with the scattering of an electron si-

multaneously by several impurities can be neglected. In [11] it is shown that the neglected terms lead to corrections $\sim 1/p_0 l \ll 1$. In addition, we have taken into consideration the fact that the parameter Δ can be regarded as constant in space with a high degree of accuracy.

If the scattering by an impurity is weak, then in the second term in Eq. (27) one can set $\hat{\varepsilon} = \hat{V}$ and obtain the AG theory.

Introducing the exact amplitude $\hat{f}_{pp'}$ for the scattering of an electron by an impurity, we write Eq. (27) in complete analogy with the case of a single impurity:

$$\hat{\mathfrak{F}}_{pp'}(\omega) = \frac{2\pi}{m} \hat{f}_{pp'} + p_0 \int \hat{f}_{pp'} \hat{\mathfrak{G}}_{\omega} \hat{\mathfrak{F}}_{p,p'}(\omega) \frac{d\omega_1}{4\pi} \quad (28)$$

The notation is the same as in Eq. (6). In order to avoid misunderstandings involving the notation, we note that in Eq. (26) the function $\hat{\varepsilon}$ is assumed to already be averaged over the directions of the impurity spin, but this has not been done in Eqs. (27) and (28); on the other hand in both cases the function $\hat{\mathfrak{G}}$ is assumed to be averaged (i.e., it does not depend on the direction of \mathbf{S}). Thus, in order to obtain the averaged $\hat{\varepsilon}$, it is necessary to solve Eq. (28) in general form and then carry out the necessary averaging.

Let us assume that the function $\hat{\mathfrak{G}}$ has the same structure as $\hat{\mathfrak{G}}^{(0)}$, namely,

$$\hat{\mathfrak{G}}_{\omega}(p) = -[\tilde{\omega}^2 + \tilde{\Delta}^2 + \tilde{\xi}^2]^{-1/2} \begin{pmatrix} i\tilde{\omega} + \tilde{\xi} & i\sigma_y \tilde{\Delta} \\ i\sigma_y \tilde{\Delta} & -i\tilde{\omega} + \tilde{\xi} \end{pmatrix}. \quad (29)$$

It is also natural to seek the averaged vertex part $\hat{\varepsilon}$ in the form (11):

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\sigma_y \varepsilon_2 \\ -i\sigma_y \varepsilon_2 & \varepsilon_1^* \end{pmatrix}. \quad (30)$$

Here we have used the following obvious property:

$\varepsilon_1 = \varepsilon_1^+ = \varepsilon_1^-$ and $\varepsilon_2 = \varepsilon_2^* = -\varepsilon_2^-$ —a result of averaging over the directions of \mathbf{S} . The final result is in agreement with the assumptions which have been made.

Substituting (29) and (30) into (26), one can easily obtain the following relations between the quantities that have been introduced:

$$\tilde{\omega} = \omega - n \operatorname{Im} \mathfrak{F}_1(0, \omega), \quad \tilde{\Delta} = \Delta + n \mathfrak{F}_2(0, \omega), \quad (31)$$

$$\tilde{\xi} = \xi + n \operatorname{Re} \mathfrak{F}_1(0, \omega),$$

where $\mathfrak{F}(0, \omega) \equiv \mathfrak{F}(\mathbf{p}, \mathbf{p}; \omega)$ is the vertex part for forward scattering.

It remains for us to solve Eq. (28); the matrix $\hat{\mathfrak{G}}_{\omega}$, in accord with (29), is given by formula (7) in which it is necessary to make the following substitutions: $\omega, \Delta \rightarrow \tilde{\omega}, \tilde{\Delta}$. In view of the cumbersome nature of the resulting expressions, we cite only the result for the averaged values of ε :

$$\begin{aligned} \operatorname{Re} \mathfrak{F}_1^{(l)}(\omega) &= \frac{\pi}{2mp_0} \frac{\omega^2 + \Delta^2}{\tilde{\omega}^2 + \tilde{\varepsilon}_l^2} \sin 2(\delta_l^+ - \delta_l^-), \\ \operatorname{Im} \mathfrak{F}_1^{(l)}(\omega) &= -\frac{\pi}{mp_0} \frac{\tilde{\omega} \sqrt{\tilde{\omega}^2 + \tilde{\Delta}^2}}{\tilde{\omega}^2 + \tilde{\varepsilon}_l^2} (\sin^2 \delta_l^+ + \sin^2 \delta_l^-), \\ \mathfrak{F}_2^{(l)}(\omega) &= \frac{2\pi}{mp_0} \frac{\tilde{\Delta} \sqrt{\tilde{\omega}^2 + \tilde{\Delta}^2}}{\tilde{\omega}^2 + \tilde{\varepsilon}_l^2} \sin \delta_l^+ \sin \delta_l^- \cos(\delta_l^+ - \delta_l^-), \end{aligned} \quad (32)$$

where $\tilde{\varepsilon}_l = \tilde{\Delta} \cos(\delta_l^+ - \delta_l^-)$. Here we have used the notation of Sec. 1. In the Born approximation ($\delta_l^{\pm} \ll 1$) formulas (32) agree with the results of the AG theory:

$$\begin{aligned} \operatorname{Re} \mathfrak{F}_1^{(l)} &= 1/2 (V_l^+ - V_l^-), \\ \operatorname{Im} \mathfrak{F}_1^{(l)}(\omega) &= -\frac{mp_0}{4\pi} [(V_l^+)^2 + (V_l^-)^2] \frac{\tilde{\omega}}{\sqrt{\tilde{\omega}^2 + \tilde{\Delta}^2}}, \\ \mathfrak{F}_2^{(l)}(\omega) &= \frac{mp_0}{2\pi} V_l^+ V_l^- \frac{\tilde{\Delta}}{\sqrt{\tilde{\omega}^2 + \tilde{\Delta}^2}}. \end{aligned} \quad (32')$$

In this case the quantity $\operatorname{Re} \varepsilon_1(0, \omega)$ in formula (31) does not depend on the frequency ω and may be included in the chemical potential μ . In general this is not so in the simultaneous presence of exchange and ordinary scattering. However, this property is unimportant for an investigation of the static properties of the system since upon integration over ξ of the expressions containing the Green's function with the same frequency, the component $n \operatorname{Re} \varepsilon_1(0, \omega)$ drops out of the answer.

Substitution of (32) into (31) finally gives

$$\begin{aligned} \tilde{\omega} &= \omega + \frac{\pi n}{mp_0} \sum_l (2l+1) (\sin^2 \delta_l^+ + \sin^2 \delta_l^-) \frac{\eta \sqrt{\eta^2 + 1}}{\eta^2 + \cos^2(\delta_l^+ - \delta_l^-)}, \\ \tilde{\Delta} &= \Delta + \frac{2\pi n}{mp_0} \sum_l (2l+1) \sin \delta_l^+ \sin \delta_l^- \cos(\delta_l^+ - \delta_l^-) \frac{\sqrt{\eta^2 + 1}}{\eta^2 + \cos^2(\delta_l^+ - \delta_l^-)}, \end{aligned} \quad (33)$$

where $\eta = \tilde{\omega}/\tilde{\Delta}$ satisfies the equation

$$\omega = \eta \left[\Delta - \frac{\pi n}{mp_0} \sum_l (2l+1) \sin^2(\delta_l^+ - \delta_l^-) \frac{\sqrt{\eta^2 + 1}}{\eta^2 + \cos^2(\delta_l^+ - \delta_l^-)} \right]. \quad (34)$$

Let us consider the case of absolute zero in more detail; here for simplicity we shall regard the scattering as isotropic ($l = 0$). The dependence of Δ on the impurity concentration is determined by the equation [11]

$$\Delta_c \ln \frac{\Delta}{\Delta_0} = \int_0^{\infty} d\omega \left[\frac{1}{\sqrt{\eta\omega^2 + 1}} - \frac{\Delta_0}{\sqrt{\omega^2 + \Delta_0^2}} \right], \quad (35)$$

where Δ_0 is the value of Δ for a pure superconductor. With the aid of a change of the integration variable $d\omega = (d\omega/d\eta)d\eta$ in formula (34), one can express the result in terms of elementary functions:

$$\begin{aligned} \ln \frac{\Delta}{\Delta_0} &= -\frac{\pi}{2(1+\varepsilon_0)} \frac{1}{\tau_s \Delta}, \quad \frac{1}{\tau_s \Delta} \leq \varepsilon_0^2; \\ \ln \frac{\Delta}{\Delta_0} &= -\ln[\eta_0 + \sqrt{\eta_0^2 + 1}] + \frac{1}{\tau_s \Delta} \frac{\eta_0}{\eta_0^2 + \varepsilon_0^2}; \\ \frac{1}{\tau_s \Delta (1 - \varepsilon_0^2)} \operatorname{arctg} \frac{\eta_0(1 - \varepsilon_0)}{\eta_0^2 + \varepsilon_0} &- \frac{1}{\tau_s \Delta (1 + \varepsilon_0)} \operatorname{arctg} \frac{\varepsilon_0}{\eta_0}, \quad \frac{1}{\tau_s \Delta} \geq \varepsilon_0^2, \end{aligned} \quad (36)$$

where

$$\begin{aligned} \eta_0^2 &= \frac{1}{2(\tau_s \Delta)^2} [1 - 2\varepsilon_0^2 (\tau_s \Delta)^2 + \sqrt{1 + 4(1 - \varepsilon_0^2) (\tau_s \Delta)^2}], \\ \frac{1}{\tau_s} &= \frac{\pi n}{mp_0} (1 - \varepsilon_0^2), \quad \varepsilon_0 = \cos(\delta_0^+ - \delta_0^-). \end{aligned}$$

The parameter τ_s represents the time it takes for an electron spin to flip during impurity scattering processes. In the Born approximation ($1 - \varepsilon_0 \ll 1$) formulas (36) go over into the formulas of the AG theory.

According to (36) the vanishing of superconductivity ($\Delta = 0$) takes place at the concentration $n = n_{\text{cr}}$ at which $\tau_{\text{cr}} \Delta_0 = 2$. We note that the non-Born nature of the scattering now develops in the terms $\sim \Delta^2$, i.e., in the region where the Ginzburg-Landau equations are not applicable. Thus, for isotropic scattering (at $T = 0$)

$$\left(\frac{\Delta}{\Delta_0} \right)^2 = \frac{3}{5 - 4\varepsilon_0^2} \frac{n_{\text{cr}} - n}{n_{\text{cr}}}, \quad n_{\text{cr}} - n \ll n_{\text{cr}}. \quad (37)$$

In the general case of anisotropic scattering and arbitrary Δ the results depend in a complicated way on various superpositions of all the harmonics f_l^{\pm} .

It is of great interest to determine the absorption threshold and the corresponding structure (i.e., the density of states) near it. The retarded Green's function $\hat{G}^R(\mathbf{p}, \omega)$ gives the answer to this question. In [11] it is shown that for its construction one must make the following substitutions in formulas (29), (33), and (34): $\omega_n \rightarrow -i\omega$, $\eta_n \rightarrow -i\eta$, $\tilde{\omega}_n \rightarrow -i\omega$, $\tilde{\Delta}_n \rightarrow \Delta$, and one must define the root as the analytic continuation of the root $+\sqrt{1-\eta^2}$ (for $|\eta| < 1$) into the upper half-plane of the variable η . Then the absorption threshold is the minimum frequency $\omega_0 > 0$ for which the roots of the equation

$$\omega = \eta \left(\Delta - \frac{1}{\tau_s} \frac{\sqrt{1-\eta^2}}{\epsilon_0^2 - \eta^2} \right), \quad 0 \leq \eta \leq \epsilon_0. \quad (38)$$

become complex for $\omega > \omega_0$. It is easy to see that for $\tau_s \Delta > 1/\epsilon_0^2$ and as $\omega \rightarrow 0$ this equation always has a real solution η lying in the interval $0 < \eta < \epsilon_0$; there is an energy gap in the spectrum of the system. The magnitude of the gap $\omega_0(n, \epsilon_0)$ is equal to the largest value of the right-hand side of Eq. (38), i.e., it is obtained from the condition $(d\omega/d\eta)_{\omega_0} = 0$. The resulting equation can only be solved numerically. In the limit of small concentrations ($n \ll n_{cr}$) the gap in the spectrum is equal to $\omega_0 = \epsilon_0 \Delta$, which reflects the presence of a discrete level (with $l = 0$) for an isolated impurity, obtained in Sec. 1.

On the other hand, for $\tau_s \Delta \leq 1/\epsilon_0^2$ Eq. (38) does not have any real solutions, i.e., $\omega_0 \equiv 0$. From (36) it follows that the gap vanishes at the concentration

$$n_{cr}' = 2\epsilon_0^2 n_{cr} \exp[-\pi\epsilon_0^2/2(1+\epsilon_0)]. \quad (39)$$

The phenomenon of gapless superconductivity occurs in the region $n_{cr}' < n < n_{cr}$. In the Born approximation ($\epsilon_0 = 1$): $n_{cr}' = 0.91 n_{cr}$, the value which follows from the AG theory. As is evident from Eq. (39) the presence of impurity levels ($\epsilon_0 < 1$) leads to a broadening of the region of gapless superconductivity. It is natural to expect that taking account of higher harmonics in the scattering leads to a further decrease of n_{cr}' .⁴⁾

As mentioned at the very beginning, in the experiments of Reif and Woolf^[3, 4] the value of n_{cr}' was approximately equal to $0.5 n_{cr}$. As alloys they used iron atoms in indium.^[3] Within the framework of the stated theory, from here one can conclude that for such an alloy of indium containing iron impurities, the exchange interaction of the conduction electrons is not weak. One can obtain a rough estimate of the magnitude of the exchange interaction with the aid of Eq. (39) if one sets $n_{cr}' = 0.5 n_{cr}$ in it. This gives $\epsilon_0 \approx 0.6$ which, according to (13), leads to a quite reasonable value for the amplitudes $f_0^+ = -f_0^- = 0.5 p_0^{-1}$ (we assumed $U \equiv 0$).

In conclusion let us calculate the heat capacity in the gapless region at low temperatures. For this it is obviously sufficient to know the density of states $N_S(\omega)$

for excitations of small energies. According to general rules of statistics,

$$N_s(\omega) = \frac{1}{\pi} \text{Im} \int G^R(\mathbf{p}, \omega) \frac{d\mathbf{p}}{(2\pi)^3} \quad (40)$$

(for a given spin direction). Taking what has been said above into account, we have

$$N_s(\omega) = N_n \text{Im} \frac{\eta}{\sqrt{1-\eta^2}} \quad (41)$$

where $N_n = mp_0/2\pi^2$ is the corresponding density of states in the normal metal. For $\omega \rightarrow 0$ and $\tau_s \Delta < 1/\epsilon_0^2$ the root (38) is purely imaginary ($\eta = i\eta_0$); an expression for η_0 is given above (see Eq. (36)). Consequently

$$N_s(\omega) = N_n \frac{\eta_0}{\sqrt{1+\eta_0^2}}. \quad (42)$$

In view of the fact that $N_S(\omega) = \text{const}$ as $\omega \rightarrow 0$, an expression for the heat capacity can be written down at once:

$$C_s(T) = \frac{\pi^2}{3} 2N_s T = \frac{mp_0}{3} \frac{\eta_0}{\sqrt{1+\eta_0^2}} T. \quad (43)$$

At the point $\tau_s \Delta = 1/\epsilon_0^2$ the coefficient in the heat capacity vanishes ($\eta_0 = 0$); this corresponds to the concentration n_{cr}' and only indicates that the heat capacity does not vary faster than linearly as a function of T . In the region $n < n_{cr}'$ the electronic heat capacity has an exponential character. However, since the investigation can be successfully carried out only in the uninteresting region of very small concentrations, we shall not cite the corresponding formulas here.

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237