CONTRIBUTION TO THE THEORY OF THRESHOLD PHENOMENA IN DIFFRACTION OF ELECTROMAGNETIC WAVES

B. M. BOLOTOVSKIĬ and K. I. KUGEL'

P. N. Lebedev Physics Institute, U.S.S.R. Academy of Sciences

Submitted November 11, 1968

Zh. Eksp. Teor. Fiz. 57, 165–174 (July, 1969)

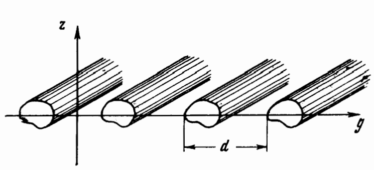
The behavior of the amplitudes and phases of electromagnetic waves at the threshold of appearance of a new electromagnetic wave (a spectrum of a new order) is considered for the case of scattering by a transparent diffraction lattice or by the open end of a cylindrical waveguide.

THE anomalous behavior of the amplitudes and phases of electromagnetic waves when a new mode is produced was considered in [1] for certain diffraction problems, and a theory of Wood's anomalies was developed for a reflecting diffraction grating. It was shown that the energy-conservation law makes it possible to determine the behavior of the phases and amplitudes of the electromagnetic field near the threshold of occurrence of the new harmonic. It is easy to see that in the problem of light scattering by a reflecting diffraction grating, the appearance of a new harmonic denotes physically the creation of a diffraction spectrum of higher order. It has been shown that the creation of the new spectrum is accompanied by characteristic changes in the amplitudes of the already existing spectra. These characteristic changes were experimentally observed at the beginning of the century by Wood[2] and are called Wood's anomalies. Wood's anomalies are a typical threshold effect.

We shall consider below two problems connected with threshold effects in electrodynamics: Wood's anomalies in the diffraction of light by a transparent diffraction grating, and the diffraction of electromagnetic wave by the open end of a waveguide.

1. WOOD'S ANOMALIES IN THE DIFFRACTION OF LIGHT BY A TRANSPARENT DIFFRACTION GRATING

Let us consider a transparent diffraction grating that is periodic along the y axis and has a period of arbitrary structure; the coordinate axes are chosen as shown



in the figure, with the x axis perpendicular to the plane of the figure. Assume that a plane electromagnetic wave in the form

$$\Psi = \exp\{-ik_z z + ik_y y - i\omega t\} \quad (1)$$

is incident on the grating from the upper half-space $x > 0$. We assume henceforth that either the electric field or the magnetic field is directed along the x axis. Therefore the quantity Ψ stands for either E_x or H_x. The remaining components of the field vectors can be easily expressed in terms of Ψ.

The total field in our problem is a superposition of waves of the type

$$\Psi_n^{\pm} = \exp\left\{\pm iz\sqrt{\frac{\omega^2}{c^2} - \left(k_y - \frac{2\pi n}{d}\right)^2} + i\left(k_y - \frac{2\pi n}{d}\right)y - i\omega t\right\}, \quad (2)$$

which diverge upwards and downwards away from the grating. In formula (2) n is an integer that can be either positive, negative, or zero. The function $\Psi_n^{\pm}$ describes a propagating wave only at definite values of the number n, satisfying the inequality

$$\frac{d}{2\pi}\left(k_y - \frac{\omega}{c}\right) < n < \frac{d}{2\pi}\left(k_y + \frac{\omega}{c}\right). \quad (3)$$

The inequality (3) is the condition for the existence of a spectrum of n-th order.

Just as in [1], we shall say that the function Ψ_n describes the channel numbered n. If n satisfies the inequality (3), i.e., if there exists a spectrum numbered n, then we shall call the channel open, otherwise the channel is called closed.

Let us consider a case when an electromagnetic wave is incident on the grating through channel k. If this wave is incident from the upper half-space, then it is described by the function Ψ_k^-. The field on both sides of the grating can then be described in the form

$$\Phi_k = \begin{cases} \Psi_k^- + \sum_n R_{nk}\Psi_n^+ & \text{above the grating} \\ \sum_n L_{nk}\Psi_n^- & \text{below the grating} \end{cases} \quad (4)$$

where the coefficients R_{nk} and L_{nk} are chosen such as to satisfy the boundary conditions on the grating. The coefficient R_{nk} can be naturally called the coefficient of the transformation of the incident wave of channel k into a reflected wave of channel n. Accordingly, L_{nk} is the coefficient of transformation of an incident wave of channel k into a transmitted wave of channel n. If the wave is incident on the grating from the lower half-space, then the field is written in the form

$$\Phi_{k'} = \begin{cases} \sum_n K_{nk'}\Psi_n^+ & \text{above the grating} \\ \Psi_{k'}^+ + \sum_n M_{nk'}\Psi_n^- & \text{below the grating} \end{cases} \quad (5)$$

Let the number of open channels (the number of diffraction spectra) be equal to N. The channels above and below the grating are assumed to be different. Accordingly, we renumber the channels in such a way that the index k in formula (4) takes on the values from 1 to N, and the index k′ in (5) takes on the values from N + 1 to 2N. In the general case the field at infinity can contain converging and diverging waves of all the open channels, i.e., it can be of the form

$$\Phi = \sum_n \alpha_n \Phi_n$$

$$= \begin{cases} \sum_{k=1}^{N} \alpha_k \Psi_k^- + \sum_{n=1}^{N}\sum_{k=1}^{N} \alpha_k R_{nk} \Psi_n^+ + \sum_{n=1}^{N}\sum_{k=N+1}^{2N} \alpha_k K_{nk} \Psi_n^+, & z \to +\infty, \\ \sum_{k=N+1}^{2N} \alpha_k \Psi_k^+ + \sum_{n=N+1}^{2N}\sum_{k=N+1}^{2N} \alpha_k M_{nk} \Psi_n^- + \sum_{k=1}^{N}\sum_{n=N+1}^{2N} \alpha_k L_{nk} \Psi_n^-, & z \to -\infty. \end{cases} \quad (6)$$

If we set up a matrix P of order 2N out of the transformation coefficients R, L, M, and K, namely

$$P = \begin{pmatrix} R & K \\ L & M \end{pmatrix}, \quad (7)$$

then relations (6) can be rewritten in simpler form

$$\Phi = \begin{cases} \sum_{k=1}^{N} \alpha_k \Psi_k^- + \sum_{n=1}^{N}\sum_{k=1}^{2N} \alpha_k P_{nk} \Psi_n^+, & z \to +\infty; \\ \sum_{k=N+1}^{2N} \alpha_k \Psi_k^+ + \sum_{n=N+1}^{2N}\sum_{k=1}^{2N} \alpha_k P_{nk} \Psi_n^-, & z \to -\infty. \end{cases} \quad (8)$$

The matrix P, made up of the coefficients P_{nk}, describes completely the diffraction of the waves in the open channels. The order of the matrix is equal to the number of open channels 2N. We recall that in the assumed notation each spectrum corresponds to two open channels (the wave can be radiated either into the upper or into the lower half-space). Therefore when a new spectrum appears, the order of the matrix increases by two.

Let us write down the energy conservation law for the field (8), taking into account the fact that the component of the Poynting vector along the z axis is proportional to $\Phi^* \partial\Phi/\partial z$. Equating the energy fluxes in the waves traveling to and from the grating, we have

$$\sum_{k=1}^{2N} |\alpha_k|^2 \varkappa_k = \sum_{n,k,k'=1}^{2N} \alpha_k \alpha_{k'}^* P_{nk} P_{nk'}^* \varkappa_n, \quad (9)$$

where

$$\varkappa_n = \left[\frac{\omega^2}{c^2} - \left(k_y - \frac{2\pi n}{d} \right)^2 \right]^{1/2}. \quad (10)$$

By virtue of the complete arbitrariness of the coefficients α_k, we can conclude from (9) that

$$\sum P_{nk} P_{nk'}^* \varkappa_n = \delta_{kk'} \varkappa_k, \quad (11)$$

or, introducing the matrix

$$S_{nk} = \sqrt{\frac{\varkappa_n}{\varkappa_k}} P_{nk}, \quad (12)$$

we obtain

$$\sum_n S_{nk} S_{nk'}^* = \delta_{kk'}, \quad (13)$$

i.e., the matrix S is unitary:

$$S^+ = S^{-1}. \quad (14)$$

The unitarity condition is in this case the consequence of the energy conservation law.

Let us assume that the number N of the allowed spectra increases by unity. This phenomenon can occur, for example, when the frequency of the waves under consideration increases. When a new spectrum appears, the number of open channels increases, and the order of the matrix S consequently increases, as already noted, by two. The matrix acquires two new rows, numbered N + 1 and 2N + 2, and two new columns with the same numbers. Let us examine the behavior of the matrix elements S_{nk} near the point of appearance of the new spectrum. At the point of appearance of the new spectrum of order N + 1, the quantity

$$\varkappa_{N+1} = \varkappa_{2N+2} = \sqrt{\frac{\omega^2}{c^2} - \left[k_y - \frac{2\pi}{d}(N+1) \right]^2} \quad (15)$$

vanishes, and the quantity κ_{N+1} is small near this point. We can therefore represent the matrix S, accurate to terms linear in κ_{N+1}, in the form

$$S = S_0 + a\varkappa_{N+1}, \quad (16)$$

where a is a matrix to be determined. To determine a, we used the unitarity of the matrix S, which is written near threshold in the form

$$\begin{aligned} &S^+S = (S_0^+ - ia^+|\varkappa_{N+1}|)(S_0 + ia|\varkappa_{N+1}|) = 1 \text{ below the threshold} \\ &S^+S = S_0^+S_0 = 1 \text{ at the threshold} \\ &S^+S = (S_0^+ + a^+\varkappa_{N+1})(S_0 + a\varkappa_{N+1}) + B\varkappa_{N+1} = 1 \text{ above the threshold} \end{aligned} \quad (17)$$

The matrix B is defined by the equation

$$B_{ij} = \frac{1}{\sqrt{\varkappa_j \varkappa_j}} [P_{N+1,i} P_{N+1,j}^* + P_{2N+2,i} P_{2N+2,j}^*]. \quad (18)$$

The indices i and j cannot assume values corresponding to the newly opened channel, i.e., the values N + 1 and 2N + 2.

With the aid of simple calculations, we obtain the following expression for the unknown matrix a:

$$a = -{}^1/_2 S_0 B. \quad (19)$$

We recall that the order of the matrix S_0 is 2N. Substituting this relation in (16), we get

$$S = S_0 - \frac{i}{2} |\varkappa_{N+1}| S_0 B \text{ below the threshold}$$

$$S = S_0 - {}^1/_2 \varkappa_{N+1} S_0 B \text{ above the threshold}$$

The result (20) signifies that, generally speaking, when a spectrum of new order appears, the amplitudes in all the previously opened channels acquire a root singularity of the form

$$A + B\sqrt{\frac{\omega^2}{c^2} - \left[k_y - \frac{2\pi}{d}(N+1) \right]^2}, \quad (21)$$

where the coefficients A and B depend on the grating parameters and on the number of the channel. The coefficients A and B are complex numbers and relation (21) describes the behavior near threshold of both the amplitudes and the phase of the wave in the open channel. The amplitude of the wave is then determined by the absolute magnitude of (21), and is equal to

$$|A| + \varkappa_{N+1} \frac{\operatorname{Re} AB^*}{|A|}. \quad (22)$$

The phase of the wave in the open channel near the threshold of appearance of the new spectrum is given by

$$\operatorname{tg}\varphi=\frac{\operatorname{Im}A}{\operatorname{Re}A}-\varkappa_{N+1}\frac{\operatorname{Im}AB^*}{(\operatorname{Re}A)^2}. \quad (23)$$

Formulas (22) and (23) are valid above the threshold of the appearance of the new spectrum. Below the threshold, the amplitude of the wave in the existing channels can be written in the form

$$|A|+|\varkappa_{N+1}|\frac{\operatorname{Im}AB^*}{|A|}, \quad (24)$$

and the phase is given by

$$\operatorname{tg}\varphi=\frac{\operatorname{Im}A}{\operatorname{Re}A}+|\varkappa_{N+1}|\frac{\operatorname{Re}AB^*}{(\operatorname{Re}A)^2}. \quad (25)$$

We note that the foregoing calculations are closely connected with those employed in the theory of threshold nuclear reactions.[3]

2. DIFFRACTION OF ELECTROMAGNETIC WAVE FROM THE OPEN END OF A WAVEGUIDE

We shall consider below the incidence of an electromagnetic wave on the open end of a waveguide. For simplicity we confine ourselves to the case of a cylindrical semi-infinite waveguide with infinitesimally thin ideally conducting walls. As is well known, the natural modes in a cylindrical single-connected waveguide are either of the electric type or of the magnetic type, depending on which field component along the z axis (which is parallel to the waveguide generatrix) differs from zero.[4] When $E_z \neq 0$ we have E waves, and when when $H_z \neq 0$ we have H waves.

Assume that a wave propagates from the interior of the waveguide to an open end. The field inside the waveguide can be written with the aid of two Hertz vectors—electric Π and magnetic $\tilde{\Pi}$:[5]

$$\Pi=\Pi_z=\Pi_n^-+\sum_m L_{mn}\Pi_m^+,\quad \tilde{\Pi}=\tilde{\Pi}_z=\sum_k M_{kn}\tilde{\Pi}_k^+. \quad (26)$$

By $\Pi_m^{\pm}$ and $\tilde{\Pi}_m^{\pm}$ we denote here the Hertz vectors of the electric and magnetic modes that can propagate in the waveguide of a given cross section:

$$\Pi_n^{\pm}=\Psi_n(x,y)e^{\pm ih_nz},\qquad \tilde{\Pi}_m^{\pm}=\tilde{\Psi}_m(x,y)e^{\pm i\tilde{h}_mz}; \quad (27)$$

Ψ_n and $\tilde{\Psi}_m$ are the solutions of the corresponding boundary-value problems, and h_n and $\tilde{h}_m$ are the propagation constants of the electric and magnetic waves. We shall assume the functions Ψ_n and $\tilde{\Psi}_m$ to be normalized in such a way that the energy flux carried through the waveguide cross section is equal to the propagation of h_n (and accordingly $\tilde{h}_m$). The index n should be taken to mean the set of indices defining the eigenfunction. The electric and magnetic fields in the waveguide are determined in terms of Π and $\tilde{\Pi}$ in the well-known manner:

$$\mathbf{E}=-\frac{1}{c^2}\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{c}\frac{\partial}{\partial t}\operatorname{rot}\tilde{\Pi},$$

$$\mathbf{H}=\frac{1}{c}\operatorname{rot}\frac{\partial\Pi}{\partial t}-\frac{1}{c^2}\frac{\partial^2\tilde{\Pi}}{\partial t^2}. \quad (28)$$

The expression for the field in the waveguide in the form (26) has the following physical meaning. The first term in the expression for the electric Hertz vector describes an E wave that travels from the interior of the waveguide to its open end. Upon striking the open end, this wave is partially reflected and is transformed into other electric and magnetic modes that travel from the open end to the interior of the waveguide. The sum over m in expression (26) for Π describes just the E waves that travel to the interior of the wave guide. The numbers L_{mn} denote the coefficients of transformation of the n-th incident E wave into the m-th reflected E wave. The vector $\tilde{\Pi}$ describes H waves. The quantity M_{kn} is the coefficient of transformation of the n-th incident E wave into the k-th reflected H wave.

We now write the field outside the waveguide at large distances R from the open end:

$$\Pi=F_n(\vartheta,\varphi)\frac{e^{ikR}}{R},$$
$$\tilde{\Pi}=\tilde{F}_n(\vartheta,\varphi)\frac{e^{ikR}}{R}, \quad (29)$$

where the functions F_n and $\tilde{F}_n$ are determined by the number of the incident E waves.

If an H wave is incident on the open end of the waveguide from its interior, the field can be written in the form

$$\Pi=\sum_n N_{nk}\Pi_n^+,\quad \tilde{\Pi}=\tilde{\Pi}_k^-+\sum_s R_{sk}\tilde{\Pi}_s^+, \quad (30)$$

where N_{nk} is the coefficient of transformation of the incident H wave numbered k into the reflected E wave numbered n, and R_{sk} is the coefficient of transformation of the incident H wave numbered k into the reflected H wave numbered s. The fields outside the waveguide (as $R\to\infty$) are given by

$$\Pi=\Phi_k(\vartheta,\varphi)\frac{e^{ikR}}{R},\quad \tilde{\Pi}=\tilde{\Phi}_k(\vartheta,\varphi)\frac{e^{ikR}}{R}. \quad (31)$$

The functions Φ_k and $\tilde{\Phi}_k$ are determined by the number of the incident H wave.

We now consider the general case, when an arbitrary superposition of electric and magnetic waves propagates from the interior of the waveguide towards its open end:

$$\Pi^-=\sum\alpha_n\Pi_n^-,\quad \tilde{\Pi}^-=\sum\beta_k\Pi_k^-. \quad (32)$$

the total field inside the waveguide can be written in the form

$$\Pi=\sum_n\alpha_n\Pi_n^-+\sum_{n,m}\alpha_nL_{mn}\Pi_m^++\sum_{k,m}\beta_kN_{mk}\Pi_m^+,$$

$$\tilde{\Pi}=\sum_k\beta_k\Pi_k^-+\sum_{n,s}\alpha_nM_{sn}\tilde{\Pi}_s^++\sum_{k,s}\beta_kR_{sk}\tilde{\Pi}_s^+. \quad (33)$$

If we consider the field inside the waveguide, far from its open end, then the summation in (33) extends only over the open channels.

We also write down the field outside the waveguide, at large distances from the open end:

$$\Pi=\Big(\sum_n\alpha_nF_n+\sum_k\beta_k\Phi_k\Big)\frac{e^{ikR}}{R},$$
$$\tilde{\Pi}=\Big(\sum_n\alpha_n\tilde{F}_n+\sum_k\beta_k\tilde{\Phi}_k\Big)\frac{e^{ikR}}{R}. \quad (34)$$

Formulas (33) and (34) can be simplified. Let there be N_1 open electric channels and N_2 open magnetic channels. We introduce a matrix P of order N_1+N_2, defined in the following manner:

$$P = \begin{pmatrix} L & N \\ M & R \end{pmatrix}. \qquad (35)$$

Obviously

$$\begin{aligned} P_{ik} &= L_{ik} && (i,\ k \leqslant N_1); \\ P_{i,\,N_1+k} &= N_{ik} && (i \leqslant N_1,\ k \leqslant N_2); \\ P_{N_1+i,\,k} &= M_{ik} && (i \leqslant N_2,\ k \leqslant N_1); \\ P_{N_1+i,\,N_2+k} &= R_{ik} && (i,\ k \leqslant N_2). \end{aligned} \qquad (36)$$

We also put

$$\begin{aligned} &\beta_k = \alpha_{N_1+k} \qquad (k \leqslant N_2); \\ &\Phi_k = F_{N_1+k}, \quad \widetilde{\Phi}_k = \widetilde{F}_{N_1+k}, \quad \widetilde{h}_k = h_{N_1+k}. \end{aligned} \qquad (37)$$

Then the law of conservation of the electromagnetic energy assumes the form

$$\sum_n |\alpha_n|^2 h_n = \sum_{m,\,n,\,n'} \alpha_n \alpha_{n'}^* \left[P_{mn} P_{mn'}^* h_m + \frac{c}{8\pi} k^4 \int (F_n F_{n'}^* + \widetilde{F}_n \widetilde{F}_{n'}^*) \sin^2 \vartheta d\Omega \right]. \qquad (38)$$

The summation over all the indices is from 1 to $N_1 + N_2$. The left side of (38) describes the energy flux of the incident waves. The right side of (38) is the sum of the energy fluxes of the respective waves and of the energy that goes out to the open space (the integral over all angles $d\Omega$).

In view of the arbitrariness of the coefficients α_n, we get from (38)

$$\sum_m S_{mn} S_{mn'}^* = \delta_{nn'} - \frac{c}{8\pi} \frac{k^4}{\sqrt{h_n h_{n'}}} \int (F_n F_{n'}^* + \widetilde{F}_n \widetilde{F}_{n'}^*) \sin^2 \vartheta d\Omega, \qquad (39)$$

where

$$S_{mn} = \sqrt{\frac{h_m}{h_n}} P_{mn}. \qquad (40)$$

The matrix S_{mn} introduced in this manner is symmetrical, as can be readily shown by using the reciprocity theorem. Relation (39), which follows from the energy conservation law, does not reduce in this case to the condition for the unitarity of the matrix S, since part of the energy goes out to the open space.

Let us examine the behavior of the elements S_{mn} at the threshold of appearance of the new channel (it is immaterial whether it is electric or magnetic). Let the propagation constant of the wave in the new channel be κ. Near the threshold of the new channel we assume, as usual, that κ is the small quantity and confine ourselves to relations of order not higher than the first in κ. We write relation (39) in the form

$$SS^+ = 1 - \Phi, \qquad (41)$$

where

$$\Phi_{nn'} = \frac{c}{8\pi} \frac{k^4}{\sqrt{h_n h_{n'}}} \int (F_n F_{n'}^* + \widetilde{F}_n \widetilde{F}_{n'}^*) \sin^2 \vartheta d\Omega. \qquad (42)$$

Near the threshold of the new channel, the following relations hold: below the threshold

$$\begin{aligned} &S = S_0 + ia|\varkappa|, \\ &F_n = F_{n0} + i\gamma_n |\varkappa|, \qquad \widetilde{F}_n = \widetilde{F}_{n0} + i\widetilde{\gamma}_n |\varkappa|; \end{aligned} \qquad (43)$$

at the threshold

$$S = S_0, \qquad F_n = F_{n0}, \qquad \widetilde{F}_n = \widetilde{F}_{n0}; \qquad (44)$$

and above the threshold

$$S = S_0 + a\varkappa, \qquad F_n = F_{n0} + \gamma_n \varkappa, \qquad \widetilde{F}_n = \widetilde{F}_{n0} + \widetilde{\gamma}_n \varkappa. \qquad (45)$$

If the values of F_n and $\widetilde{F}_n$ below the threshold, given in formula (43), are substituted in formula (42), then we obtain

$$\Phi = \Phi_0 + i|\varkappa| A \quad \text{below the threshold} \qquad (46)$$

where

$$A_{nn'} = \frac{c}{8\pi} \frac{k^4}{\sqrt{h_n h_{n'}}} \int (F_{n'0}^* \gamma_n - F_{n0} \gamma_{n'}^* + \widetilde{F}_{n'0}^* \widetilde{\gamma}_n - \widetilde{F}_{n0} \widetilde{\gamma}_{n'}^*) \sin^2 \vartheta d\Omega. \qquad (47)$$

We note that the matrix A is anti-Hermitian, i.e., $A^+ = -A$.

We now write the expansion in terms of κ for the matrix Φ above the threshold. From (42) and (45) we get

$$\Phi = \Phi_0 + \varkappa H \quad \text{above the threshold} \qquad (48)$$

where

$$H_{nn'} = \frac{ck^4}{8\pi \sqrt{h_n h_{n'}}} \int (F_{n'0}^* \gamma_n + F_{n0} \gamma_{n'}^* + \widetilde{F}_{n'0}^* \widetilde{\gamma}_n + \widetilde{F}_{n0} \widetilde{\gamma}_{n'}^*) \sin^2 \vartheta \, d\Omega. \qquad (49)$$

From (49) we see that H is a Hermitian matrix.

The obtained relations enable us to write down the electromagnetic-energy conservation law (41) near the threshold of the new channel:

$$\begin{aligned} (S_0 + ia|\varkappa|)(S_0^+ - ia^+|\varkappa|) &= 1 - \Phi_0 - iA|\varkappa| && \text{below the threshold} \\ S_0 S_0^+ &= 1 - \Phi_0 && \text{at the threshold} \\ (S_0 + a\varkappa)(S_0^+ + a^+\varkappa) + B\varkappa &= 1 - \Phi_0 - H\varkappa && \text{above the threshold} \end{aligned} \qquad (50)$$

where

$$B_{ik} = \frac{1}{\sqrt{h_i h_k}} P_{N,\,i} P_{N,\,k}^*, \qquad (51)$$

and N denotes the number of the channel. From (50) we obtain two relations

$$\begin{aligned} S_0 a^+ - a S_0^+ &= A, \\ S_0 a^+ + a S_0^+ &= -B - H. \end{aligned} \qquad (52)$$

Adding and subtracting these two equations, we obtain

$$\begin{aligned} S_0 a^+ &= {}^1/_2 (A - B - H), \\ a S_0^+ &= -{}^1/_2 (A + B + H). \end{aligned} \qquad (53)$$

Recognizing that B and H are Hermitian matrices, and A is anti-Hermitian, we can show that one of the relations (53) is the consequence of the other. From (53) we obtain

$$a = -{}^1/_2 (A + B + H)(S_0^{-1})^+. \qquad (54)$$

Substituting this value of a in formulas (43) and (45) and taking (40) into account, we obtain the law of the behavior of the transformation coefficients P_{mn} near the threshold of appearance of the new mode.

Let us estimate the relative order of magnitude of the threshold anomalies. To this end we note that the matrix S defined by formula (12) is symmetrical, as can be readily shown with the aid of the reciprocity theorem. Taking the symmetry of the S matrix into account, expression (19) can be rewritten in the form

$$a_{ik} = \frac{1}{2}\left[\frac{P_{N+1,i}P_{N+1,k}}{\sqrt{\varkappa_i\varkappa_k}}P^*_{N+1,N+1} + \frac{P_{N+1,i}P_{2N+2,k}}{\sqrt{\varkappa_i\varkappa_k}}P^*_{N+1,2N+2} + \frac{P_{2N+2,i}P_{2N+2,k}}{\sqrt{\varkappa_i\varkappa_k}}P^*_{2N+2,2N+2} + \frac{P_{2N+2,i}P_{N+1,k}}{\sqrt{\varkappa_i\varkappa_k}}P^*_{2N+2,N+1}\right]. \quad (19a)$$

To estimate the value of a_{ik}, it is necessary to know the order of magnitude of the coefficients P_{ik} that determine the transformation of the waves in the newly opened channels numbered $N + 1$ and $2N + 2$.

To estimate the transformation coefficients P, we shall use the exact solution of the problem of diffraction of an electromagnetic wave that is normally incident on a plane grating made up of ideally conducting ribbons. It is assumed here that the width of the slits is equal to the width of the ribbons. Obviously, the exact solution of such a problem[5] gives expressions for the transformation coefficients P_{0n}. Knowing these expressions, we can determine the form of the threshold singularities by expanding P_{0n} in powers of κ_{N+1} at the threshold of currents of the new channel. In particular, in the case of a first-order spectrum, the coefficient P_{00}, as follows from the exact formulas, behaves in the following manner:

$$|P_{00}| = 1 - 2\varkappa_1 / k,$$

and the coefficient P_{01} can be written in the form

$$|P_{01}| = \sqrt{2}(1 - \varkappa_1 / k).$$

It is seen from these formulas that in the region $2\pi < kd < 4\pi$ (i.e., near the threshold for the first-order spectrum) the magnitude of the threshold anomaly is of the same order as the first term of the expansion, which does not depend on κ_1. From the exact formulas it is also seen that the relative magnitude of the anomaly in the behavior of the coefficient $|P_{00}|$ decreases with increasing order of the occurring spectrum. In addition, P_{0n} decreases in inverse proportion to n at high frequencies.

The problem of diffraction by a grating has not yet found an exact solution capable of giving the complete matrix of the transformation coefficients P_{ik}. The value of P_{ik} can be estimated with the aid of the Kirchhoff approximation. This yields

$$P_{ik} \sim (i - k)^{-1}$$

at a sufficiently large difference $i - k$. Therefore the first coefficient in (20), which gives the behavior of the matrix coefficients at the threshold, is proportional to $(i - k)^{-1}$, and the coefficient of κ_{N+1} is proportional to $(N + 1 - i)^{-1}(N + 1 - k)^{-1}$. These rough estimates can give an idea of the relative magnitude of the threshold anomaly of the coefficient P_{ik}. It must be remembered, however, that all this reasoning does not apply to diffraction by a grating having a more complicated profile. The magnitude of the anomalies depend strongly on the profile of the grating.

Let us consider now the question of the useful information that can be extracted by observing threshold anomalies. We note first that, as a rule, diffraction problems cannot be solved exactly and quantitative results are obtained mostly with the aid of electronic computers. Knowledge of the law governing the threshold anomalies facilitates the calculations. Indeed, since all the coefficients P_{ik} have near the threshold the form $A + B\kappa$, it is sufficient to determine the two constants A and B in order to ascertain the behavior of the amplitudes and phases of the waves near the threshold.

In addition to this circumstance, we note also the following. The matrix a_{ik} (19), which contains the coefficients of κ in the expressions for S_{ik}, is determined only by the coefficients of transformations from the existing channels to the newly created ones (see formula (19a)). By measuring the threshold anomalies of the transformation coefficients in the "old" channels, we can determine with the aid of formulas (20), (19), and (19a) all the coefficients of wave transformation from the "old" channels into the "new" ones. Conversely, knowing all the transformation coefficients between the "old" and "new" channels, we can determine the threshold anomalies in the "old" channels. (The entire reasoning is applicable also to a waveguide.) The symmetry of the matrix S also reduces the number of measurements.

[1] B. M. Bolotovskiĭ and A. N. Lebedev, Zh. Eksp. Teor. Fiz. **53**, 1349 (1967) [Sov. Phys.-JETP **26**, 784 (1968)].

[2] R. W. Wood, Phil. Mag. **23**, 310 (1912); Phys. Rev. **48**, 928 (1935).

[3] A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Rasseyanie, reaktsii i raspady v nerelyativistskoĭ kvantovoĭ mekhanike (Scattering, Reactions, and Decays in Nonrelativistic Quantum Mechanics), Nauka, 1966.

[4] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred, Fizmatgiz, 1959 (Electrodynamics of Continuous Media, Addison-Wesley, 1960).

[5] L. A. Vaĭnshteĭn, Teoriya difraktsii i metod faktorizatsii (Theory of Diffraction and the Factorization Method), Soviet Radio, 1966.

Translated by J. G. Adashko
21