SOVIET PHYSICS JETP VOLUME 30, NUMBER 1 JANUARY, 1970

CONTRIBUTION TO THE THEORY OF PARAMETRIC INTERACTION BETWEEN AN ULTRA-HIGH FREQUENCY FIELD AND A PLASMA

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Submitted December 19, 1968

Zh. Eksp. Teor. Fiz. 57, 183–195 (July, 1969)

Equations are formulated for the evolution of the particle distribution and of the nonequilibrium field created by parametric action of a strong ultrahigh frequency field in a turbulent plasma. Application of the theory to the case of the kinetic instability of a nonisothermal plasma, due to the Cerenkov effect on electrons, indicates an anomalous increase of dissipative high frequency conductivity of the plasma, due to the increased intensity of the plasma oscillations.

1. Recent investigations have shown that when a plasma interacts with a strong microwave field, peculiar parametric instability effects are produced as the result of the oscillatory motion of the electrons relative to the ions (see the review[1]). The existing theory of parametric resonance in a plasma considers the nonlinear influence of a strong external field on the plasma, and in this sense it is nonlinear. On the other hand, such a theory, being confined to the linear approximation relative to the amplitudes of the growing perturbations in the unstable plasma, leaves unanswered the question of the influence of the developing instability on the plasma.

In this article we consider a parametric theory, nonlinear in the growing perturbations, of the interaction between a microwave field and a plasma. The theory described below corresponds to the description of the weakly-turbulent state of the plasma. It takes into account the influence of the perturbations of the field on the particle distribution, and neglects at the same time, for example when kinetic instability sets in[2], the processes of induced scattering of the waves[3].

The general premises of the theory are obtained by investigating the consequences that follow from the equation for the paired correlation function of the system of charged particles[4]. Detailed equations are obtained for the description of the kinetics of the plasma instability when low-frequency oscillations develop as a result of the high-frequency Cerenkov effect on the electrons[2]. In addition, we reveal, for such an instability of a non-isothermal plasma, an effect wherein the dissipative high-frequency conductivity of the plasma increases, corresponding to anomalous acceleration of plasma heating as the result of the increase of the plasma-oscillation intensity.

As usual[1], in the theory of parametric resonance in a plasma, we assume that the external electric field is spatially homogeneous and monochromatic

$$\mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t. \tag{1.1}$$

Then, bearing in mind the development of instabilities connected with the potential field of the perturbations, we can construct the plasma theory with the aid of kinetic equations for the particle distribution functions f_a and the equations for the correlation functions. Moreover, neglecting effects of the interaction on the plasma waves with one another, we confine ourselves only to allowance for pair correlations (cf.[3]). To simplify the derivations, we confine ourselves to spatially-homogeneous distributions. We can then write the following system of equations describing the plasma kinetics[4]:

$$\frac{\partial f_a(\mathbf{p}_a, t)}{\partial t} + e_a \mathbf{E}(t) \frac{\partial f_a(\mathbf{p}_a, t)}{\partial \mathbf{p}_a} = \left[\frac{\partial f_a}{\partial t}\right]_{\mathrm{st}}$$

$$= \frac{\partial}{\partial \mathbf{p}_a} \sum_b \int d\mathbf{p}_b \frac{d\mathbf{k}}{(2\pi)^3} \frac{4\pi e_a e_b}{ik^2} \mathbf{k} G_{ab}(\mathbf{k}, \mathbf{p}_a, \mathbf{p}_b, t), \tag{1.2}$$

$$\left\{\frac{\partial}{\partial t} + i(\mathbf{k}, \mathbf{v}_a - \mathbf{v}_b) + \mathbf{E}(t)\left(e_a \frac{\partial}{\partial \mathbf{p}_a} + e_b \frac{\partial}{\partial \mathbf{p}_b}\right)\right\} G_{ab}(\mathbf{k}, \mathbf{p}_a, \mathbf{p}_b, t)$$

$$= i\mathbf{k} \frac{\partial f_a}{\partial \mathbf{p}_a} \frac{4\pi e_a}{k^2} \sum_c e_c \int d\mathbf{p}_c G_{bc}(-\mathbf{k}, \mathbf{p}_b, \mathbf{p}_c, t)$$

$$- i\mathbf{k} \frac{\partial f_b}{\partial \mathbf{p}_b} \frac{4\pi e_b}{k^2} \sum_c e_c \int d\mathbf{p}_c G_{ac}(\mathbf{k}, \mathbf{p}_a, \mathbf{p}_c, t), \tag{1.3}$$

where e_a is the charge of particle of type a, G_{ab} is the component of the Fourier expansion of the pair-correlation function. In (1.3) we have omitted the inhomogeneous part describing the usual particle collisions.

The system (1.2) and (1.3) makes it possible to describe the weakly-turbulent state that occurs when parametric instabilities develop, and also the process of anomalous interaction between the microwave field and the plasma.

2. Unlike the theory of collision integrals, for a stable plasma in a strong field[4] we are interested in time intervals within which the pair correlations have not yet relaxed to their asymptotic values that are fully determined by the single-particle distributions. On the other hand, we are interested in plasma states which, from the point of view of the linear theory, are unstable and therefore lead to the occurrence of high-intensity oscillations. It is precisely these oscillations which determine the pair-correlation function. The latter means that the inhomogeneous part of (1.3) is relatively small. This is why we have omitted it.

The solution of (1.3) can be represented in the form

$$G_{ab}(\mathbf{k}, \mathbf{p}_a, \mathbf{p}_b, t) = \psi_a(\mathbf{k}, \mathbf{p}_a, t)\psi_b(-\mathbf{k}, \mathbf{p}_b, t), \tag{2.1}$$

where the functions ψ satisfy an equation of the type[1)]

$$\left\{\frac{\partial}{\partial t}+i\mathbf{k}\mathbf{v}_a+e_a\mathbf{E}(t)\frac{\partial}{\partial \mathbf{p}_a}\right\}\psi_a(\mathbf{k},\mathbf{p}_a,t)=ie_a\varphi_{\mathbf{k}}(t)\mathbf{k}\frac{\partial f_a(\mathbf{p}_a,t)}{\partial \mathbf{p}_a}, \quad (2.2)$$

and the function φ is defined by

$$\varphi_{\mathbf{k}}(t)=\frac{4\pi}{k^2}\sum_c e_c\int d\mathbf{p}_c\psi_c(\mathbf{k},\mathbf{p}_c,t). \quad (2.3)$$

In accordance with (2.1), the collision integral of the kinetic equation (1.2) takes the form

$$\left[\frac{\partial f_a}{\partial t}\right]_{\text{st}}=-\frac{\partial}{\partial p_a}\int\frac{d\mathbf{k}}{(2\pi)^3}e_a i\,\mathbf{k}\varphi_{-\mathbf{k}}(t)\psi_a(\mathbf{k},\mathbf{p}_a,t). \quad (2.4)$$

From the kinetic equation there follow here the following energy and momentum conservation laws:

$$\frac{d}{dt}\sum_a\int d\mathbf{p}_a\mathbf{p}_a f_a(\mathbf{p}_a t)$$

$$=\int\frac{d\mathbf{k}}{(2\pi)^3}i\mathbf{k}\frac{k^2}{4\pi}\varphi_{-\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)+\mathbf{E}(t)\sum_a e_a\int d\mathbf{p}_a f_a, \quad (2.5)$$

$$\frac{d}{dt}\left\{\sum_a\int d\mathbf{p}_a f_a(\mathbf{p}_a,t)\frac{p_a^2}{2m_a}+\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k^2}{8\pi}\varphi_{-\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\right\}$$

$$=\mathbf{E}(t)\sum_a e_a\int d\mathbf{p}_a\mathbf{v}_a f_a. \quad (2.6)$$

It is easy to see that $\psi_{\mathbf{k}}(t)$ is the potential of the electric field of the perturbations that develop when turbulent states occur.

Neglecting the initial value of ψ, and also considering times that are separated from the initial time by many periods of the oscillations, we can write (2.2) in the form

$$\psi_a(\mathbf{k},\mathbf{p}_a,t)=\int_{-\infty}^{t}dt'\exp\left\{i\mathbf{k}\mathbf{v}_a(t'-t)+i\frac{e_a}{m_a}\int_{t'}^{t}dt'''\mathbf{E}(t''')\mathbf{k}(t'''-t')\right\}$$

$$\times ie_a\varphi_{\mathbf{k}}(t')\mathbf{k}\frac{\partial}{\partial \mathbf{p}_a}f_a\left(\mathbf{p}_a+e_a\int_{t}^{t'}dt''\mathbf{E}(t''),t'\right). \quad (2.7)$$

Substituting this expression in (2.3) and (2.4), we obtain the following system of equations for the potential of the turbulent field and particle distributions:

$$\varphi_{\mathbf{k}}(t)=\int_{-\infty}^{t}dt'\varphi_{\mathbf{k}}(t')\sum_a\frac{4\pi e_a^2}{k^2}\int d\mathbf{p}_a\,i\mathbf{k}\frac{\partial}{\partial \mathbf{p}_a}f_a\left(\mathbf{p}_a+e_a\int_{t}^{t'}dt''\mathbf{E}(t''),t'\right)$$

$$\times\exp\left\{i\mathbf{k}\mathbf{v}_a(t'-t)+i\frac{e_a}{m_a}\int_{t'}^{t}dt'''(t'''-t')k\mathbf{E}(t''')\right\}, \quad (2.8)$$

$$\frac{\partial f_a(\mathbf{p}_a,t)}{\partial t}+e_a\mathbf{E}(t)\frac{\partial f_a(\mathbf{p}_a,t)}{\partial \mathbf{p}_a}=\frac{\partial}{\partial p_{a,i}}e_a^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\varphi_{-\mathbf{k}}(t)$$

$$\times\int_{-\infty}^{t}dt'\varphi_{\mathbf{k}}(t')\frac{\partial}{\partial p_{a,j}}f_a\left(\mathbf{p}_a+e_a\int_{t}^{t'}dt''\mathbf{E}(t''),t'\right)$$

$$\times\exp\left\{i\mathbf{k}\mathbf{v}_a(t'-t)+i\frac{e_a}{m}\int_{t'}^{t}dt'''(t'''-t')\mathbf{k}\mathbf{E}(t''')\right\}. \quad (2.9)$$

It will be convenient in what follows to introduce in place of f_a the distribution F_a, which is determined by the relation

$$f_a\left(\mathbf{p}_a-\frac{e_a\mathbf{E}_0}{\omega_0}\cos\omega_0 t,t\right)=F_a(\mathbf{p}_a,t),$$

where account is taken of formula (1.1). In addition, for the case of a monochromatic dependence of the external field on the time (1.1), it is convenient to represent the potential field and the distributions in the form of expansions in terms of harmonics:

$$\varphi_{\mathbf{k}}(t)=\sum_{n=-\infty}^{+\infty}e^{-in\omega_0 t}\varphi^{(n)}(\mathbf{k},t), \quad (2.11)$$

$$F_a(\mathbf{p}_a,t)=\sum_{n=-\infty}^{+\infty}e^{-in\omega_0 t}F_a^{(n)}(\mathbf{p}_a,t). \quad (2.12)$$

In this case, for example for the oscillations that grow during the parametric resonance, called the low-frequency oscillations, the amplitudes of such expansions of $\varphi^{(n)}(\mathbf{k},t)$ and $F_a^{(n)}(\mathbf{p}_a,t)$ will change little within the period of the oscillations of the external electric field. Accordingly, we have for the amplitudes the following system of equations:

$$\varphi^{(n)}(\mathbf{k},t)=\int_{-\infty}^{0}dt'\sum_{l,m,s=-\infty}^{+\infty}\sum_a\frac{4\pi e_a^2}{k^2}\int d\mathbf{p}_a\,e^{i\mathbf{k}\mathbf{v}_a t'}i\mathbf{k}\frac{\partial F_a^{(s)}(\mathbf{p}_a,t+t')}{\partial \mathbf{p}_a}$$

$$\times J_{l-s}(A_a)J_{l+m-n}(A_a)e^{-i(l+m)\omega_0 t'}\varphi^{(m)}(\mathbf{k},t+t'), \quad (2.13)$$

$$-in\omega_0F_a^{(n)}(\mathbf{p}_a,t)+\frac{\partial F_a^{(n)}(\mathbf{p}_a,t)}{dt}=\frac{\partial}{\partial p_{a,i}}e_a^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j$$

$$\times\sum_{m,l,s,u=-\infty}^{+\infty}J_{m+u-n}(A_a)J_{u-s-l}(A_a)\varphi^{(m)}(-\mathbf{k},t)$$

$$\times\int_{-\infty}^{0}dt'e^{-iu\omega_0t'+i\mathbf{k}\mathbf{v}_a t'}\varphi^{(s)}(\mathbf{k},t+t')\frac{\partial F_a^{(l)}(\mathbf{p}_a,t+t')}{\partial p_{a,j}}, \quad (2.14)$$

where J_n are Bessel functions and $A_a=e_a\mathbf{E}_0\cdot\mathbf{k}/m_a\omega_0^2$. This system of equations can be used to justify the estimate presented in[7] for the energy contained in the high-frequency parts of the particle distributions at parametric resonance in a plasma situated in a strong field of frequency close to the electron Langmuir frequency $\omega_{Le}=\sqrt{4\pi e^2n_e/m_e}$. We note that the average work performed by the external field on the plasma (averaged over the period of the microwave-field oscillations) is determined by the amplitudes $F^{(\pm1)}$ and is given by

$$\langle\mathbf{E}(t)\mathbf{j}(t)\rangle=\sum_a e_a\frac{1}{2i}\int d\mathbf{p}_a\mathbf{E}_0\mathbf{v}_a[F_a^{(1)}(\mathbf{p}_a,t)-F_a^{(-1)}(\mathbf{p}_a,t)]. \quad (2.15)$$

This expression characterizes the rate of plasma heating.

3. A much more detailed study of the system (2.13) and (2.14) can be performed for the case of oscillations whose frequency turns out to be much larger than the increment. Such a situation occurs both under parametric-resonance conditions in a cold plasma near the threshold frequency of the external field[7], and in kinetic buildup of oscillations when the frequency of the external field differs greatly from the electron Langmuir frequency[2]. However, before we proceed to derive the corresponding averaged equations, we note that in accordance with (2.14), at least during a relatively long time, the expansion amplitudes $F_a^{(n)}$ for $n\neq 0$ will remain small compared with $F_a^{(0)}$. We can use for these amplitudes the expression:

$$F_a^{(n)}(\mathbf{p}_a,t)=\frac{ie_a^2}{n\omega_0}\frac{\partial}{\partial p_{a,i}}\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_{m,s,l=-\infty}^{+\infty}J_{m+l-n}(A_a)J_{l-s}(A_a)$$

[1)]A similar equation is obtained also in the theory that uses the Prigogine-Balescu diagram technique for the construction of a kinetic theory of a stable plasma [5]. In the absence of an external field – see [6].

$$\times\varphi^{(m)}(-\mathbf{k},t)\int_{-\infty}^{0}dt'\,e^{i(\mathbf{kv}_a-l\omega_0)t'}\varphi^{(s)}(\mathbf{k},t+t')\frac{\partial F_a^{(0)}(\mathbf{p}_a,t+t')}{\partial p_{a,j}}. \quad (3.1)$$

We shall subsequently assume that these quantities are small, and expression (3.1) can be used to check on such an assumption. Accordingly, we shall henceforth retain only $F_a^{(0)}$ in formula (2.14) with n = 0 and in formula (2.13).

Denoting by $\omega_r(\mathbf{k}, t)$ the frequency of the plasma oscillations of the mode r, we represent the amplitude of the field $\varphi^{(n)}$ in the form

$$\varphi^{(n)}(\mathbf{k},t)=\sum_r\exp\left\{-i\int^t dt'\omega_r(\mathbf{k},t')\right\}\varphi_n(\omega_r(\mathbf{k},t),\mathbf{k},t), \quad (3.2)$$

where ω_r and φ_n depend slowly on the time.

After substituting (3.2) in the right side of (2.14) with n = 0, we have, accurate to the first time derivatives of the slowly varying quantities,

$$\frac{\partial F_a^{(0)}(\mathbf{p}_a,t)}{\partial t}=\frac{\partial}{\partial p_{a,i}}e_a^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_{m\ l,s=-\infty}^{+\infty}J_{l+m}(A_a)J_{l-s}(A_a)$$
$$\times\sum_{r,u}\exp\left\{-i\int^t dt''[\omega_r(-\mathbf{k},t'')+\omega_u(\mathbf{k},t'')]\right\}\varphi_m(\omega_r(-\mathbf{k},t),-\mathbf{k},t)$$
$$\times\left\{\frac{i}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}-\left[\frac{\partial}{\partial\omega_u(\mathbf{k},t)}\frac{1}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}\right]\right.$$
$$\times\left[\frac{\partial}{\partial t}+\frac{\partial\omega_u(\mathbf{k},t)}{\partial t}\frac{\partial}{\partial\omega_u(\mathbf{k},t)}\right]-\frac{1}{2}\frac{\partial\omega_u(\mathbf{k},t)}{\partial t}$$
$$\left.\times\left[\frac{\partial^2}{\partial\omega_u^2(\mathbf{k},t)}\frac{1}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}\right]\right\}\left\{\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\frac{\partial F_a^{(0)}(\mathbf{p}_a,t)}{\partial p_{a,j}}\right\}. \quad (3.3)$$

Here $(x + i0)^{-1} = -i\pi\delta(x) + P/x$, where P denotes the principal value. The right side of formula (3.3) contains both a slowly varying part and a part oscillating at the frequencies of the plasma oscillations. Neglecting the small oscillating part after averaging, we obtain

$$\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial t}=\frac{\partial}{\partial p_{a,i}}e_a^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_{m,l,s=-\infty}^{+\infty}J_{l-m}(A_a)J_{l-s}(A_a)$$
$$\sum_u\left\{\pi\delta(l\omega_0+\omega_u(\mathbf{k},t)-\mathbf{kv}_a)\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\right.$$
$$\times\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial p_{a,j}}-\left[\frac{\partial}{\partial\omega_u(\mathbf{k},t)}\frac{P}{l\omega_0+\omega_u(\mathbf{k},t)-\mathbf{kv}_a}\right]\frac{\partial^2\bar F_a(\mathbf{p}_a,t)}{\partial t\,\partial p_{a,j}}$$
$$\times\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)-\frac{1}{2}\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial p_{a,j}}$$
$$\times\frac{d}{dt}\left[\left(\frac{\partial}{\partial\omega_u(\mathbf{k},t)}\frac{P}{l\omega_0+\omega_u(\mathbf{k},t)-\mathbf{kv}_a}\right)\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\right.$$
$$\left.\times\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\right]-i\frac{\pi}{2}\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial p_{a,j}}\frac{\partial\delta(l\omega_0+\omega_u(\mathbf{k},t)-\mathbf{kv}_a)}{\partial\omega_u(\mathbf{k},t)}$$
$$\times\left[\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\frac{d}{dt}\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\right.$$
$$\left.\left.-\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\frac{d}{dt}\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\right]\right\}, \quad (3.4)$$

where $\bar F_a$ is the slowly-varying part of the function F_a.

We note that neglect of the oscillating part of the function $F_a^{(0)}$ is permissible because we have assumed in this section that the increment of the plasma oscillations is small compared with their frequency. Expression (3.4) obtained here is the analog of the kinetic equation of the so called quasilinear theory[8,9,10]. For the investigated type of plasma instability in a strong microwave field, it is the fundamental equation describing, together with the field equations (3.6) derived below, the relaxation of the particles and of the plasma oscillations. However, for a plasma in a microwave field, the particle distribution is described not only by the zeroth harmonic of the distribution function. In this connection, averaging (3.1) in analogy with the averaging used in the derivation of (3.4), we get

$$\bar F_a^{(n)}(\mathbf{p}_a,t)=\frac{i}{n\omega_0}\frac{\partial}{\partial p_{a,i}}e_a^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_{m,l,s=-\infty}^{+\infty}\sum_u\varphi_{-m}(-\omega_u(\mathbf{k},t),-\mathbf{k},t)\cdot$$
$$\times J_{l-s}(A_a)J_{l-n-m}(A_a)\left\{\frac{i}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}\right.$$
$$-\left[\frac{\partial}{\partial\omega_u(\mathbf{k},t)}\frac{1}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}\right]\frac{d}{dt}$$
$$\left.-\frac{1}{2}\frac{\partial\omega_u(\mathbf{k},t)}{\partial t}\left[\frac{\partial^2}{\partial\omega_u(\mathbf{k},t)^2}\frac{1}{l\omega_0+\omega_u(\mathbf{k},t)+i0-\mathbf{kv}_a}\right]\right\}\cdot$$
$$\times\left\{\varphi_s(\omega_u(\mathbf{k},t),\mathbf{k},t)\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial p_{a,j}}\right\}. \quad (3.5)$$

Finally, substituting (3.2) and (2.13) we obtain, likewise accurate to the first derivatives with respect to the time

$$\varphi_n(\omega_r(\mathbf{k},t),\mathbf{k},t)+\sum_a\sum_{s,l=-\infty}^{+\infty}J_{l-s}(A_a)J_{l-n}(A_a)$$
$$\times\delta\varepsilon_a(\omega_r(\mathbf{k},t)+l\omega_0,\mathbf{k},t)\varphi_s(\omega_r(\mathbf{k},t),\mathbf{k},t)$$
$$=i\sum_{l,s=-\infty}^{+\infty}J_{l-s}(A_a)J_{l-n}(A_a)$$
$$\times\left\{-\frac{d}{dt}\left[\frac{\partial\delta\varepsilon_a(\omega_r(\mathbf{k},t)+l\omega_0,\mathbf{k},t)}{\partial\omega_r(\mathbf{k},t)}\varphi_s(\omega_r(\mathbf{k},t),\mathbf{k},t)\right]\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2\delta\varepsilon_a(\omega_r(\mathbf{k},t)+l\omega_0,\mathbf{k},t)}{\partial\omega_r^2(\mathbf{k},t)}\frac{\partial\omega_r(\mathbf{k},t)}{\partial t}\varphi_s(\omega_r(\mathbf{k},t),\mathbf{k},t)\right\}, \quad (3.6)$$

where

$$\delta\varepsilon_a(\omega,\mathbf{k},t)=\frac{4\pi e_a^2}{k^2}\int d\mathbf{p}_a\frac{1}{\omega+i0-\mathbf{kv}_a}\mathbf{k}\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial\mathbf{p}_a}. \quad (3.7)$$

The last quantity is the contribution of the particles of type a to the longitudinal dielectric constant of the plasma.

4. Under conditions when kinetic instability develops[2] in a plasma situated in a strong microwave field, for example with frequency greatly exceeding the electron Langmuir frequency, the growing oscillations have a frequency close to the ion Langmuir frequency $\omega_{Li} = \sqrt{4\pi e^2 n_i/m_i}$. In this connection, the dispersion properties of the oscillations vary slowly in time. In similar cases we can neglect in Eqs. (3.4)—(3.6) the time derivatives of the frequency and of the distribution function. Assuming these quantities negligible, we consider in greater detail the case of a plasma consisting of electrons and one type of ions with an electron temperature much higher than the ion temperature. To simplify the derivations, we neglect from the outset the influence of the electric field on the ions, bearing in mind the fact that the decisive factor is the relative motion of the electrons and ions in the microwave field[11]. Then (3.6) takes the form

$$\varphi_n\{1+\delta\varepsilon_i'(\omega+n\omega_0,\mathbf{k})\}+\sum_{l=-\infty}^{+\infty}J_{l-n}(a)\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})\Phi_l$$
$$+i\left\{\frac{\partial\delta\varepsilon_i'(\omega+n\omega_0,\mathbf{k})}{\partial\omega}\frac{d\varphi_n}{dt}+\delta\varepsilon_i''(\omega+n\omega_0,\mathbf{k})\varphi_n\right.$$

$$+\sum_{l=-\infty}^{+\infty} J_{l-n}(a)\left[\frac{\partial\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}{\partial\omega}\frac{d\Phi_l}{dt}+\delta\varepsilon_e''(\omega+l\omega_0,\mathbf{k})\Phi_l\right]\Big\}=0, \quad (4.1)$$

where

$$a=e\mathbf{E}_0\mathbf{k}/m_e\omega_0^2, \quad \omega=\omega_r(\mathbf{k},t) \text{ и}$$

and

$$\Phi_l(\omega,\mathbf{k},t)=\sum_{s=-\infty}^{+\infty} J_{l-s}(a)\varphi_s(\omega,\mathbf{k},t). \quad (4.2)$$

In writing out (4.1), we took account of the fact that the imaginary part $\delta\epsilon_a''$ of the partial dielectric constants are small compared with the real parts $\delta\epsilon_a'$.

Multiplying (4.1) by $J_{m-n}(a)$ and summing over n from $-\infty$ to $+\infty$, we obtain

$$\Phi_m\{1+\delta\varepsilon_e'(\omega+m\omega_0,\mathbf{k})\}+i\frac{\partial\delta\varepsilon_e'(\omega+m\omega_0,\mathbf{k})}{\partial\omega}\frac{d\Phi_m}{dt}$$

$$+i\delta\varepsilon_e''(\omega+m\omega_0,\mathbf{k})\Phi_m=-\sum_{s=-\infty}^{+\infty}J_{m-s}(a)\Big\{\delta\varepsilon_i'(\omega+s\omega_0,\mathbf{k})\varphi_s$$

$$+i\frac{\partial\delta\varepsilon_i'(\omega+s\omega_0,\mathbf{k})}{\partial\omega}\frac{d\varphi_s}{dt}+i\delta\varepsilon_i''(\omega+s\omega_0,\mathbf{k})\varphi_s\Big\}. \quad (4.3)$$

With the aid of this relation, Eq. (4.1) can now be rewritten in the form

$$\varphi_n\{1+\delta\varepsilon_i'(\omega+n\omega_0,\mathbf{k})\}-\sum_{l,s=-\infty}^{+\infty}J_{l-n}(a)J_{l-s}(a)\frac{\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}{1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}$$

$$\times\delta\varepsilon_i'(\omega+s\omega_0,\mathbf{k})\varphi_s=i\sum_{l=-\infty}^{+\infty}J_{l-n}(a)\frac{1}{1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}$$

$$\times\Big\{\frac{\partial\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}{\partial\omega}\frac{d\Phi_l}{dt}+\delta\varepsilon_e''(\omega+l\omega_0,\mathbf{k})\Phi_l$$

$$+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})\sum_{s=-\infty}^{+\infty}J_{l-s}(a)$$

$$\times\left[\frac{\partial\delta\varepsilon_i'(\omega+s\omega_0,\mathbf{k})}{\partial\omega}\frac{d\varphi_s}{dt}+\delta\varepsilon_i''(\omega+s\omega_0,\mathbf{k})\varphi_s\right]\Big\}$$

$$+i\frac{\partial\delta\varepsilon_i'(\omega+n\omega_0,\mathbf{k})}{\partial\omega}\frac{d\varphi_n}{dt}+i\delta\varepsilon_i''(\omega+n\omega_0,\mathbf{k})\varphi_n. \quad (4.4)$$

For oscillations with frequency ω much lower than the frequency ω_0 of the external field, we can use the smallness of $\delta\epsilon_i(\omega+s\omega_0,\mathbf{k})$ when $s\neq 0$. Then

$$\Phi_m\cong-\frac{\delta\varepsilon_i'(\omega,\mathbf{k})J_m(a)}{1+\delta\varepsilon_e'(\omega+m\omega_0,\mathbf{k})}\varphi_0, \quad (4.5)$$

$$\varphi_n\cong\varphi_0\frac{\delta\varepsilon_i'(\omega,\mathbf{k})}{1+\delta\varepsilon_i'(\omega+n\omega_0,\mathbf{k})}\cdot$$

$$\times\sum_{l=-\infty}^{+\infty}J_l(a)J_{l-n}(a)\frac{\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}{1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}. \quad (4.6)$$

The spectrum of the low-frequency plasma oscillations in the microwave field is then determined by the dispersion relation (see[7])

$$1+\frac{1}{\delta\varepsilon_i'(\omega,\mathbf{k})}=\sum_{l=-\infty}^{+\infty}J_l^2(a)\frac{\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}{1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}. \quad (4.7)$$

Finally, for the temporal evolution of the field amplitude φ_0 we have the following equation

$$\frac{d\varphi_0(\omega,\mathbf{k},t)}{dt}-\gamma(\omega,\mathbf{k},t)\varphi_0(\omega,\mathbf{k},t)=0, \quad (4.8)$$

where

$$\gamma(\omega,\mathbf{k},t)=-\Big\{\frac{\delta\varepsilon_i''(\omega,\mathbf{k},t)}{[\delta\varepsilon_i'(\omega,\mathbf{k})]^2}+\sum_{l=-\infty}^{+\infty}J_l^2(a)\frac{\delta\varepsilon_e''(\omega+l\omega_0,\mathbf{k},t)}{[1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})]^2}\Big\}$$

$$\times\Big\{\frac{\partial}{\partial\omega}\Big[\frac{1}{\delta\varepsilon_i'(\omega,\mathbf{k})}+\sum_{l=-\infty}^{+\infty}J_l^2(a)\frac{1}{1+\delta\varepsilon_e'(\omega+l\omega_0,\mathbf{k})}\Big]\Big\}^{-1}. \quad (4.9)$$

Using formulas (4.5) and (4.6), we write the equations of evolution of the electron and ion distributions in the form

$$\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial t}=\frac{\partial}{\partial p_{a,i}}D_{ij}^a(\mathbf{p}_a)\frac{\partial\bar F_a(\mathbf{p}_a,t)}{\partial p_{a,j}} \quad (4.10)$$

where the diffusion coefficients in momentum space for the electrons and ions have the following respective forms:

$$D_{ij}^i(\mathbf{p})=e_i^2\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_u|\varphi_0(\omega_u(\mathbf{k}),\mathbf{k},t)|^2$$

$$\times\Big\{\pi\delta(\omega_u(k)-\mathbf{k}\mathbf{v})-\gamma(\omega_u,\mathbf{k},t)\Big[\frac{\partial}{\partial\omega_u(\mathbf{k})}\frac{\mathrm{P}}{\omega_u(\mathbf{k})-\mathbf{k}\mathbf{v}}\Big]\Big\}, \quad (4.11)$$

$$D_{ij}^e(\mathbf{p})=e^2\int\frac{\partial\mathbf{k}}{(2\pi)^3}k_ik_j\sum_u|\varphi_0(\omega_u(\mathbf{k}),\mathbf{k},t)|^2[\delta\varepsilon_i'(\omega_u(\mathbf{k}),\mathbf{k})]^2$$

$$\times\sum_{l=-\infty}^{+\infty}J_l^2(a)\frac{1}{[1+\delta\varepsilon_e'(\omega_u(\mathbf{k})+l\omega_0,\mathbf{k})]^2}\Big\{\pi\delta(l\omega_0+\omega_u(\mathbf{k})-\mathbf{k}\mathbf{v})$$

$$-\gamma(\omega_u(\mathbf{k}),\mathbf{k},t)\Big[\frac{\partial}{\partial\omega_u(\mathbf{k})}\frac{\mathrm{P}}{l\omega_0+\omega_u(\mathbf{k})-\mathbf{k}\mathbf{v}}\Big]\Big\}. \quad (4.12)$$

By virtue of the fact that the influence of the electric field on the motion of the ions can be neglected, the ion diffusion coefficient coincides in form with the one usually encountered in the theory that takes into account the reaction of the oscillations on the particle distribution in the unstable plasma[10]. To the contrary, the influence of the microwave electric field determines essentially the form of the electron diffusion coefficient (4.12).

A similar transformation should be carried out also for the amplitudes of the harmonics of the electron distribution function (3.5). We note that the occurrence of higher harmonics in the electron distribution function can be likened to the situation occurring for a plasma in a strong electric field when account is taken of the usual charged-particle collisions[11]. It is possible in this case, in particular, to obtain an expression for the product of the current density by the electric field intensity, averaged over the period of the oscillations of the external field:

$$\langle\mathbf{E}(t)\mathbf{j}(t)\rangle=\langle\mathbf{E}_0\sin\omega_0t\mathbf{j}\rangle=e\int d\mathbf{p}\mathbf{v}\mathbf{E}_0\frac{1}{2i}[\bar F_e^{(1)}-\bar F_e^{(-1)}]$$

$$=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{e\mathbf{E}_0\mathbf{k}}{m_e\omega_0}\frac{k^2}{4\pi}\sum_r|\varphi_0(\omega_r(\mathbf{k}),\mathbf{k},t)|^2[\delta\varepsilon_i'(\omega_r(\mathbf{k}),\mathbf{k})]^2$$

$$\times\sum_{l=-\infty}^{+\infty}J_l(a)\frac{\delta\varepsilon_e''(l\omega_0+\omega,\mathbf{k},t)+\gamma(\omega_r(\mathbf{k}),\mathbf{k},t)(\delta\varepsilon_e'(l\omega_0+\omega,\mathbf{k}))_\omega}{1+\delta\varepsilon_e'(\omega_r(\mathbf{k})+l\omega_0,\mathbf{k})}$$

$$\times\Big\{\frac{J_{l-1}(a)}{1+\delta\varepsilon_e'(\omega_r(\mathbf{k})+[l-1]\omega_0,\mathbf{k})}+\frac{J_{l+1}(a)}{1+\delta\varepsilon_e'(\omega_r(\mathbf{k})+[l+1]\omega_0,\mathbf{k})}\Big\}, \quad (4.13)$$

where $(X)_\omega=\partial X/\partial\omega$. Expression (4.13), as follows from the energy conservation law (2.6), determines the growth of the energy of the particles and of the waves in the plasma under the influence of the external microwave field.

5. We shall now show that it is possible to predict, on the basis of the described theory, the effects of an anomalously strong interaction of a microwave field with a plasma, leading to a relatively fast field-energy absorption due to the anomalous increase of the dissi-

pative high-frequency conductivity of the plasma. To this end we turn to a consideration of the kinetic instability of a strongly non-isothermal plasma with $T_e \gg T_i$, the linear theory of which was developed in[2].

Let the frequency of the external field ω_0 greatly exceed the electron Langmuir frequency ω_{Le}, but let it be at the same time bounded from above in such a way that the following inequalities are satisfied:

$$1 \ll \frac{\omega_0^2}{\omega_{Le^2}} \ll \frac{|e|T_e}{e_i T_i} \Big/ \ln\left[\frac{T_e^3 e_i^2 m_i}{T_i^3 e^2 m_e}\right] \equiv I. \tag{5.1}$$

Then, as shown in[2], the instability becomes possible at an external high-frequency electric field intensity larger than the threshold value at which the velocity of the electron oscillations in the external field $v_E = eE_0/m_e\omega_0$ is approximately twice as large as the thermal velocity $v_{Te} = \sqrt{\kappa T_e/m_e}$. We confine ourselves to a discussion of the case when v_E, while exceeding the threshold value $\sim 2v_{Te}$, still does not differ from it in order of magnitude. In this case the growing oscillations lie in a large range of wavelengths that are small compared with the electron Debye radius, but at the same time are larger than v_E/ω_0. The frequency of the growing oscillations practically coincides with the ion Langmuir frequency $\omega_{Li} = \sqrt{4\pi e_i^2 n_i/m_i}$, and the following formula can be written for the estimate of the increment:

$$\gamma \sim \omega_{Li}^2 \omega_{Le}^2 / k^3 v_E^3. \tag{5.2}$$

Finally, on the short-wave side, the region of instability is bounded by the inequality

$$\lambda > 1/k_0 = r_{De} I^{-1/2}, \tag{5.3}$$

Just as in the usual theory of electron-electron collision integral due to the interaction with the ion-acoustic oscillations (see, for example,[4]).

Simple calculations now make it possible to write down the following estimate for the average work (4.13) performed by the external field on the plasma:

$$\langle \mathbf{E}(t)\mathbf{j}(t)\rangle \sim \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{k^2}{4\pi}|\varphi_0(\mathbf{k},t)|^2 \frac{k v_{Te}}{k^2 r_{De}^2}. \tag{5.4}$$

In practice the same expression determines the time rate of change of the average kinetic energy of the electrons:

$$\frac{d}{dt}\int d\mathbf{p}\frac{p^2}{2m_e}\bar{F}_e \approx \langle \mathbf{E}(t)\mathbf{j}(t)\rangle. \tag{5.5}$$

This agrees with the fact that the plasma oscillation energy constitutes a small fraction in the total energy balance, and the rate of change of the ion energy is much smaller than given by the expression (5.5).

The fact that the properties of the growing oscillations do not depend on the fine details of the electron distribution makes it possible to conclude that it is possible to employ for the field growth the exponential law

$$|\varphi_0(\mathbf{k},t)|^2 = |\varphi(\mathbf{k},0)|^2 e^{-2\gamma t}, \tag{5.6}$$

which follows from the linear theory, until the increase of the kinetic energy of the electrons becomes comparable with their initial temperature. Assuming as the initial value of the square of the field the one determined by the thermal noise,

$$|\varphi(\mathbf{k},0)|^2 \sim \varkappa T_e/(2\pi)^3 k^2, \tag{5.7}$$

we can now write (5.5) in the form

$$\frac{dn_e\varkappa T_e}{dt} \approx \langle \mathbf{E}(t)\mathbf{j}(t)\rangle \sim \varkappa T_e \frac{v_{Te}}{r_{De}} \int_{\omega_0/v_E}^{k_0} k\,dk \exp\left\{2\frac{\omega_{Li}^2\omega_{Le}^2}{k^3 v_E^3}t\right\}. \tag{5.8}$$

For small values of the time, when

$$t \lesssim \omega_0^3 \omega_{Le}^{-2}\omega_{Li}^{-2}, \tag{5.9}$$

which corresponds to the start of the development of the instability, we have at small values of the wave vectors ($k \sim \omega_0/v_E$)

$$\frac{dn_e\varkappa T_e}{dt} \sim n_e\varkappa T_e \nu_0 I \sim E^2 \frac{e^2 n_e}{m_e\omega_0^2}\nu_0 I, \tag{5.10}$$

where

$$\nu_0 \cong e^4 n_e / \sqrt{m_e}(\varkappa T_e)^{3/2}. \tag{5.11}$$

Formula (5.10), in particular, means that during the initial stage of the development of the instability the conductivity of the plasma is of the order of magnitude of

$$\sigma \sim \frac{e^2 n_e}{m_e \omega_0^2}\nu_0 I. \tag{5.12}$$

We note that in the initial stage of development of the instability, the energy density of the oscillations is approximately $\kappa T_e k_0^3$, which makes it possible to understand the agreement between formula (5.12) and the result of[12], which, to be sure, pertains to the case of essentially lower electromagnetic-field frequencies. Expression (5.12) exceeds the usual conductivity of the fully ionized plasma, due to the paired collisions at $T_e > 100\,T_i$. In the opposite limits when $T_e < 100\,T_i$ and in the case of time values given by (5.9), the pair collisions are more important and decisively influence the conductivity of the plasma.

In the case of large values of the time, when

$$t \gtrsim \frac{\omega_0^3}{2\omega_{Le}^2\omega_{Li}^2}\ln\frac{\omega_{Le}^2 I}{\omega_0^2}, \tag{5.13}$$

formula (5.8) assumes the following relatively simple form:

$$\frac{d(n_e\varkappa T_e)}{dt} \sim \varkappa T_e \frac{\omega_0^5}{v_{Te}^3\omega_{Li}^2 t}\exp\left\{\frac{2\omega_{Li}^2\omega_{Le}^2 t}{\omega_0^3}\right\}. \tag{5.14}$$

This formula leads to an increase of the electron temperature by an amount comparable with the initial value, after a time of the order of

$$t_T \sim \frac{\omega_0^3}{2\omega_{Li}^2\omega_{Le}^2}\ln\left[n_e r_{De}^3 \frac{\omega_{Li}^2\omega_{Le}^3}{\omega_0^5}\right]. \tag{5.15}$$

Since the right side of formula (5.14) becomes equal by that time to

$$\frac{n_e\varkappa T_e \omega_{Le}^2\omega_{Li}^2}{\omega_0^5 \ln[n_e r_{De}^3 \omega_{Li}^2 \omega_{Le}^3 \omega_0^{-5}]}, \tag{5.16}$$

we can state, taking into account the fact that $v_{Te} \sim v_E$, that the plasma conductivity reaches a value

$$\sigma \sim \frac{e^2 n_e}{m_e\omega_0^2}\frac{\omega_{Le}^2\omega_{Li}^2}{\omega_0^3 \ln[n_e r_{De}^3 \omega_{Li}^2\omega_{Le}^3\omega_0^{-5}]}. \tag{5.17}$$

For the time given by (5.15), the inequality (5.13) is satisfied if the following relation holds for the number of particles in the Debye sphere:

$$n_e r_{De}^3 \gg \omega_0^3 I / \omega_{Li}^2 \omega_{Le}. \tag{5.18}$$

Satisfaction of this relation causes the turbulent conductivity of the plasma (5.17) to exceed greatly the value given by (5.12). On the other hand, if I in the right side of (5.18) is replaced by the Coulomb logarithm $\Lambda \sim \ln(n_e r_{De})$, which occurs in the usual Landau collision integral, then the turbulent conductivity turns out to be much larger than the usual one. This can be readily satisfied for a plasma with a Coulomb logarithm $\Lambda \gtrsim 15-20$.

We note that the energy density of the plasma oscillations reaches by the time t_T an order of magnitude of

$$\frac{n_e \varkappa T_e \omega_{Li}^2}{\omega_{Le}\omega_0 \ln[n_e r_{De}^3 \omega_{Li}^2 \omega_{Le}^3 \omega_0^{-5}]} \tag{5.19}$$

and amounts to a rather small fraction of the electron energy, which obviously is necessary for the obtained estimates.

Simple estimating formulas that follow from (5.8) can be used only in the case of slight heating of the electrons. In the case $v_E \sim 2v_{Te}$ considered by us, an increase of the temperature of the electronic component by an amount of the order of the initial value can lead to stabilization of the kinetic instability. Yet the onset of an anisotropic distribution of the electrons is also possible, accompanied by a growth of the intensity of the field oscillations. The choice between the two alternatives can be decided by investigating the consequences that follows from the equations of the fourth and third sections of the present article, in which equations were derived describing the kinetics of a weakly-turbulent plasma situated in a strong microwave field, and the possibility was revealed of the occurrence of anomalous plasma conductivity even in the early stage of turbulence development.

In conclusion we note that the analysis presented in the last section makes it possible to carry out a definite generalization to the case of a hydrodynamic instability which occurs in parametric resonance in a plasma when $\omega_0 \cong \omega_{Le}$. The maximum growth increment of the oscillations $\gamma \sim (m_e/m_i)^{1/3}\omega_{Le}$, just like (5.2), does not depend on the details of the particle distribution. One can therefore speak of an exponential growth of the oscillations until the thermal velocities of the electrons turn out to be close to the velocities of their oscillations in the strong microwave fields. Just as in the case of formula (5.17), the conductivity reaches, in order of magnitude, a value determined by an expression in which the role of the effective collision frequency is played by the growth increment of the oscillations. In other words, the conductivity turns out to be proportional to $(m_e/m_i)^{1/3}\omega_{Le}$.

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Translated by J. G. Adashko
23