ABSORPTION AND DISPERSION OF SOUND IN A SUPERFLUID LIQUID NEAR THE λ POINT

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The absorption and dispersion of sound in a superfluid liquid is investigated near the λ point. A new derivation is given of the equations of hydrodynamics of a superfluid liquid near the λ point, which contains some refinements in comparison with the original derivation of L. P. Pitaevskiĭ. The mechanism of sound attenuation is the transfer of energy to second sound quanta with a wavelength of the order of the correlation length. Formulas are derived for the dispersion of the velocity of first sound and the attenuation coefficient of second sound.

THE problem of the absorption of sound near the λ point in superfluid helium was considered in [1] on the basis of the thermodynamic theory of Landau for second order phase transitions.[2] In the theory[2] of second order phase transitions, the expandibility of the thermodynamic potential in a series in powers of the small "order" parameter is assumed; this has not been confirmed at recent date[3] —the heat capacity at the λ point has a singularity in contradiction with this assumption. However, it can be shown that the study of the problem of sound propagation near the λ point can be carried out without the use of the Landau thermodynamic theory.

For the study of the problem of sound propagation near the λ point, we use the hydrodynamic equations obtained by Pitaevskiĭ.[4] Since the derivation of these equations and their form require some refinement, we shall begin with their derivation.

1. THE HYDRODYNAMIC EQUATIONS OF A SUPERFLUID LIQUID NEAR THE λ POINT

Closeness to the λ point is characterized by a small parameter—the density of the superfluid part of the liquid ρ_s. Introduction of a certain complex function $\psi(\mathbf{r}, t) = \eta \exp i\varphi$ is convenient; this function is so defined that[1)]

$$\rho_s = m|\psi|^2, \quad \mathbf{v}_s = \frac{\hbar}{m}\nabla\varphi. \tag{1.1}$$

For small values of $\mathbf{v}_n$ and $\mathbf{v}_s$, we expand the energy per unit volume E in a series in $\mathbf{v}_n$ and $\nabla\psi$:

$$E = (\rho - m|\psi|^2)\frac{\mathbf{v}_n^2}{2} + \frac{\hbar^2}{2m}|\nabla\psi|^2 + E_0(\rho, S, |\psi|^2). \tag{1.2}$$

Further, expressing ψ in terms of ρ_s and $\mathbf{v}_s$, we get

$$E = \rho_n\frac{\mathbf{v}_n^2}{2} + \rho_s\frac{\mathbf{v}_s^2}{2} + \frac{\hbar^2}{8m^2}\frac{(\nabla\rho_s)^2}{\rho_s} + E_0, \quad \rho_n = \rho - m|\psi|^2. \tag{1.3}$$

We transform to a coordinate system moving with the velocity of normal motion $\mathbf{v}_n$, and introduce the momentum of the relative motion of the liquid in this system:

$$\mathbf{p} = \mathbf{j} - \rho\mathbf{v}_n = \frac{i\hbar}{2}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi) - m|\psi|^2\mathbf{v}_n. \tag{1.4}$$

The energy E is then, in accord with the Galilean transformation, equal to[5]

$$E = \rho\frac{\mathbf{v}_n^2}{2} + \mathbf{p}\mathbf{v}_n + E_{\text{rel}} \tag{1.5}$$

where E_{rel} is the sum of the internal energy and the energy of relative motion. Comparing (1.5) with (1.2), we find

$$E_{\text{rel}} = \frac{m}{2}\left|\left(-\frac{i\hbar}{m} - \mathbf{v}_n\right)\psi\right|^2 + E_0(\rho, S, |\psi|^2). \tag{1.6}$$

For finding the equilibrium value of ψ it is necessary to minimize the thermodynamic potential of the system Φ which, in accord with (1.6), is equal to

$$\Phi = \frac{m}{2}\left|\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)\psi\right|^2 + \Phi_0(T, p, |\psi|^2). \tag{1.7}$$

The minimum must be sought for fixed values of the thermodynamic variables T, p and the relative velocity $\mathbf{v}_s - \mathbf{v}_n = \mathbf{p}/\rho_s$. We introduce the Lagrangian multiplier $\mathbf{u}$ and vary the integral $\int(\Phi + \rho_s^{-1}\,\mathbf{u}\cdot\mathbf{p})dV$ successively in ψ and $\mathbf{v}_n$. In this way we find the two conditions

$$\frac{m}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2\psi + \frac{\partial\Phi_0}{\partial\rho_s}m\psi - \frac{i\hbar}{2}\left(\nabla\frac{\mathbf{u}}{\rho_s}\psi + \frac{\mathbf{u}}{\rho_s}\nabla\psi\right) \tag{1.8}$$

$$-\frac{1}{\rho_s}m\mathbf{v}_n\mathbf{u}\psi - \frac{\mathbf{u}\mathbf{p}}{\rho_s^2}m\psi = 0,$$

$$\mathbf{p} + \mathbf{u} = 0. \tag{1.9}$$

Eliminating the Lagrangian multiplier $\mathbf{u}$ from these two conditions, we obtain an equation defining the equilibrium value of ψ:

$$\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2 + \mu_s + \frac{i\hbar}{2m\rho_s}\operatorname{div}\mathbf{p}\right]m\psi = 0, \tag{1.10}$$

where

$$\mu_s = \left(\frac{\partial\Phi_0}{\partial\rho_s}\right)_{T,p} = \left(\frac{\partial E_0}{\partial\rho_s}\right)_{\rho,S}. \tag{1.11}$$

Equation (1.10) recalls the fundamental equation of the Ginzburg–Landau theory for superconductors. However, in the theory of superconductivity, the function ψ (its modulus and phase) is completely determined by the

1)The notation is the same as in [4].

given magnetic field and therefore the minimum of the thermodynamic potential is found for fixed values of the vector potential of the electromagnetic field **A**.

In the work of Pitaevskiĭ,[4] the equation defining the equilibrium value of ψ was found by variation of the total energy E in terms of ψ for fixed values of the momentum $\mathbf{j} = \mathbf{p} + \rho\mathbf{v}_n$. In the equation thus obtained, the term $(i\hbar/2m\rho_s)$ div **p** is absent in comparison with (1.10). The equation obtained in [4] is equivalent to two conditions: its real part is identical with Eq. (1.10) and the imaginary part gives the extraneous condition div **p** = 0, for which there is no foundation. Such a result appeared as the result of an unfounded assumption on the extremum of E relative to ψ for fixed **j**. The equilibrium value of ρ_s can be found from the condition of minimum energy. Here it is necessary to choose the density ρ, the entropy S, the momentum **j** and the velocity of superfluid motion $\mathbf{v}_s$ as independent variables, the values of which ought to be fixed. The fact that such a choice of variables appears natural can be established in the limiting case when the gradients of ρ_s and specific quantum effects are small. Here the equations of hydrodynamics can be obtained from the law of energy conservation. The energy in this case is expressed as a function of ρ, S, **j** and $\mathbf{v}_s$ and the laws of conservation are used for these quantities.[5]

In the given problem, a new independent variable ρ_s is present, the equilibrium value of which can be found from the condition of the minimum of the total energy $\int E dV$ relative to ρ_s for fixed values of all the remaining variables. By varying $\int E dV$ in ρ_s, we obtain the desired condition

$$-\frac{\hbar^2}{4m^2}\Delta\rho_s + \frac{\hbar^2}{8m^2\rho_s}(\nabla\rho_s)^2 + \rho_s\frac{(\mathbf{v}_s - \mathbf{v}_n)^2}{2} + \rho_s\mu_s = 0.$$

Expressing ρ_s and $\mathbf{v}_s$ in these equations as functions of ψ with the help of (1.1), we write down the resultant condition in the form of an equation for ψ:

$$\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right) + \mu_s + \frac{i\hbar}{2m\rho_s}\operatorname{div}\mathbf{p}\right] m\psi = 0.$$

Thus we again obtain condition (1.10). The given derivation indicates that the equilibrium value of the modulus of the function ψ (the density ρ_0) corresponds to the minimum of the energy for the given phase of φ (the velocity $\mathbf{v}_s$).

In the nonstationary case, it is assumed that the state of the system is defined by furnishing ψ (the same as for the other thermodynamic variables), i.e., ψ satisfies the linear differential equation in t. By analogy with quantum mechanics, the equation for ψ is written in the form

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{L}\psi, \tag{1.12}$$

where $\hat{L}$ is some linear operator. Since the value of ρ_s can relax, the operator $\hat{L}$ contains a non-Hermitian part. The Hermitian part of the operator $\hat{L}$, by analogy with the Schrödinger equation, is written in the form

$$-\frac{\hbar^2}{2m}\Delta + U, \tag{1.13}$$

where

$$U = \left(\frac{\partial E_0}{\partial|\psi|^2}\right)_{\rho_n S} = (\mu + \mu_s)m. \tag{1.14}$$

Here μ and μ_s are defined by the thermodynamic identity for E_0:

$$dE_0 = TdS + \mu d\rho + \mu_s d\rho_s, \tag{1.15}$$

U represents the potential energy of the superfluid part of the liquid.

So far as the anti-Hermitian part is concerned, which describes the approximation of ρ_s for the equilibrium value, for small departures from equilibrium we can write

$$i\Lambda\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2 + \mu_s + \frac{i\hbar}{2m\rho_s}\operatorname{div}\mathbf{p}\right] m\psi, \tag{1.16}$$

where Λ is some dimensionless kinetic coefficient. Finally, we have the following equation for ψ:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + (\mu + \mu_s)m\psi - i\Lambda\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2 + \mu_s + \frac{i\hbar}{2m\rho_s}\operatorname{div}\mathbf{p}\right] m\psi. \tag{1.17}$$

The coefficient Λ should be real, since in the opposite case a transport of the superfluid part of the liquid with normal velocity would have been possible. The remaining equations of hydrodynamics are written as usual in the form of conservation laws:

Mass

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{1.18}$$

Momentum

$$\frac{\partial j_i}{\partial t} + \frac{\partial\Pi_{ik}}{\partial x_k} = 0, \tag{1.19}$$

$$\Pi_{ik} = \frac{\hbar^2}{2m}\left(\frac{\partial\psi}{\partial x_i}\frac{\partial\bar{\psi}}{\partial x_k} - \psi\frac{\partial^2\bar{\psi}}{\partial x_i\,\partial x_k} + \text{к. с.}\right) + \rho_n v_{ni} v_{nk} + p\delta_{ik},$$

$$p = -E_0 + TS + \mu\rho + \mu_s\rho_s, \tag{1.20}$$

and, finally, the law of entropy growth

$$\frac{\partial S}{\partial t} + \operatorname{div} S\mathbf{v}_n = \frac{R}{T}, \tag{1.21}$$

in which the dissipative function is found from the energy conservation law in the form

$$R = \frac{2\Lambda}{\hbar}\left|\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2 + \mu_s + \frac{i\hbar}{2m\rho_s}\operatorname{div}\mathbf{p}\right] m\psi\right|^2. \tag{1.22}$$

In Eqs. (1.19) and (1.21), the dissipative terms that are due to the viscosity and the thermal conductivity are omitted. These terms have the usual form.[5] As will be seen from the following, they do not play important roles in processes of dissipation of sound in superfluid helium near the λ point.

In the case of small gradients we replace $-i\hbar m^{-1}\nabla\psi$ everywhere by $\mathbf{v}_s$. As a result, Eqs. (1.17) and (1.21) take the following simple form:

$$\dot{\mathbf{v}}_s + \nabla\left(\frac{v_s^2}{2} + \mu + \mu_s\right) = 0, \tag{1.23}$$

$$\dot{\rho} + \operatorname{div}(\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n) = 0, \tag{1.24}$$

$$\frac{\partial}{\partial t}(\rho_s v_{si} + \rho_n v_{ni}) + \frac{\partial}{\partial x_k}(\rho_n v_{ni} v_{nk} + \rho_s v_{si} v_{sk} + p\delta_{ik}) = 0, \tag{1.25}$$

$$\dot{S} + \operatorname{div} S\mathbf{v}_n = \frac{2\Lambda m}{\hbar}\left[\mu_s + \frac{(\mathbf{v}_n - \mathbf{v}_s)^2}{2}\right]^2\rho_s, \tag{1.26}$$

$$\dot{\rho}_s + \operatorname{div}(\rho_s \mathbf{v}_s) = -\frac{2\Lambda m}{\hbar}\left[\mu_s + \frac{(\mathbf{v}_n - \mathbf{v}_s)^2}{2}\right]\rho_s. \quad (1.27)$$

We apply the equations given above to the study of the problem of sound propagation. They differ from those obtained in [4] by the fact that in Eqs. (1.23) and (1.26) dissipative terms of the type of second viscosity, containing the coefficient Λ, are absent.

2. THE RELAXATION MECHANISM NEAR THE λ POINT

The relaxation of the density of the superfluid component of ρ_s is described by the right hand side of Eq. (1.27). Here the velocity of approach to equilibrium is determined by the Λ kinetic coefficient. We consider the mechanism of the dissipation of sound energy near the λ point. The system is characterized by a certain correlation length ξ. At lengths of the order of ξ the correlation of phase φ is damped out. According to Eq. (1.23), this length will also be characteristic for the temperature correlations. It is therefore evident that second sound can be propagated only in the case for which $k\xi \ll 1$, where k is the wave vector of second sound, i.e., when the wavelength of second sound is greater than the correlation length ξ. Waves of second sound with wavelengths of the order of ξ will be completely dissipated. Thus there is some characteristic time $\tau = \xi/u_2$ (u_2 is the speed of second sound) which determines the rate of energy dissipation of waves of second sound.[6,7] The energy dissipation in first sound then takes place by decay into waves of second sound and is characterized by the same time.

By giving such a mechanism of dissipation, we easily find the temperature dependence of the dimensionless coefficient Λ. It is evident that Λ can depend only on u_2 and ξ, while the rate of relaxation should be proportional to the speed of second sound. Using the dimensional constants ħ and m that are at our disposal, we find

$$\Lambda \approx m u_2 \xi / \hbar. \quad (2.1)$$

Such a determination of the value of Λ is equivalent to the introduction of the characteristic time $\tau = \xi/u_2$. Actually, it follows from (1.19) that the characteristic relaxation time τ is determined by the relation

$$\frac{1}{\tau} = \frac{2\Lambda m}{\hbar}\frac{\partial \mu_s}{\partial \rho_s}\rho_s. \quad (2.2)$$

The basic part of the potential μ_s, by definition of ξ, is of the order of $\hbar^2/m^2\xi^2$. It therefore follows from (2.1) and (2.2) that

$$\frac{1}{\tau} \cong \frac{u_2 \xi m^2}{\hbar^2}\frac{\hbar^2}{m^2\xi^2} \sim \frac{u_2}{\xi}. \quad (2.3)$$

It has been shown in [8] that such a definite relaxation time has a general dependence on the temperature, not depending on the critical indices of similarity theory,[9] i.e., not depending on the form of the singularity in the thermodynamic functions near the λ point. Actually, from the expression for u_2,[5]

$$u_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{\sigma^2 T}{C}},$$

and the obvious relations for the principal part of the heat capacity C

$$C \sim \frac{T_c}{\rho}\frac{\partial^2 \Phi}{\partial T^2} \sim \frac{\rho_s}{\rho}\frac{\hbar^2}{m^2\xi^2}\frac{1}{T_c \varepsilon^2}, \qquad \varepsilon = \frac{T_c - T}{T_c}$$

we get

$$\frac{1}{\tau} \sim \frac{m\sigma T_c}{\hbar}\varepsilon \sim \varepsilon. \quad (2.4)$$

Thus the relaxation time τ is inversely proportional to the difference $T_c - T$. It is curious that Eq. (2.4) gives the correct order of magnitude of the time τ, in excellent agreement with experiment ($1/\tau \sim 10^{11}\epsilon$).

We note that the ordinary viscosity and thermal conductivity, which are due to collisions of the excitations, will be characterized by very small times of an entirely different order[5] and therefore their contribution to the energy dissipation of the sound will be insignificant.

3. SOUND DISPERSION

For the study of sound propagation, we transform the set of equations (1.23)–(1.26) to linearized form, after which we eliminate the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$. As a result, we obtain two wave equations:

$$\ddot{\rho} - \Delta p = 0, \quad (3.1)$$

$$\ddot{\sigma} - \frac{\rho_s}{\rho}\sigma^2 \Delta T + \frac{\rho_s \sigma}{\rho}\Delta\mu_s = 0 \quad (3.2)$$

in which, in view of the smallness of ρ_s, we have set $\rho_n = \rho$ everywhere. Equation (1.27), which describes the relaxation of ρ_s, takes the following form here:

$$\dot{\rho}_s + \frac{\rho}{\sigma}\dot{\sigma} = -\frac{2\Lambda m}{\hbar}\mu_s'\rho_s, \quad (3.3)$$

where μ_s' is the variable part of the potential μ_s, since the condition of equilibrium in the linear approximation changes to the requirement $\mu_s = 0$.

The thermodynamic identity for the potential $w = \mu + T\sigma$, according to (1.15), is written in the form

$$dw = T d\sigma + \frac{1}{\rho}dp - \frac{\rho_s}{\rho}d\mu_s.$$

We use this identity for establishing the connection between the derivatives of the thermodynamic functions. If we choose p and the entropy per unit mass σ as independent variables, then Eqs. (3.1) and (3.2) are completely uncoupled, thanks to the extraordinary smallness of the difference $C_p - C_v$ and the ratio of the squares of the velocities u_2^2/u_1^2. We emphasize that this does not mean the neglect of the difference $C_p - C_v$ in the final formulas. Only terms of relative order $(C_p/C_v - 1)u_2^2/u_1^2$ are thrown away. With such accuracy, we find an expression from Eq. (3.1) for the square of the speed of first sound

$$\frac{1}{u_1^2} = \frac{k^2}{\omega^2} = \left(\frac{\partial \rho}{\partial p}\right)_\sigma, \quad (3.4)$$

and an expression for the square of the speed of second sound from (3.2)

$$u_2^2 = \frac{\rho_s}{\rho}\sigma^2\left[\left(\frac{\partial T}{\partial \sigma}\right)_p - \frac{1}{\sigma}\left(\frac{\partial \mu_s}{\partial \sigma}\right)_p\right]. \quad (3.5)$$

We now make use of Eq. (3.3). From Eqs. (3.4) we find the expression for the dispersion of the speed of first sound

$$\frac{1}{u_1^2}=\left(\frac{\partial\rho}{\partial p}\right)_{\sigma,\mu_s}+\left(\frac{\partial\rho}{\partial\mu_s}\right)_{\sigma,p}\frac{\partial\mu_s}{\partial p}=\frac{1}{u_{10}^2}-\left(-\frac{1}{u_{10}^2}+\frac{1}{u_{1\infty}^2}\right)\frac{i\omega\tau}{1-i\omega\tau},\tag{3.6}$$

where u_{10} is the equilibrium speed of sound as $\omega\to 0$, equal to

$$\frac{1}{u_{10}^2}=\left(\frac{\partial\rho}{\partial p}\right)_{\sigma,\mu_s}=\frac{C_p}{C_v}\left(\frac{\partial\rho}{\partial p}\right)_{T,\mu_s},\tag{3.7}$$

$u_{1\infty}$ is the speed of sound in the limit $\omega\tau\gg 1$, when the equilibrium value of ρ_s lags behind the sound wave. The difference $u_{1\infty}-u_{10}$ is found from the relation

$$\frac{1}{u_{10}^2}-\frac{1}{u_{1\infty}^2}=\left(\frac{\partial\rho_s}{\partial p}\right)_\sigma\frac{\partial\rho}{\partial\mu_s}=\rho^2\frac{\partial\rho_s}{\partial p}\frac{\partial}{\partial p}\left(\frac{\rho_s}{\rho}\right)\left(\frac{\partial\mu_s}{\partial\rho_s}\right)_{p,\sigma}.\tag{3.8}$$

Here we have used the identity (3.4), according to which

$$\frac{\partial\rho}{\partial\mu_s}=\rho^2\frac{\partial}{\partial p}\left(\frac{\rho_s}{\rho}\right).\tag{3.9}$$

The time τ is determined by the relation

$$\frac{1}{\tau}=\frac{2\Lambda m}{\hbar}\left(\frac{\partial\mu_s}{\partial\rho_s}\right)_{p,\sigma}\rho_s.\tag{3.10}$$

In the variables p, T, Eq. (3.8) takes the form

$$\frac{1}{u_{10}^2}-\frac{1}{u_{1\infty}^2}=\left[\frac{\partial}{\partial T}\left(\frac{\rho_s}{\rho}\right)\frac{\partial\rho}{\partial T}\Big/\frac{\partial\sigma}{\partial T}-\rho\frac{\partial}{\partial p}\left(\frac{\rho_s}{\rho}\right)\right]^2\frac{1}{\rho}\left(\frac{\partial\mu_s}{\partial\rho_s}\right)_{p,\sigma}.\tag{3.11}$$

The singular part of the derivative $(\partial/\partial p)(\rho_s/\rho)$ is evidently equal to $(\partial/\partial T)(\rho_0/\rho)(\partial T_c/\partial p)$, i.e., it is expressed in terms of the derivative along the curve $T_\lambda(p)$. Thanks to this fact, the singular parts of both components in the square bracket in (3.11) are reduced and the bracket is of the order of $\epsilon^\alpha(\rho_s/\epsilon)$, where α is an index characterizing the singularity in the heat capacity $C\sim\epsilon^{-\alpha}$. The derivative $\partial\mu_s/\partial\rho_s$ has the order of $\Phi_0/\rho_s^2\sim\epsilon^{2-\alpha}/\rho_s^2$. Thus, the difference $u_{10}-u_{1\infty}$ is of the order of ϵ^α, i.e., it tends to zero.

Equation (3.6) describes the dispersion of first sound, i.e., it is valid in the region $\omega\tau\gtrsim 1$. Actually, in this region, where $\omega\tau\sim ku_1\xi/u_2\sim 1$, we have $k\xi\ll 1$, inasmuch as $u_2/u_1\ll 1$. Thus, in the region where $\omega\tau\gtrsim 1$, the wavelength of the sound is still greater than the correlation length.

We now return to second sound. In this case, the condition $\omega\tau\sim 1$ is identical with the condition $k\xi\sim 1$; therefore, it is legitimate to consider only the case $\omega\tau\ll 1$, i.e., the case of small damping. From Eq. (3.5) and Eq. (3.3), we get in this case

$$u_2^2=u_{20}^2\left[1-i\omega\tau\left(\frac{\partial\mu_s}{\partial\rho_s}\right)\left(\frac{\partial\rho_s}{\partial\sigma}+\frac{\rho}{\sigma}\right)\left(\frac{\partial}{\partial\sigma}\left(\frac{\rho_s}{\rho}\right)+\frac{1}{\sigma}\right)\right].\tag{3.12}$$

The function ρ_s, as follows from the theory of similarity, changes with temperature according to the law $1/\epsilon\sim\epsilon\exp(2-\alpha)/3$; therefore, $\partial\rho_s/\partial\sigma\gg\rho/\sigma$.[2)]

We introduce the heat capacity $C_{\rho_s}=T(\partial\sigma/\partial T)_{\rho_s}$, which is characteristic for fast processes, when as the temperature changes the density ρ_s does not have time to change. The connection between C_{ρ_s} and the equilibrium value of the heat capacity $C_0=T(\partial\sigma/\partial T)_{\mu_s}$ follows from the thermodynamic identity (3.4):

$$\frac{C_0}{C_{\rho_s}}-1=\left(\frac{\partial\rho_s}{\partial\sigma}\right)\frac{\partial}{\partial\sigma}\left(\frac{\rho_s}{\rho}\right)\left(\frac{\partial\sigma}{\partial T}\right)_{\mu_s}\left(\frac{\partial\mu_s}{\partial\rho_s}\right)_\sigma.\tag{3.13}$$

With the help of (3.13), we can rewrite Eq. (3.12) in the following simple form:

$$u_2^2=u_{20}^2\left[1-i\omega\tau\left(\frac{C_0}{C_{\rho_s}}-1\right)\right].\tag{3.14}$$

Equations (3.6) and (3.14) allow us to compute the damping coefficients of first and second sound, respectively. We have

$$\alpha_1=\mathrm{Im}\frac{\omega}{u_1}=\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{1}{2u_{10}}\left(1-\frac{u_{10}^2}{u_{1\infty}^2}\right),\tag{3.15}$$

$$\alpha_2=\mathrm{Im}\frac{\omega}{u_2}=\omega^2\tau\frac{1}{2u_{20}}\left(\frac{C_0}{C_{\rho_s}}-1\right).\tag{3.16}$$

Since the difference $u_{1\infty}-u_{10}\sim\epsilon^\alpha$ and $\tau\sim 1/\epsilon$, then $\alpha_1\sim\epsilon\exp(-1+\alpha)$, which agrees well with the temperature dependence of α_1 observed in the experiments of Barmatz and Rudnick.[10][3)]

Experiments on the observation of the damping of second sound below the λ point, carried out by Tyson,[6] confirm the temperature dependence of α_2 which follows from (3.16). The difference $C_0/C_{\rho_s}-1$ does not depend on the temperature for small ϵ. Actually, according to (3.13), we have

$$\frac{C_0}{C_{\rho_s}}-1\sim\frac{1}{C_0}\left(\frac{\partial\rho_s}{\partial T}\right)^2\frac{\partial\mu_s}{\partial\rho_s}\sim\frac{1}{\varepsilon^{-\alpha}}\frac{\rho_s^2}{\varepsilon^2}\frac{\varepsilon^{2-\alpha}}{\rho_s^2}\sim\text{const.}$$

In [6] the damping of second sound is characterized by the damping constant, which is determined in the following way:

$$D_2=\mathrm{Im}\,\omega/k^2$$

and, in accord with (3.16), is equal to

$$D_2=u_{20}^2\tau(C_0/C_{\rho_s}-1).\tag{3.17}$$

The temperature dependence follows from (3.17):

$$D_2\sim\frac{\rho_s}{C_0}\frac{1}{\varepsilon}\sim\frac{\varepsilon^{(2-\alpha)/3}}{\varepsilon^{1-\alpha}}\sim\varepsilon^{(-1+2\alpha)/3},$$

and agrees well with the experimental data[6] and with the predictions of the theory of dynamic similarity. We note that for damping of first sound the theory of dynamic similarity is generally not applicable. The dependence $\alpha_1\sim\epsilon\exp(-1+\alpha)$ cannot be obtained from considerations of similarity only.

The problem of the dispersion of first sound is entirely analogous to the problem of the dispersion of sound in the presence of slow processes of approach to the state of equilibrium in ordinary hydrodynamics. Equation (3.6) has the typical form which the theory of Mandel'shtam–Leontovich gives in ordinary hydrodynamics.[11] So far as second sound is concerned, here, thanks to the presence of a third term in Eq. (3.2) containing μ_s explicitly, the situation differs somewhat

2)Only in the limiting case of the thermodynamic theory of Landau is $\rho_s\sim\epsilon$ and the given terms have the same order of magnitude.

3)The experiments of Buckingham and Fairbank [3] indicate that the singularity in the heat capacity is logarithmic or a power series with a very small exponent α, so that one can set $\alpha = 0$ in comparison with experiment.

from the ordinary. It is true that this term is small in the case of superfluid helium; however, in the limiting case of the classical theory of phase transitions of Landau, it has the same order of magnitude as the first two terms in (3.2).

The system of hydrodynamic equations for a superfluid liquid near the λ point contain an extraneous equation for the function ρ_s in comparison with the usual equations of two-component hydrodynamics. It might appear that this should lead to the possibility of propagation of a new type of oscillation. However, Eq. (1.27) has the form of the equation of continuity and, eliminating div $\mathbf{v}_s$ from it and Eqs. (1.24) and (1.26), we obtain only the connection between the changes of the thermodynamic quantities and ρ_s in sound waves. A new wave equation does not appear and therefore there are no new types of oscillation.

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