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NONLINEAR HIGH-FREQUENCY PROPERTIES OF THIN SUPERCONDUCTING FILMS

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Nonlinear effects in thin superconducting films at microwave frequencies are investigated, viz., the dependence of the impedance on the microwave field amplitude, the decrease of critical superconducting current under the influence of the microwaves, and harmonics generation and mixing of frequencies in the presence of a stationary current component. The critical power leading to the destruction of superconductivity is determined. It is shown that for films with a thickness $d \ll \delta_L$ (δ_L is the London penetration depth) destruction of superconductivity is controlled by the electric component of the rapidly alternating field. The critical field amplitude E_0 is defined by the condition $eE_0\xi(T) \sim \hbar\omega$, where $\xi(T)$ is the temperature-dependent coherence length. The limits of stability of an homogeneous current state in the superconducting channel in the presence of a microwave field are determined in the extreme cases of high and low frequencies ($\hbar\omega \gg (T_c - T)$ and $\hbar\omega \ll (T_c - T)$ respectively). The dependence of the film impedance on the power is shown to exhibit hysteresis. Thus destruction of superconductivity by a microwave field occurs at a power much greater than that required to restore the superconducting state in a decreasing field.

1. INTRODUCTION

A number of recent papers are devoted to nonlinear effects in thin superconducting films at microwave frequencies. Such effects are a dependence of the impedance on the transport current[1], a nonlinear dependence of the transmission of the film on the power of the incident microwave and the appearance of harmonics in the spectrum of the transmitted radiation,[2] a dependence of the critical current on the intensity of the high-frequency irradiation[3], the occurrence of singularities in the impedance at discrete values of the high-frequency power[4,5], and others. A consistent theoretical description of effects of this type is possible under conditions when the nonstationary (differential with respect to time) Ginzburg-Landau equations[6-8] are valid. In particular, as follows from[7], such equations can be obtained for superconducting alloys with paramagnetic impurities if $\Delta\tau_s < 1$ and $T \to T_c$. Equations of this type were used in[6] to study resistive effects in alloys with nonmagnetic impurities at $H \to H_{c2}$. One might think that the situation should be analogous in this respect for all gapless superconductors, regardless of whether the gapless superconductivity is due to the presence of paramagnetic impurities or to nonmagnetic impurities in the presence of a strong magnetic field.

The purpose of the present paper is to investigate the nonlinear effects in superconductors in a high-frequency field on the basis of equations obtained in[7,8]. It turns out that even within the framework of such an approach, the solution of the problem of the nonlinear behavior of superconductors at microwave frequencies is a very complicated matter. For this reason, we do not analyze here the question of the refinement of the fundamental equations, assuming that this scheme gives a qualitatively correct description of the nonlinear effects even in a wider range of physical cases[9]. In particular, the nonlinear properties of superconductors in the so-called adiabatic approximation (i.e., at sufficiently low frequencies), and also at extremely high frequencies, should be the same, regardless of the concrete structure of the temporal terms, although the adiabaticity criterion itself will depend on the concrete physical situation[9,10].

As was already noted, the rigorous solution of the problem of the nonlinear behavior of superconductors at microwave frequencies encounters considerable difficulties. The situation simplifies in the case of sufficiently superconducting films, for in this case, as will be shown below, an important role in the destruction of the superconductivity is played only by the electric component of the electromagnetic field. At the same time, under conditions $d \ll \delta_L$ (d—thickness, δ_L—London depth of penetration), the electric field is homogeneous over the cross section of the film, and therefore the problem reduces to an investigation of the simpler question of the nonlinear properties of a superconducting channel at microwave frequencies. We note that the condition $d \ll \delta_L$ does not mean at all that the film is transparent to the electromagnetic radiation. Only much thinner films, for which the parameter $\delta_L^2/d\lambda$ (λ—length of the electromagnetic wave) is of the order of unity, become transparent.

In Sec. 2 of this paper we consider the question of the electrodynamics of superconducting films in the linear approximation, making it possible to determine the orders of magnitude of the transparency coefficients, and also to establish the limits of applicability of such an approximation. We obtain an impedance boundary condition for films, generalizing the usual relation $E_t = \zeta H_t \times n$[11]. In Sec. 3 we investigate the nonlinear properties of a superconducting channel in a given electric field in the presence of a transport current. We obtain the dependence of the critical magnitude of the superconducting current on the microwave power. We consider the question of generation of harmonics and the frequency shift in the presence of a dc component of the current. Finally, in Sec. 4 we formulate the problem of a consistent calculation of the nonlinear properties of thin films at microwave frequencies with allowance of both the nonlinear properties of the superconducting channel and of the impedance of the film itself, which leads to an essentially nonlinear dependence of the electromagnetic-field power penetrating into the film on the intensity of the incident microwave.

2. HIGH-FREQUENCY ELECTRODYNAMICS OF FILMS

The electromagnetic properties of superconductors are described by introducing the complex conductivity $\sigma_1 + i\sigma_2$ (see, for example,[12]), or, which is equivalent, by specifying the connection between the current **j** and the vector potential **A**:

$$\mathbf{j} = -\frac{c}{4\pi\delta^2}\mathbf{A}, \tag{2.1}$$

where the complex "depth of penetration" δ takes into account both the screening of the field by the Meissner currents, and the skin effect in the normal state:

$$\frac{1}{\delta^2} = \frac{1}{\delta_L^2} - \frac{2i}{\delta_{sk}^2}, \quad \delta_L^2 = \frac{mc^2}{4\pi N_s e^2}, \quad \delta_{sk}^2 = \frac{c^2}{2\pi\sigma\omega} \tag{2.2}$$

(σ—conductivity of the normal metal, ω—frequency).

By virtue of Maxwell's equations, we obtain the following connection between the electric field intensity $\mathbf{E} = i\omega\mathbf{A}/c$ and the magnetic field $\mathbf{H} = \operatorname{curl}\mathbf{A}$:

$$\mathbf{E} = -\frac{c\zeta_0^2}{i\omega}\operatorname{rot}\mathbf{H}, \tag{2.3}$$

where ζ_0 is the impedance (using the notation of[12]):

$$\zeta_0 = -i\omega\delta / c. \tag{2.4}$$

The components of the field satisfy the equations

$$\Delta\mathbf{H} - \frac{1}{\delta^2}\mathbf{H} = 0, \quad \Delta\mathbf{E} - \frac{1}{\delta^2}\mathbf{E} = 0. \tag{2.5}$$

If an electromagnetic wave is incident on the surface, then its characteristic depth of penetration into the metal (in a direction normal to its surface) amounts to ~δ, which is much smaller than the characteristic distances of variation of the field along the surface of the metal, which equals the wavelength λ of the electromagnetic wave in vacuum. For this reason, we can neglect the variation of the field along the surface and write

$$\mathbf{H}(z) = \mathbf{A}e^{\varkappa z} + \mathbf{B}e^{\varkappa z}, \quad \varkappa = 1/\delta, \tag{2.6}$$

where the constants **A** and **B** are expressed in terms of the values of **H** at z = 0 and z = d:

$$\mathbf{A} = \frac{\mathbf{H}(d) - \mathbf{H}(0)e^{-\varkappa d}}{e^{\varkappa d} - e^{-\varkappa d}}, \quad \mathbf{B} = \frac{\mathbf{H}(0)e^{\varkappa d} - \mathbf{H}(d)}{e^{\varkappa d} - e^{-\varkappa d}}. \tag{2.7}$$

In (2.6), the coordinate z is reckoned along the normal to the film surface. Calculating the electric field with the aid of (2.3) and using relations (2.6) and (2.7) we obtain, putting z = 0 and z = d,

$$\mathbf{E}(0) = \zeta_1[\mathbf{H}(0)\mathbf{n}] - \zeta_2[\mathbf{H}(d)\mathbf{n}],$$
$$\mathbf{E}(d) = -\zeta_1[\mathbf{H}(d)\mathbf{n}] + \zeta_2[\mathbf{H}(0)\mathbf{n}]. \tag{2.8*}$$

We have introduced here the quantities

$$\zeta_1 = \zeta_0 \operatorname{cth}\frac{d}{\delta}, \quad \zeta_2 = \zeta_0\frac{1}{\operatorname{sh}(d/\delta)}, \tag{2.9}$$

and **n** represents a unit vector normal to the surface of the film (directed to the interior of the superconductor at z = 0). Equations (2.8) connect the tangential components of the electric and magnetic fields inside the superconductor near its surface. Inasmuch, on the other hand, as the tangential components of the field must be continuous on the interface between the media, Eqs. (2.8) can be regarded as boundary conditions for the electromagnetic field in vacuum (the film itself can in this case be allowed to tend to zero):

$$\mathbf{E}_t(-0) = \zeta_1[\mathbf{H}_t(-0)\mathbf{n}] - \zeta_2[\mathbf{H}_t(+0)\mathbf{n}],$$
$$\mathbf{E}_t(+0) = \zeta_2[\mathbf{H}_t(-0)\mathbf{n}] - \zeta_1[\mathbf{H}_t(+0)\mathbf{n}]. \tag{2.10}$$

We note that in the case of bulk metal (d → ∞), the quantity ζ_2 vanishes, and ζ_1 tends to ζ_0, as a result of which (2.10) goes over into the usual boundary condition on the surface of the metal[11]:

$$\mathbf{E}_t(0) = \zeta_0[\mathbf{H}_t(0)\mathbf{n}].$$

A condition of the same type holds for a film if we introduce "impedance" matrix

$$\hat{\zeta} = \begin{pmatrix} \zeta_1 & -\zeta_2 \\ \zeta_2 & -\zeta_1 \end{pmatrix}, \tag{2.11}$$

where ζ_1 and ζ_2 are defined by formulas (2.9). Equations (2.10) are valid so long as ζ_1 and ζ_2 are small compared with unity, thus ensuring slowness of the variation of the field along the surface of the metal compared with its variation in the direction normal to the film. In addition, it is required (in the case when the film is not plane) that the characteristic radii of its curvature be large compared with the thickness d.

We shall use (2.10) to solve the problem of passage of an electromagnetic wave through the film. Let the wave be incident on the film normally, so that **k** ∥ **n**, where **k** is the wave vector, and the components of the electric and magnetic field are parallel to the surface of the film. Putting $A = A_x$, we obtain for $z < 0$

$$A = A_0 e^{ikz} + A_1 e^{-ikz}, \quad k = \omega / c, \tag{2.12}$$

which corresponds to the incident (A_0) and reflected (A_1) waves, and at $z > 0$ there is only the transmitted wave

$$A = A_2 e^{ikz}, \quad z > 0. \tag{2.13}$$

Expressing E_x and H_y in terms of A and using the boundary conditions (2.10), we readily obtain the amplitudes of the reflected and transmitted wave[1)]

$$A_1 = A_0\frac{1 - \zeta_1^2 + \zeta_2^2}{(\zeta_1 + \zeta_2 - 1)(\zeta_2 - \zeta_1 + 1)},$$
$$A_2 = A_0\frac{2\zeta_2}{(\zeta_1 + \zeta_2 - 1)(\zeta_2 - \zeta_1 + 1)} \tag{2.14}$$

It is seen from (2.14) that in the case when $\zeta_1, \zeta_2 \ll 1$, the coefficient of transmission of the wave through the film is

*$[\mathbf{H}(0)\mathbf{n}] \equiv \mathbf{H}(0) \times \mathbf{n}$.

1)We note that although in the general case the relations (2.10) are valid only when $\zeta_1, \zeta_2 \ll 1$, at the chosen geometry (**k** ∥ **n**) they are exact for all values d, i.e., formulas (2.14) remain valid for arbitrary values of ζ_1 and ζ_2.

$$T = \left| \frac{A_2}{A_0} \right|^2 \cong 4|\zeta_2|^2. \tag{2.15}$$

For films with thickness $d \ll \delta$, this yields

$$T \approx \left(\frac{2\omega|\delta|^2}{cd} \right)^2 \ll 1. \tag{2.16}$$

Assuming that $\delta \sim \delta_L$, we find that the transparency coefficient becomes of the order of unity only for films of thickness $d \sim 4\pi\delta_L^2/\lambda$, corresponding to exceedingly small thicknesses. However, if it is recognized that the quantity δ_L itself increases with decreasing thickness (as a result of the decrease of the mean free path), then we obtain for the characteristic thickness d_0, at which T becomes of the order of unity, the estimate

$$d_0 \sim 2\delta_{L0}(\xi_0/\lambda)^{1/2}(1-t)^{-1/2} \tag{2.17}$$

(δ_{L0} and ζ_0 are the values of the depth of penetration and the coherence length of the pure superconductor at $T = 0$, and $t = T/T_c$). A typical value of d_0 is several dozen Angstrom (or less).

A simple analysis shows that the field distribution inside the superconductor has the following form when $z \ll \delta_L$. The magnetic field varies linearly like $H(z) \approx H_0(d-z)/d - \zeta_2 H_0$, where H_0 is the amplitude of the field in the incident wave, whereas the electric field is constant over the thickness of the film and its order of magnitude is $E(z) \approx \text{const} \approx -H_0\zeta_1$. Consequently, for thin films the ratio E/H has an order of magnitude $|\zeta_1| \sim \delta_L^2/\lambda d$.

Thus, the electric field in the film is practically always weak compared with the magnetic field. Nonetheless, the action of the electric field on the "superconducting electrons" is more significant than the action of the magnetic field. Let us examine, for example, a geometry in which **E** and **H** are parallel to the surface. The criterion for a "strong" magnetic field is the condition $H \sim H_c$ [7], where H_c is the critical field, equal to $\sim \Phi_0/d\xi(T)$ for thin films [13], and $\Phi_0 = hc/2e$ is the magnetic-flux quantum. The criterion of a "strong" electric field is the condition $eE\xi(T) \sim \hbar\omega$ (a detailed justification of this criterion is given in Sec. 3), which can be rewritten also in the form $E \sim E_c \sim \Phi_0/\lambda\xi(T)$. Taking into account the fact that the amplitudes of the electric and magnetic fields in the film are connected by the relation $E/H \sim \delta_L^2/\lambda d$, we find that when $d \ll \delta_L$ the destruction of the superconductivity of the film will be controlled by the electric component of the electromagnetic field, for when $E \sim E_c$ the effective magnetic field is $H/H_c \sim (d/\delta_L)^2 \ll 1$. When $d \gtrsim \delta_L$, the effects of the electric and magnetic fields turn out to be of the same order of magnitude, so that the problem is essentially no longer one-dimensional[2)].

3. NONLINEAR PROPERTIES OF A SUPERCONDUCTING CHANNEL

The already mentioned circumstance, namely the possibility of neglecting the magnetic component of a rapidly-alternating field compared with its electric component when $d \ll \delta_L$ makes it possible to use the following model for the analysis of the behavior of films at microwave frequencies. We consider a long superconducting channel, i.e., a narrow superconducting gap, along the surface of which a time-varying electric field $E_x = E(t)$ is applied. The variation of all the quantities with thickness can be neglected, making the problem one-dimensional. The equation for the ordering parameter $\Delta(x, t)$ assumes in this case the form ($\hbar = 1$)[7,8]:

$$\frac{\partial\Delta}{\partial t} - D\left(\frac{\partial}{\partial x} + ip_s(t)\right)^2 \Delta = \varepsilon_0(T)\Delta - \varepsilon_1(T)|\Delta|^2\Delta, \tag{3.1}$$

where $D = v_0 l/3$ is the diffusion coefficient in the normal state, and the parameters ϵ_0 and ϵ_1 are defined near T_c by the formulas

$$\varepsilon_0 = C_1(T_c - T), \quad \varepsilon_1 = C_2/T_c \tag{3.2}$$

(C_1 and C_2 are dimensionless (positive) constants of the order of unity). The quantity p_s in (3.1) stands for $-2eA/c$, where A is the vector potential, i.e.,

$$\partial p_s/\partial t = 2eE. \tag{3.3}$$

The last relation is analogous to the well known Josephson formula $\partial\varphi/\partial t = 2eV$, where φ is the phase of the ordering parameter of the "weak" superconductivity and V is the voltage across the tunnel barrier. In this case V is replaced by the electric field intensity E and φ by a quantity analogous to the gradient of the phase: $p_s \sim \nabla\varphi$ (indeed, by virtue of the gauge invariance of (3.1), we can choose for a homogeneous current state a gauge in which p_s is replaced by $\nabla\varphi$).

The expression for the current in the superconducting channel is

$$j(t) = \sigma E(t) + C_0 \operatorname{Im}\left[\Delta^*\left(\frac{\partial}{\partial x} + ip_s(t)\right)\Delta\right], \tag{3.4}$$

where $\sigma = Ne^2\tau/m$ is the conductivity of the normal state, and the constant C_0 is proportional to the critical current j_c (see below).

1. Let us investigate first the stability of the normal state ($\Delta = 0$) against infinitesimally small fluctuations in a specified alternating field $E(t) = E_0\cos\omega t$. In accordance with (3.3), we obtain $p_s(t) = 2eE_0\omega^{-1}\sin\omega t$. Writing down the general solution of the linearized equation (3.1) in the form $\Delta(x, t) = e^{iqx}\Delta_0(t)$, we have

$$\Delta_0(t) = \Delta_0(0)\exp\left\{\int_0^t \left[\varepsilon_0 - D\left(q + \frac{2eE_0}{\omega}\sin\omega t\right)^2\right] dt\right\}. \tag{3.5}$$

We see therefore that if $D[q^2 + \frac{1}{2}(2eE_0/\omega)^2] > \epsilon_0$, the normal phase is stable against infinitesimally small fluctuations: $\Delta_0(t) \to 0$ as $t \to \infty$. With decreasing amplitude of the alternating field, loss of stability occurs, accompanied by a transition to the superconducting state. This occurs first when $q = 0$ and when E_0 is given by

$$2eE_0 = (2\varepsilon_0/D)^{1/2}\omega. \tag{3.6}$$

Introducing the temperature-dependent coherence radius of the superconductor $\xi(T) = \hbar(D/\epsilon_0)^{1/2}$ (see [14,15]), we can rewrite the condition (3.6) in the form

$$eE_0\xi(T)\sqrt{2} = \hbar\omega. \tag{3.7}$$

Formula (3.7) means that the energy acquired by the electron over the coherence length becomes comparable

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[2)]Gor'kov and Éliashberg [7] have considered the case of destruction of superconductivity of a film by an alternating magnetic field. This is in fact a film placed in a resonator at the node of the electric field. The total current parallel to the surface then vanishes [7]. The situation considered in the present paper corresponds to destruction of superconductivity by current, leading to lower critical-power levels.

with the magnitude of the electromagnetic quantum $\hbar\omega$. Introducing the quantum of the magnetic flux Φ_0, we can rewrite expression (3.7) for the critical value of the field amplitude also in the form

$$E_0^{cr} = \sqrt{2}\Phi_0 / \xi\lambda, \tag{3.8}$$

thus justifying the estimate used in Sec. 2.

We note that in the static case, i.e., when $\omega = 0$, the normal phase is stable against a transition into the superconducting state in the presence of an arbitrarily weak electric field. In the latter case, however, it is necessary also to analyze the question of the role of the thermal (and other) fluctuations, for if these fluctuations are sufficiently large, the magnitude of the mean-squared "gap" $\overline{\Delta^2}$ may differ from zero and may be quite large, since the "stability" in the sense indicated above takes place only asymptotically as $t \to \infty$.

2. We proceed further to investigate the nonlinear properties of a superconducting channel when $E_0 < E_0^{cr}$, turning to the nonlinear equation (3.1) for this purpose. We formulate the problem as follows. Assume that a certain constant ("transport") current $j_T = \overline{j}$ is made to pass through the film and an alternating voltage $E = E_0 \cos\omega t$ is applied. Then the maximum value of the current $\overline{j}$ that can flow without the appearance of resistance will be a certain function of the field E_0, decreasing with increasing E_0 and vanishing when $E_0 = E_0^{cr}$. To determine the corresponding function, we must choose $p_s(t)$ in the form[3)]

$$p_s^0(t) = q + \frac{2eE_0}{\omega}\sin\omega t, \tag{3.9}$$

solve Eq. (3.1) for $\Delta = \Delta_0(t)$, and then determine the value of q from the condition that the average current $\overline{j}$ be maximal.

We shall find it convenient to changeover to dimensionless variables, expressing Δ in units of $(\epsilon_0/\epsilon_1)^{1/2}$, x and p_s^{-1} in units of $\xi(T)$, and the time t in units of ϵ_0^{-1} (the frequency ω is in this case measured in units of ϵ_0). In addition, we introduce a dimensionless field amplitude $z = E_0(\Phi_0/\xi\lambda)^{-1}$ so that the critical value of z is $\sqrt{2}$ (see formula (3.8)). In terms of the new variables, Eqs. (3.1) and (3.4) take the form

$$\frac{\partial\Delta}{\partial t} - \left(\frac{\partial}{\partial x} + ip_s\right)^2 \Delta = \Delta - |\Delta|^2\Delta, \tag{3.10}$$

$$j = \mathrm{Im}\left[\Delta^*\left(\frac{\partial}{\partial x} + ip_s\right)\Delta\right] + \gamma\dot{p}_s, \quad \frac{\partial j}{\partial x} = 0, \tag{3.11}$$

where γ is a dimensionless parameter, equal to

$$\gamma = \frac{\sigma\varepsilon_1}{2eC_0} = \frac{\sigma\varepsilon_0}{3\sqrt{3}\xi j_c}. \tag{3.12}$$

Near T_c, the quantity γ does not depend on the temperature and (in the limit as $l \to 0$) on the mean free path, i.e., it is a certain constant of the order of unity (for a contaminated alloy). We shall henceforth regard γ as an independent phenomenological parameter[4)].

We call attention to the last relation of (3.11), namely $\partial j/\partial x = 0$. As shown in[8], this relation expresses the continuity condition for the superconductor, and is necessary for self-consistency of the equations of the nonstationary model (3.10) and (3.11). We note immediately that for a solution independent of the coordinates, namely $\Delta = \Delta_0(t)$, this relation is automatically satisfied.

The equation for the function $\Delta_0(t)$ takes the form

$$d\Delta_0/dt + (q + z\sin\omega t)^2\Delta_0 = \Delta_0(1 - \Delta_0^2). \tag{3.13}$$

Its general solution is given in[7]

$$\Delta_0^2(t) = \Delta_0^2(0)e^{2f(t)} \Big/ \left\{1 + 2\Delta_0^2(0)\int_0^t e^{2f(t')}dt'\right\}, \tag{3.14}$$

where

$$f(t) = \int_0^t [1 - (q + z\sin\omega t)^2]\,dt. \tag{3.15}$$

If $q^2 + z^2/2 < 1$, there is no sensitivity to the initial conditions, and the asymptotic form of $\Delta_0^2(t)$ as $t \to \infty$ becomes

$$\Delta_0^2(t) = e^{2f(t)} \Big/ 2\int_0^t e^{2f(t)}\,dt. \tag{3.16}$$

We obtain further, with the aid of (3.13), the current j(t):

$$j(t) = (q + z\sin\omega t)\Delta_0^2(t) + \gamma\omega z\cos\omega t. \tag{3.17}$$

It is convenient to investigate the obtained formulas in the limiting cases of high and low frequencies.

a) In the case $\omega \gg 1$, the function f(t) is equal to $(1 - q^2 - z^2/2)t$ plus a small addition proportional to the parameter $1/\omega$. Omitting this addition, we obtain on the basis of (3.16)

$$\Delta_0^2 \approx 1 - q^2 - z^2/2, \quad \omega \gg 1, \tag{3.18}$$

which shows that Δ_0 is independent of the time and is determined by the square of the mean value of the field E^2. Calculating the current, we obtain in the same approximation

$$\overline{j} = q(1 - q^2 - z^2/2). \tag{3.19}$$

The maximum of (3.19) is reached at $q = q_m = (1 - z^2/2)^{1/2}/\sqrt{3}$. From the ratio of the maximum current to the critical current of the superconducting channel at z = 0, which equals $\frac{2}{3}\sqrt{3}$ in the chosen units, we obtain the dependence of the critical current on the microwave power:

$$\overline{j}_{max}(z)/\overline{j}_{max}(0) = (1 - z^2/2)^{3/2}. \tag{3.20}$$

A plot of the corresponding dependence is shown in Fig. 1. When the critical value of the high frequency power ($z = \sqrt{2}$) is reached, the superconducting current vanishes.

b) In the adiabatic case ($\omega \ll 1$), we integrate the expression in the denominator of (3.16) by parts, obtaining

$$\Delta_0^2(t) \approx \begin{cases} f'(t), & f'(t) > 0 \\ 0, & f'(t) < 0, \end{cases} \quad (\omega \ll 1) \tag{3.21}$$

(in the case when $f'(t) < 0$, Δ_0^2 is exponentially small in terms of the parameter $1/\omega$). Thus, in this case Δ_0 depends on the time in an essentially anharmonic manner. At the point corresponding to the transition to the normal state ($z = \sqrt{2}$), Δ_0^2 vanishes jumpwise. This points to the possibility of delaying the transition to the normal phase above the value of the field $z = \sqrt{2}$.

We present expressions for the average current $\overline{j}$ and for the amplitudes of the harmonics j_n (j_n are the

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3)The introduction of the constant q is equivalent, apart from a gauge, to choosing the solution for Δ in the form $\Delta = e^{iqx}\Delta_0(t)$.

4)The quantity γ can be represented also in the form $\gamma = 2\delta_L^2\epsilon_0/\delta_{sk}^2\omega$, where δ_{sk} is the skin depth in the normal state (2.2).

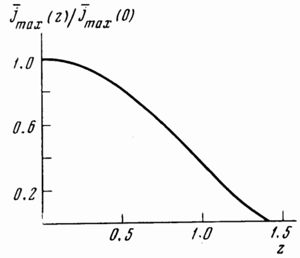

FIG. 1

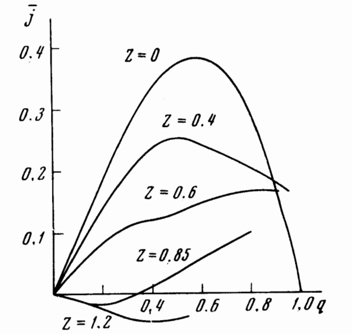

FIG. 2

FIG. 1. Dependence of critical current on the power at high frequencies ($\omega \gg \epsilon_0$). The parameter $z = \sqrt{2E_0/E_0^{cr}}$.

FIG. 2. Superconducting current as a function of the superfluid velocity in the presence of a high frequency field (in the adiabatic case).

Fourier coefficients of the function j(t)). In the case $q + z < 1$ (see below) they take the form

$$\bar{j} = q(1 - q^2 - {}^3/_2 z^2), \tag{3.22}$$

$$j_1 = \{[z(1 - 3q^2 - {}^3/_4 z^2)]^2 + \gamma^2\omega^2 z^2\}^{1/2},$$
$$j_2 = {}^3/_2 q z^2, \quad j_3 = {}^1/_4 z^3 \text{ etc.} \tag{3.23}$$

We see therefore, for example, that the second harmonic appears only in the presence of transport current. Its intensity is proportional to the square of the current and to the square of the intensity of the microwave power at the fundamental frequency:

$$W_2 \sim j_T^2 W_1^2, \quad \omega \ll T_c - T. \tag{3.24}$$

We note that in the extremely non-adiabatic case, a similar analysis shows that the intensity of the harmonics is much lower than in the case of "low" frequencies. Thus, from the second harmonic we obtain, with accuracy to the same coefficients as in (3.24):

$$W_2 \sim j_T^2 W_1^2 \left(\frac{T_c - T}{\omega}\right)^4, \quad \omega \gg T_c - T.$$

Thus, the frequency-conversion efficiency decreases with increasing ω.

Figure 2 shows examples of the dependence of the current $\bar{j}$ on the parameter of the "superfluid velocity" q in the adiabatic case, at various conditions of high-frequency power. The quantity q in Fig. 2 ranges from 0 to $q_m = (1 - z^2/2)^{1/2}$. As seen from the plots, in the presence of a microwave field the form of the function $\bar{j}(q)$ differs greatly from the corresponding dependence at z = 0. With increasing z, the absolute value of $\bar{j}$ decreases and vanishes at $z = \sqrt{2}$. The dependence of the crystal value of the current on z will be determined below.

3. The question of the stability of the obtained solutions against infinitesimally small fluctuations is far from trivial. The need for analyzing the stability follows already from the ambiguity of the dependence of the parameter q on the current $\bar{j}$ (see Fig. 2). It turns out that only one branch of the function $q(\bar{j})$ is stable; this makes it possible to determine uniquely the superconducting state at a given current. At sufficiently large z there may be no stable solutions of (3.10) at all, depending on the frequency region.

Proceeding to an investigation of the stability, we put $\Delta = \Delta_0(t) + \Delta_1(x, t)$, where Δ_0 is given by (3.14) or (3.16), and Δ_1 is an infinitesimally small perturbation. Similarly, taking into account the condition $\partial j/\partial x = 0$, we should write the addition to the electric field in the form $p_s = p_s^0(t) + p_s^1(x, t)$, where $p_s^0(t) = q + z\sin\omega t$. Using (3.10) and (3.11), we obtain a system of coupled equations for the quantities Δ_1 and p_s^1:

$$\frac{\partial\Delta_1}{\partial t} - \left(\frac{\partial}{\partial x} + ip_s^0\right)^2 \Delta_1 - \Delta_1 + \Delta_0^2(t)(2\Delta_1 + \Delta_1^*) = -\Delta_0\left(2p_s^0 - i\frac{\partial}{\partial x}\right)p_s^1,$$

$$p_s^0\Delta_0\frac{\partial}{\partial x}(\Delta_1 + \Delta_1^*) + \frac{1}{2i}\Delta_0\frac{\partial^2}{\partial x^2}(\Delta_1 - \Delta_1^*) + \left(\Delta_0^2\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x\,\partial t}\right)p_s^1 = 0. \tag{3.25}$$

The general solution of the foregoing equations can be represented in the form

$$\Delta_1(x, t) = \varphi(t)e^{ikx} + \psi^*(t)e^{-ikx}, \quad p_s^1(x, t) = \zeta(t)e^{ikx} + \zeta^*(t)e^{-ikx}, \tag{3.26}$$

from which we obtain for the functions φ, ψ, ζ:

$$\frac{d\varphi}{dt} = [1 - (p_s^0 + k)^2 - 2\Delta_0^2]\varphi - \Delta_0^2\psi - \Delta_0(2p_s^0 + k)\zeta,$$
$$\frac{d\psi}{dt} = [1 - (p_s^0 - k)^2 - 2\Delta_0^2]\psi - \Delta_0^2\varphi - \Delta_0(2p_s^0 - k)\zeta,$$
$$\frac{d\zeta}{dt} = -\frac{\Delta_0^2}{\gamma}\zeta - \frac{\Delta_0}{2\gamma}(2p_s^0 + k)\varphi - \frac{\Delta_0}{2\gamma}(2p_s^0 - k)\psi. \tag{3.27}$$

The obtained equations can be investigated in the limiting cases of high and low frequencies: $\omega \gg 1$ and $\omega \ll 1$.

At high frequencies, the change of φ, ψ, and ζ within one period is small. In this case we can regard φ, ψ, and ζ as slowly-varying functions and average the coefficients (3.27) with respect to the time. As a result we obtain a system of equations with constant coefficients, the characteristic indices of which λ_i (i = 1, 2, 3) are given by

$$\lambda_{1,2} = -\left(\frac{\Delta_0^2}{2\gamma} + \Delta_0^2 + k^2\right) \pm \left[\left(\frac{\Delta_0^2}{2\gamma} + \Delta_0^2 + k^2\right)^2 - \left(\frac{\Delta_0^2}{\gamma} + k^2\right)(2\Delta_0^2 - 4q^2 + k^2)\right]^{1/2},$$
$$\lambda_3 = 0. \tag{3.28}$$

Here Δ_0^2 is given by (3.18). We see therefore that the instability ($\lambda > 0$) appears when $2\Delta_0^2 - 4q^2 + k^2 < 0$. When $q > (1 - z^2/2)^{1/2}/\sqrt{3}$, there are always values of k corresponding to instability. To the contrary, when $q < (1 - z^2/2)^{1/2}/\sqrt{3}$, stability obtains at all values of k. This condition determines the choice of the solution of Eq. (3.19) for a state with a specified current $\bar{j}$. As shown by the analysis, for all values $z < \sqrt{2}$ the solutions corresponding to the values of the current up to $\bar{j} = \bar{j}_{max}(z)$ (Fig. 1) are stable.

In the case of low frequencies ($\omega \ll 1$), we seek a solution of the system (3.23) in the form

$$(\varphi, \psi, \zeta) = (u, v, w)\exp\int_0^t \lambda(t)\,dt,$$

assuming u, v, and w to be slow functions of the time. Omitting the derivatives of these functions, we obtain a system of algebraic equations for the adiabatic exponents $\lambda = \lambda_i(t)$. The stability criterion lies in the fact that the integral $\int \lambda(t)dt$, taken over the period, be negative. Omitting the complicated intermediate steps, we present here only the result, confining ourselves for simplicity to the limiting cases of large and small values of the dimensionless parameter λ introduced in (3.12) above (we note that γ is the only dimensionless quantity characterizing the system of equations (3.10) and (3.11); as will be shown, the final result depends little on the concrete value of γ).

In the case when $\gamma \gg 1$, the region of values of z and q corresponding to a stable state is determined by the inequality

$$\frac{1}{\sqrt{(1-q)^2-z^2}}+\frac{1}{\sqrt{(1+q)^2-z^2}} \leqslant 3 \tag{3.29}$$

(see Fig. 3a). To the contrary, when $\gamma \ll 1$, it is given by (Fig. 3b):

$$q^2+{}^1\!/_2 z^2 \leqslant {}^1\!/_3. \tag{3.30}$$

In both cases, when z = 0 the critical value of q is $q = 1/\sqrt{3}$, which agrees with the result obtained above for the extremely non-adiabatic case.

When q = 0, the critical value of z is $z = z_{c1} = \sqrt{5}/3 = 0.75$ in the case when $\gamma = \infty$, and $z = z_{c2} = (2/3)^{1/2} = 0.82$ in the case when $\gamma = 0$. At values $z > z_c$, the superconducting state is unstable for all values of q. At the same time, we have seen that the normal phase is also unstable when $z < \sqrt{2} = 1.41$, i.e., a transition to the superconducting state should occur. The result signifies actually that in the interval of z between z_c and $\sqrt{2}$ the employed homogeneous (independent of the coordinates) solution (3.14) becomes unsuitable, and the true solution must depend on both x and t. When $\gamma \gg 1$, the problem reduces to finding a stable solution of the equation

$$\frac{\partial\Delta}{\partial t}-\left(\frac{\partial}{\partial x}+iz\sin\omega t\right)^2\Delta=\Delta(1-|\Delta|^2). \tag{3.31}$$

We note that such a solution can exist (and be stable) also at field values exceeding $z = \sqrt{2}$, this being analogous to the effects of "superheating" in the theory of superconductivity[16]. We were unable, however, to obtain the corresponding solution, and this question remains open.

In the range $z < z_c$, it is easy to construct the dependence $\bar{j}_{max}(z)$, in analogy with the corresponding dependence for the extremely nonadiabatic case shown in Fig. 1. Plots of such a dependence in the limiting cases $\gamma \to 0$ and $\gamma \to \infty$ are shown in Fig. 4. The dependence of the shape of the curve on γ is quite weak. The dashed curve in the same figure shows, for comparison, the maximum critical current obtained with the aid of a solution of type (3.21) without analyzing the stability problem. On the whole, the question of the behavior of a superconducting channel at microwave frequencies in the adiabatic case calls for further investigation.

4. In concluding this section, let us examine the question of the frequency mixing. Assume that the electric field in the film has two monochromatic components of frequency ω_1 and ω_2 (for concreteness, $\omega_1 < \omega_2$):

$$p_s^0(t) = q + z_1\sin\omega_1 t + z_2\sin(\omega_2 t+\delta). \tag{3.32}$$

Then the nonlinear properties of the system become manifest in the generation of an intermediate frequency $\Omega = \omega_2 - \omega_1$, and also of combination frequencies

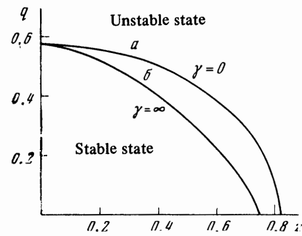

FIG. 3

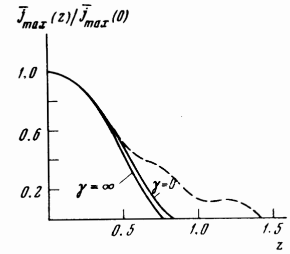

FIG. 4

FIG. 3. Regions of stability of the homogeneous solution at $\gamma \gg 1$ and $\gamma \ll 1$.

FIG. 4. Dependence of the critical current on the power in the adiabatic limit ($\omega \ll \epsilon_0$).

$\omega_1 - n\Omega$ and $\omega_2 + n\Omega$, n = 1, 2, 3, ... To determine the intensities of the corresponding signals, we substitute expression (3.32) in formulas (3.15)–(3.17). We shall consider only the extremely non-adiabatic case: $\omega_{1,2} \gg 1$. As to the difference frequency Ω, we assume that it satisfies the adiabatic condition $\Omega \ll 1$ (we recall that all the frequencies are measured in units of the quantity $\epsilon_0 \sim (T_c - T)$). The function f(t) takes the form

$$f(t)\approx\left[1-q^2-\frac{1}{2}(z_1^2+z_2^2)\right]t-\frac{z_1z_2}{\Omega}[\sin(\Omega t+\delta)-\sin\delta], \tag{3.33}$$

and we have discarded the terms containing in the denominators the sums $\omega_1 + \omega_2 \gg 1$. The characteristic value of the time t in (3.33) is equal to unity. We see therefore that the time-oscillating term is large compared with the non-oscillating in a ratio $1/\Omega \gg 1$. As a result we find that the situation with respect to the difference frequency is the same as in the adiabatic limit discussed in Sec. 2. We have

$$\Delta^2(t)\approx 1-q^2-\frac{1}{2}(z_1+z_2)^2+2z_1z_2\sin^2\frac{\Omega t+\delta}{2} \tag{3.34}$$

with $\Delta^2(t)$ vanishing when the quantity in the right side of this formula becomes negative. Let us write out the expressions for the average current and for the amplitudes of the oscillating components corresponding to the frequencies $\omega_2 - \omega_1$, $2\omega_1 - \omega_2$, and $2\omega_2 - \omega_1$:

$$\bar{j} = q(1-q^2-{}^1\!/_2 z_1^2-{}^1\!/_2 z_2^2), \tag{3.35}$$

$$j_\Omega = qz_1z_2,\quad j_{\omega_1-\Omega} = {}^1\!/_2 z_1^2 z_2,\quad j_{\omega_2+\Omega} = {}^1\!/_2 z_1 z_2^2. \tag{3.36}$$

As seen from these formulas, the intensity of the intermediate frequency is proportional to the square of the transport current (as $j_T \to 0$) and to the intensities of the mixed frequencies W_1 and W_2. At larger values of the amplitudes z_1 and z_2, we are faced with the same problems as in the adiabatic case discussed in Sec. 2.

4. NONLINEAR ELECTRODYNAMICS OF A FILM

In the preceding section we have assumed the electric field to be specified. Actually, the field penetrating into the film is determined by the impedance of the film, which itself depends on the amplitude. This leads to an essentially nonlinear dependence of the transmission coefficient of the film on the incident power; this dependence can also have hysteresis, i.e., it can be different in increasing and decreasing fields.

Let us consider for simplicity a plane electromagnetic wave incident normally on the surface of the film. We consider only the limiting cases of high and low frequencies ($\omega \gg (T_c - T)$ and $\omega \ll (T_c - T)$, respectively). Denoting by A(t) the vector-potential component parallel to the surface of the film, we obtain by virtue of the results of Sec. 3 the following expression for the current:

$$j(t) = -\frac{\sigma}{c}\frac{\partial A}{\partial t} - \frac{c}{4\pi\delta_L^2} A(t)\langle 1 - sA^2(t)\rangle, \tag{4.1}$$

where

$$s = \frac{D}{\varepsilon_0}\left(\frac{2e}{c}\right)^2 = \left(\frac{2e}{c}\xi(T)\right)^2,$$

and δ_L denotes the London depth of penetration in a zero field. The angle brackets in (4.1) have the following meaning. In the adiabatic limit ($\omega \ll \epsilon_0$) we have, by definition

$$\langle 1 - sA^2(t)\rangle = (1 - sA^2(t))\theta(1 - sA^2(t)), \quad \omega \ll \varepsilon_0, \tag{4.2}$$

where $\theta(x)$ is a function equal to unity when $x > 0$ and to 0 when $x < 0$. The presence of the θ-function is an expression of the fact that when $\omega \ll (T_c - T)$ the ordering parameter vanishes for those values of t, at which the superfluid velocity exceeds the critical value p_s^{cr}/m, where $p_s^{cr} = \xi^{-1}(T)$ (see (3.21)).

In the extremely non-adiabatic case ($\omega \gg \epsilon_0$), the value of the current is determined by the mean square of the field, and accordingly

$$\langle 1 - sA^2(t)\rangle = (1 - s\overline{A}^2)\theta(1 - s\overline{A}^2), \quad \omega \gg \varepsilon_0, \tag{4.3}$$

the θ-function being added here in order that expression (4.1) remain valid also in the normal state, when the superconducting part of the current in (4.1) vanishes.

Inserting (4.1) in Maxwell's equation curl $\mathbf{H} = 4\pi \mathbf{j}/c$, we obtain an equation for the function A (the z axis is normal to the film)

$$\frac{\partial^2 A}{\partial z^2} = \frac{4\pi\sigma}{c^2}\frac{\partial A}{\partial t} + \frac{1}{\delta_L^2} A(t)\langle 1 - sA^2(t)\rangle. \tag{4.4}$$

The boundary condition for Eq. (4.4) is the continuity of the tangential components of the electric and magnetic fields ($E_x = -c^{-1}\partial A/\partial t$ and $H_y = \partial A/\partial z$) on the interface with the vacuum. In free space A satisfies the wave equation

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0.$$

Choosing at $z < 0$ the quantity A in the form of a sum of incident and reflected waves

$$A(z, t) = A_0 \cos(kz - \omega t) + A_1(z, t), \quad \omega = ck,$$

$$A_1(z, t) = \sum_{m=1}^{\infty} A_m^{(1)} \cos[m(kz + \omega t) + \alpha_m], \tag{4.5}$$

and at $z > d$ in the form of a transmitted wave

$$A(z, t) = A_2(z, t) = \sum_{m=1}^{\infty} A_m^{(2)} \cos[m(k(z - d) - \omega t) + \beta_m] \tag{4.6}$$

and eliminating $A_1(z, t)$ and $A_2(z, t)$, we obtain the following boundary conditions for Eq. (4.4):

$$\left(\frac{\partial A}{\partial z} - \frac{1}{c}\frac{\partial A}{\partial t}\right)_{z=0} = \frac{2\omega}{c} A_0 \sin\omega t,$$

$$\left(\frac{\partial A}{\partial z} + \frac{1}{c}\frac{\partial A}{\partial t}\right)_{z=d} = 0, \tag{4.7}$$

where A_0 is the amplitude of the incident wave. Solving the boundary-value problem (4.4) and (4.7), we can determine the intensities and the harmonic compositions of the reflected and transmitted waves with the aid of the relations

$$\frac{\partial}{\partial t}\sum_{m=1}^{\infty} A_m^{(1)}\cos(m\omega t + \alpha_m) = \omega A_0 \sin\omega t + \left(\frac{\partial A}{\partial t}\right)_{z=+0}, \tag{4.8}$$

$$\frac{\partial}{\partial t}\sum_{m=1}^{\infty} A_m^{(2)}\cos(m\omega t - \beta_m) = \left(\frac{\partial A}{\partial t}\right)_{z=d-0}. \tag{4.9}$$

Proceeding to the solution of Eqs. (4.4) and (4.7), we assume that the film is thin compared with the penetration depth δ_L. In this case we can expand the function A(z, t) with $0 < z < d$ in powers of z/δ_L:

$$A(z, t) = A_0(t) + A_1(t)\frac{z}{\delta_L} + \frac{1}{2}A_2(t)\frac{z^2}{\delta_L^2} + \ldots \tag{4.10}$$

Substituting this expansion in (4.4) and (4.7) and discarding terms that are small in the parameter d/λ, we obtain

$$\dot{A}_0(t) - \frac{c}{\delta_L}A_1(t) = -2\omega A_0 \sin\omega t, \tag{4.11}$$

$$\dot{A}_0(t) + \frac{c}{\delta_L}A_1(t) + \frac{cd}{\delta_L^2}A_2(t) = 0, \tag{4.12}$$

$$A_2(t) = \frac{4\pi\sigma}{c^2}\delta_L^2\frac{\partial A_0}{\partial t} + A_0(t)\langle 1 - sA_0^2(t)\rangle. \tag{4.13}$$

The first two relations were obtained from the boundary conditions (4.7), and the last is the first term of the expansion of (4.4). From (4.11)–(4.13) we obtain an equation in closed form for the function $A_0(t)$. Introducing for convenience the dimensionless quantity $\varphi(\tau) = A_0(t)\sqrt{s}$, where $\tau = \omega t$ is the dimensionless time, we can represent this equation in the form

$$Z\frac{d\varphi}{d\tau} + \varphi\langle 1 - \varphi^2\rangle = -\varphi_0 \sin\tau, \tag{4.14}$$

where the angle brackets have the same meaning as in (4.2) and (4.3), while φ_0 and Z are defined as

$$\varphi_0 = A_0\sqrt{s}T_s^{1/2}, \quad Z = (T_s/T_n)^{1/2}. \tag{4.15}$$

T_n and T_s are the transparency coefficients of the film in the linear approximation for the normal and superconducting phases (the latter only in the frequency region $\omega \ll \omega_0$):

$$T_n = \left(\frac{\omega\delta_{sk}^2}{cd}\right)^2, \quad T_s = \left(\frac{2\omega\delta_L^2}{cd}\right)^2 \tag{4.16}$$

(In the derivation of (4.14) it was assumed that the film is not very transparent to the radiation, i.e., T_n and T_s are small compared with unity).

We note that the introduced quantity Z can be represented also in the form $Z = \gamma\omega/\epsilon_0$, where γ is determined by formula (3.12) from Sec. 3. Since $\gamma \sim 1$, the value of the parameter Z is determined by the frequency region $Z \ll 1$ in the adiabatic limit and $Z \gg 1$ in the region of high frequencies.

Solving Eq. (4.14), we can find the amplitude and the harmonic composition of the wave passing through the film with the aid of a relation that follows from (4.9):

$$\sum_{m=1}^{\infty} A_m^{(2)} \cos(m\omega t - \beta_m) = A_0(t) = \frac{1}{\sqrt{s}} \varphi(\omega t), \quad (4.17)$$

from which we see that the amplitudes of the harmonics of the radiation passing through the film represent, apart from a certain factor, the coefficients of the Fourier expansion of the function $\varphi(\tau)$. The problem thus reduces to a solution of the nonlinear equation (4.14).

Let us consider first the case when the field amplitude is sufficiently ($\varphi_0 \to \infty$), so that the metal is in the normal state. Then the function $\varphi\langle 1 - \varphi^2\rangle$ in (4.14) vanishes, and we get

$$\varphi(\tau) = \frac{\varphi_0}{Z} \cos\tau.$$

With the aid of (4.17) we obtain in this case for the amplitudes of the harmonics of the transmitted wave the expressions

$$A_1^{(2)} = \frac{\varphi_0}{Z\sqrt{s}} = A_0 \frac{T_s^{1/2}}{Z} = A_0 T_n^{1/2}; \quad A_m^{(2)} = 0, \quad m > 1. \quad (4.18)$$

It is clear therefore that the transparency coefficient in the normal state is indeed equal to T_n. With decreasing microwave power, the transition to the superconducting phase will occur, as shown in Sec. 3, at that value of the amplitude A_0, at which the quantity $\overline{\varphi^2}$ becomes equal to unity. This corresponds to the critical power

$$W_1^{cr} = \frac{\hbar^2\omega^2}{4\pi c s T_n} S, \quad (4.19)$$

where S is the surface area of the film (in the experiment—the area of the waveguide section covered by the film).

Now, to the contrary, let us consider the case of extremely small amplitudes, when the film is certainly in the superconducting state, and the nonlinear effects can be neglected ($\varphi_0 \ll 1$). In this case the solution of (4.14) takes the form

$$\varphi = -\frac{\varphi_0}{\sqrt{1+Z^2}} \sin(\tau - \theta), \quad \theta = \operatorname{arctg} Z. \quad (4.20)$$

We obtain further, in analogy with the foregoing (m = 1),

$$A_1^{(2)} = \frac{\varphi_0}{\sqrt{s}} (1 + Z^2)^{-1/2} = A_0 T^{1/2}, \text{where } T^{-1} = T_s^{-1} + T_n^{-1}. \quad (4.21)$$

Consequently, the role of the transmission coefficient of the film is played in this case by the quantity $T = (T_n^{-1} + T_s^{-1})^{-1}$, which coincides with formula (2.16) of Sec. 2.

Proceeding to the solution of (4.14) at intermediate values of φ_0 ($\varphi_0 \sim 1$), let us examine separately the cases of low and high frequencies. In the adiabatic limit ($\omega \ll \epsilon_0$) we neglect the term $Z\varphi$ in (4.14), as a result of which we obtain a cubic equation for φ:

$$\varphi(1 - \varphi^2) = -\varphi_0 \sin\tau. \quad (4.22)$$

Solution of this equation by Cardan's formulas yields

$$\varphi(\tau) = -\frac{2}{\sqrt{3}} \operatorname{sign}(\sin\tau) \cos\left[\frac{\pi}{3} + \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\varphi_0 |\sin\tau|\right)\right]. \quad (4.23)$$

This solution is valid when $\varphi_0 < \max|(1 - \varphi^2)| = 2/3\sqrt{3}$. We see therefore that the function $\varphi(\tau)$ contains only odd harmonics: m = 1, 3, 5, ...

Figure 5 shows the amplitudes of the harmonics as functions of the relative incident power W/W_2^{cr}, where W_2^{cr} is defined as the power at which the critical value

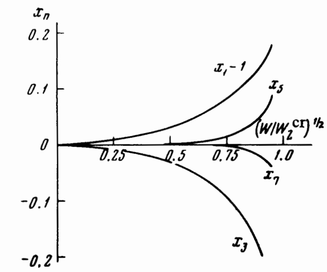

FIG. 5

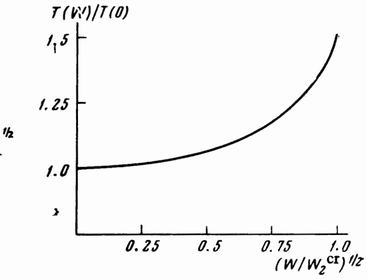

FIG. 6

FIG. 5. Amplitudes of harmonics of radiation passing through the film (when $\omega \ll \epsilon_0$) as functions of the relative power $(W/W_2^{cr})^{1/2}$. The curves are normalized in such a way that $x_1 = 1$ at W = 0 ($x_n = -n\varphi_n/\varphi_0$, where φ_n are the Fourier components of the function $\varphi(\tau)$). The true amplitudes of the harmonics are obtained by multiplying the curves presented here by T, where T is the transmission coefficient of the superconducting film in the linear approximation.

FIG. 6. Dependence of the transmission coefficient of the film at the fundamental frequency on the power of the incident microwave in the adiabatic case.

of φ_0, equal to $2/3\sqrt{3}$, is reached:

$$W_2^{cr} = \frac{\hbar^2\omega^2}{2\pi \cdot 3^3 c s T_s} S. \quad (4.24)$$

Figure 6 shows the dependence of the transparency coefficient of the film on the power when $W < W_2^{cr}$.

At values $\varphi_0 > 2/3\sqrt{3}$, an analysis of (4.14) becomes much more complicated, for it is no longer possible to neglect the term $Z\varphi$, in spite of the fact that $Z \ll 1$. Moreover, the question arises of the stability of the corresponding solution (see Sec. 3). For this reason, we shall not consider here the region of fields for which φ_0 exceeds $2/3\sqrt{3}$.

Since the quantity T_s is much smaller than T_n in the adiabatic limit, the critical power W_2^{cr} greatly exceeds the critical power W_1^{cr}. Thus, an appreciable hysteresis arises: the destruction of the superconducting state when the field amplitude is increased, occurs at a much higher power than the transition to the superconducting phase in a decreasing field.

We can investigate analogously Eq. (4.14) in the limiting case of high frequencies, $\omega \gg \epsilon_0$. However, since $Z \gg 1$ in this case, all the nonlinear effects constitute only small corrections in $1/Z$. Nor is there any hysteresis in the field dependence of the impedance, for in the limit when $\omega \gg \epsilon_0$ the value of T coincides with T_n (see (4.21)). This is connected with the fact that at high frequencies the transmission of the film changes little following a transition to the superconducting state.

In conclusion, we present estimates for the critical powers W_1^{cr} and W_2^{cr}. Putting $S \sim 1\ \text{cm}^2$ and $\omega \sim 10^{10}\ \text{sec}^{-1}$, we obtain in the adiabatic case

$$W_1^{cr} \sim 10^{-5}\ \text{W} \cdot \frac{1-t}{T}, \quad W_2^{cr} \sim 10^{-6} \cdot \frac{1-t}{T} \quad (4.25)$$

(it should be remembered here T_n and T_s are usually quite small quantities). As follows from (4.25), near T_c the quantity W_1^{cr} vanishes in linear fashion, $W_1^{cr} \sim (T_c - T)$. As to W_2^{cr}, to determine its temperature dependence we must take into account the fact that T_s also depends on the temperature. As an estimate, it can be roughly assumed that $T_s \sim (d_0/d)^4$, where d_0 is the characteristic film thickness introduced in Sec. 2. Since, according to formula (2.17), $d_0 \sim (1-t)^{-1/2}$, we find that the second critical power is proportional to $(T_c - T)^3$. Of course, this remains valid only so long as $W_2^{cr} \gg W_1^{cr}$, for in the opposite case (sufficiently close to T_c) the adiabatic approximation ceases to work.

In conclusion, I am deeply grateful to L. P. Gor'kov and G. M. Éliashberg for a discussion of this work and valuable remarks. I am also grateful to V. M. Dmitriev, I. M. Dmitrenko, and G. E. Churilov for a discussion of the experimental aspects of the questions touched upon here.

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