NONLINEAR THEORY OF CERENKOV EXCITATION OF REGULAR OSCILLATIONS BY A MODULATED BEAM OF CHARGED PARTICLES

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The nonlinear nonstationary problem of excitation of quasimonochromatic resonator oscillations by a density-modulated charged particle beam is considered. Equations are derived for the dependence of the field amplitude and phase on time in the case when the flight time is small compared with the characteristic time of growth of the field. The maximal values of the amplitude are calculated in the cases of strong and weak beam modulation.

1. AS is well known, the linear theory of plasma-beam interaction has by now been thoroughly worked out, so that the main problem is the investigation of the nonlinear characteristics of this interaction.[1] The greatest progress in this direction was attained with the aid of the quasilinear theory.[2] However, the applicability of the latter is limited, in particular, by the requirement that the characteristic distance $\Delta\omega \approx V_0 \Delta k$ (V_0 is the beam-particle velocity) between the natural frequencies of the plasma be small compared with the growth increment ϵ of the installations ($\epsilon \gg \Delta\omega$). The fields excited in this case have a relatively broad frequency spectrum consisting of a large number of oscillations with independent random phases. Such fields are of interest for a stochastic acceleration and heating of the plasma. At the same time, in many applications[3] it is necessary to obtain regular electromagnetic fields, i.e., fields with fixed phases.

The nonlinear theory of the interaction between a beam of charged particles and regular waves has been the subject of [4–8]. The closest to the experimental conditions is the problem in which account is taken of the injection of the beam into a plasma layer from the outside. It is precisely in such a formulation that the Cerenkov amplification of longitudinal waves was investigated in [6]. It was assumed there that the stabilization is ensured by the dependence of the slowing-down properties of the plasma on the field amplitude, and that the reaction of the field on the motion of the beam particles is insignificant.

In this paper we consider the nonstationary problem of nonlinear Cerenkov interaction of an injected external charged-particle beam with a slow-wave resonator, under conditions when the field in the resonator is regular, and the growth of its amplitude is stabilized by the reaction of this field on the motion of the beam particles.

2. The spectrum of the oscillations excited by the beam in the resonator can be regarded as discrete if the growth increment ϵ of the oscillations is small compared with the distance $\Delta\omega \approx V_0\Delta k$ between the natural frequencies of the resonator. Since $\Delta k \sim \pi/L$, where L is the length of the resonator, it follows that the condition $\epsilon \ll \Delta\omega$ is equivalent to the requirement that the time of flight $T \equiv L/V_0$ be small compared with the characteristic growth time $\tau_g \sim \epsilon^{-1}$ of the field of the resonator. In such a case, an appreciable increase of the field amplitude in the resonator can be ensured only by accumulating in the resonator the energy lost by the consecutively entering beam particles. To this end it is necessary that the Q of the resonator be sufficiently high ($Q \gg 1$).

The quasilinear theory of the effect of accumulation of the energy of the longitudinal oscillations excited in a plasma half-space by a beam continuously injected from the outside has been developed in [9,10]. We consider below this effect for a thin ($\epsilon T \ll 1$) resonant layer in which regular oscillations are excited.

3. The self-consistent system of equations describing the interaction of the beam with the resonator, under the conditions considered here, consists of the equation for the field and the equations of motion in terms of the Lagrange variables $Z(\tau, t_0)$

$$\frac{\partial^2 E_\|}{\partial t^2} + 2Q^{-1}\Omega\frac{\partial E_\|}{\partial t} + \Omega^2 E_\| = -4\pi\frac{\partial}{\partial t}j_b, \tag{1}$$

$$j_b \equiv \frac{e}{2\pi r}\delta(r)\int dt_0 I(t_0)\,\delta[z - Z(\tau, t_0)]\,\dot{Z}(\tau, t_0), \tag{2}$$

$$\frac{\partial}{\partial\tau}p(\tau, t_0) = eE_\|[Z(\tau, t_0);\ t(\tau, t_0)],\quad p(\tau, t_0) \equiv \frac{m_0\dot{Z}(\tau, t_0)}{[1 - \dot{Z}^2/c^2]^{1/2}}. \tag{3}$$

The dot denotes here the derivatives with respect to τ at a fixed instant of entry t_0; the beam is assumed focused in a radial direction; Ω is the natural frequency of the resonator, for which $V_{ph} = V_0$, where V_0 is the velocity of the beam particles at the entrance to the resonator. Since effective excitation of one resonant oscillation can take place only in the case of a small thermal scatter, we shall neglect this scatter throughout, assuming the beam to be modulated only in density. In this case the particle current $I(t_0)$ at the entrance to the resonator is periodic in time. From the equality $V_{ph} = V_0$ it follows that the current-modulation frequency ω_M is equal to the resonant frequency Ω of the oscillation under consideration.

It should be noted that the system of equations in terms of Lagrange variables (3) is equivalent to the system of characteristics of the kinetic equation for the beam particle distribution function

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + eE\frac{\partial f}{\partial p} = 0,$$

so that the solution of (3) makes it possible to find, in the kinetic approximation, the amplitude of the field excited by the beam.

Assuming the right side of (2) to be known, we can integrate (1) and find the time dependence of the amplitude E(t) of the field of the sought oscillation

$$E_{\|}(r, z, t) = E(t) \cos k_{\|} z \Phi(r), \quad \Phi(0) = 1. \tag{4}$$

Recognizing that the field amplitude in the resonator at the initial instant t = 0, when there are no particles in the resonator, is zero, we get

$$F(\tau) = -2\mu \int_0^{\tau} d\tau' \exp[-(\tau - \tau') Q^{-1}] \cos(\tau - \tau') \int_{n(\tau')} d\tau'' \psi(\tau'') \cos k_{\|} Z(\tau', \tau''), \tag{5}$$

where we have introduced the dimensionless variables

$$\tau \equiv \Omega t, \quad F \equiv \frac{eEk_{\|}}{m_0 \Omega^2}, \quad \mu \equiv \frac{2e^2 I_0}{m_0 \Omega^3 SL},$$

$$S \equiv \int \Phi^2(r) r dr, \quad I(t) = I_0 \psi(t), \quad I_0 \equiv \bar{I}(t),$$

and the dimensionless function $\psi(t)$ describes the form of the current at the entrance to the resonator. Integration with respect to τ'' in (5) at each fixed instant τ' is carried out only over trajectories that do not go outside the resonator $(0 < Z(\tau', \tau'') < L)$.

Reversing the order of integration with respect to τ' and τ'' in (5), we obtain

$$F(\tau) = -2\mu \int_0^{\tau} d\tau' \psi(\tau') \exp[-(\tau - \tau') Q^{-1}] \int_0^{\Theta(\tau')} d\tau'' \exp[\tau'' Q^{-1}] \times \cos[\tau - \tau' + \Delta(\tau'', \tau')], \tag{6}$$

where $\Theta(\tau')$ is the time needed for the particle entering the resonator at the instant τ' to traverse the resonator, and

$$\Delta(\tau', \tau'') \equiv k_{\|}[Z(\tau'', \tau') - V_0 \Omega^{-1} \tau'']$$

is the interaction-induced deviation of the trajectory of this particle from the straight line (in terms of the variables Z and τ'') corresponding to the motion without the field.

Relation (6) together with the equations of motion (3) is a system of integro-differential equations of the field in the resonator. It follows from (6) that at each instant of time the field in the resonator is determined by the result of the interaction of all the previously passing particles. In particular, the increments of the amplitude R and of the phase φ of the field, which are connected with the function F by the relation $F(\tau) = -R(\tau) \times \cos[\tau - \varphi(\tau)]$, taken over one period of the modulation, are determined by the following expressions

$$\Delta R \approx \frac{dR}{dn} = \mu_* \int_{-\pi}^{+\pi} \psi(\tau') \int_0^{d(n, \tau')} \frac{\cos[x + \varphi(n) - \tau'] dx}{\dot{\Delta}[R(n), \varphi(n), \tau', x]} d\tau'; \tag{7}$$

$$\Delta \varphi \approx \frac{d\varphi}{dn} = \frac{\mu_*}{R(n)} \int_{-\pi}^{+\pi} d\tau' \psi(\tau') \int_0^{d(n, \tau')} \frac{\sin[x + \varphi(n) - \tau'] dx}{\dot{\Delta}[R(n), \varphi(n), \tau', x]}; \tag{8}$$

$$R(0) = \varphi(0) = 0, \quad Q^{-1} = 0,$$

where $\mu_* \equiv \mu / 2\pi$, and $d(n, \tau')$ is the displacement, at the exit from the resonator, of the particle that entered the resonator at the instant of time $\tau_0 \equiv 2\pi n + \tau'$. When we replace the effective values of the amplitude and phase in the right sides of (7) and (8) by the average values, we use the assumption that R and φ vary slowly in time, which is equivalent to the requirement that the parameter $q \equiv \epsilon T$ be small $(q \ll 1)$. As will be shown below, the condition $q \ll 1$ makes it possible to integrate the equation of motion and to find equations for the field in the resonator. We shall examine this problem in detail in two limiting cases, of strong and weak modulation of the current exciting the resonator.

4. Let us consider the case of a strongly modulated beam (a sequence of charged bunches $\psi(\tau) = \delta(\tau)$).

In this case each bunch excites the resonator with its Cerenkov radiation. Since the repetition frequency of the bunches is equal to the natural frequency of the resonator, the fields of all the bunches turn out to be coherent at the initial stage of the interaction, as the result of which the amplitude of the field in the resonator increases linearly in time. With increasing field amplitude, its reaction on the motion of the bunches increases. To take this effect into account, it is necessary to find the first integrals of the equations of motion of the bunches:

$$\frac{\ddot{\Delta}}{[1 - \beta_0^2 (1 + \dot{\Delta})^2]^{3/2}} = -\frac{1}{2} R(n) \cos[\varphi(n) + \Delta], \tag{9}$$

which do not depend on the entry phase and can be easily integrated in the limiting cases of nonrelativistic beams $(\beta_0^2 \ll 1)$ and relativistic $(\gamma \equiv (1 - \beta_0^2)^{-1/2} \gg 1)$ beams.

In the nonrelativistic case, multiplying both sides of (9) by $\dot{\Delta}$ and integrating with the initial condition $\dot{\Delta}(\Delta = 0) = 0$, we get

$$\dot{\Delta} = -\sqrt{R}(n) \{\sin \varphi(n) - \sin[\varphi(n) + \Delta]\}^{1/2}. \tag{10}$$

Substituting this expression in (7) and (8) we obtain after straightforward but complicated algebraic transformations a final system of equations for the amplitude and phase of the field in the following form:[11]

$$\frac{dP}{dn} = 3\sqrt{2} \mu_* \sin \xi \cos \alpha(P, \xi),$$

$$P \frac{d\xi}{dn} = \frac{\mu_*}{\sqrt{2}} [2I_{1/2}(k, \alpha) - I_{-1/2}(k, \alpha)], \tag{11}$$

where

$$P \equiv R^{3/2}, \quad \xi \equiv \frac{\pi}{4} + \frac{\varphi}{2} \quad k \equiv \sin^2 \xi,$$

$$I_s(k, \alpha) \equiv \int_{\alpha}^{\pi/2} dx [1 - k \sin^2 x]^s,$$

and the dependence of α on P and ξ is determined by the following transcendental equation that follows from the condition $Z[\Theta(\tau'), \tau'] = L$:

$$I_{-1/2}(k, \alpha) = \sqrt{2} \pi N P^{1/3} \left[1 + \frac{\xi - \arcsin(\sin \xi \sin \alpha)}{\pi N}\right]. \tag{12}$$

The maximum of the amplitude and the corresponding value of the phase can be found by equating the right sides of (11) to zero. We thus obtain

$$\alpha_{max} = -\pi/2, \quad 2E(k_m) - K(k_m) = 0,$$
$$k_m^{1/2} = \sin \xi_m = 0.826, \quad \xi_m \approx 1.14, \quad K(k_m) \approx 2.32, \tag{13}$$

where E and K are respectively complete elliptic integrals of the first and second kind.

Substituting these relations in (12), we obtain a final expression for the amplitude of the field at the maximum:

$$R_{max} = \frac{2K^2(k_m)}{\pi^2 N^2 [1 + 2\xi_m/\pi N]^2} \tag{14}$$

or in the dimensional variables

$$E_{max} = \frac{m_0 V_0^2}{2|e|L} \frac{8}{\pi N} \frac{K^2(k_m)}{\{1 + 2\xi_m/\pi N\}^2}. \tag{14a}$$

The physical meaning of the result is as follows: at large field amplitudes, the phase shift of the particles under the influence of the decelerating field is so large, that the particle enters the accelerating phase of this field, corresponding to the region of negative α. The rate of growth of the field amplitude is equal to zero when the field amplitude is so large that the particle has time to execute half the phase oscillation within the time of flight ($\alpha_{min} = -\pi/2$). Putting $\Omega_{ph}T = \pi$, where Ω_{ph} is the frequency of the phase oscillations and T is the time of flight, we get

$$E_{max} = \frac{m_0 V_0^2}{2|e|L} \frac{\pi}{N}, \tag{14b}$$

which coincides, apart from a numerical factor, with Eq. (14a) obtained from an analytic solution of the problem.

It must be emphasized that the field E_{max} is excited as the result of Cerenkov loss of a large number of bunches, each of which loses only a fraction of its energy. The maximum energy loss corresponds to a field approximately one quarter as large as E_{max}. In such a field, the particle executes one quarter of the phase oscillations during the time of flight ($\Omega_{ph}T = \pi/2$).

The effective pulse duration necessary to obtain maximum field (14) can be estimated with the aid of (11): $n_m \approx R_m^{3/2}/3\sqrt{2}\,\mu_*$. It is seen from this relation that during the linear stage of the interaction the field amplitude increases in proportion to $n^{2/3}$. The reduction in the field growth rate, compared with the initial stage in which the field amplitude increased in proportion to the number of bunches passing through the resonator, is due to the loss of coherence of the interaction between the bunches as the result of the deceleration under the influence of the field.

Knowing the total energy of the resonator at the maximum and the corresponding pulse duration, let us estimate the efficiency of the system as a ratio of the field energy in the resonator to the energy of the particles passing through it:

$$\eta \equiv W_f / W_r \simeq 1/N,$$

where $W_f \equiv E_n^2 \pi SL/8\pi$ is the field energy and $W_r \equiv \pi I_0 \Omega^{-1} n_m m_0 V_0^2$ is the energy of the particles passing through the resonator. We have omitted here a numerical factor of the order of unity, since we have determined only the order of magnitude of the number of particles n_m.

As seen from this estimate, the efficiency of the generator depends only on the relative length of the resonator.

5. Let us consider a relativistic modulated beam ($\gamma \gg 1$; $\psi(\tau) = \delta(\tau)$).

In this case the first integral of the equation of motion (8) is given by

$$\dot{\Delta}(R, \varphi, \Delta) = -{}^1\!/_2 \gamma^{-2} \{[\varkappa(2+\varkappa)]^{1/2} [\varkappa^{1/2} + (2+\varkappa)^{1/2}]^2\},$$
$$\varkappa(R, \varphi, \Delta) \equiv {}^1\!/_2 \gamma R [\sin\varphi - \sin(\varphi + \Delta)]. \tag{15}$$

Substituting this expression in (7), we obtain the following equations for the amplitude and phase of the field

$$R^{1/2} \frac{dR}{dn} = 8\gamma\mu_* \left\{1 - \exp\left[-2 \operatorname{Arsh} \frac{\varkappa}{2}\right]\right\}, \tag{16}$$

$$R^{3/2} \frac{d\xi}{dn} = -2\gamma^{3/2}\mu_* \int_\alpha^{\pi/2} \frac{1 - 2\sin^2\xi \sin^2 x}{(1 - \sin^2\xi \sin^2 x)^{1/2}} \frac{[2+\varkappa]^{-1/2} dx}{[\varkappa^{1/2} + (2+\varkappa)^{1/2}]^2},$$
$$\varkappa(R, \xi, x) \equiv \gamma R \sin^2\xi \cos^2 x, \tag{17}$$

where $\xi \equiv \pi/4 + \varphi/2$, and the dependence of α on ξ and R is determined by the equation

$$\pi N + \xi - \arcsin[\sin\xi \sin\alpha] = \frac{2\gamma^{3/2}}{R^{1/2}} \cdot$$
$$\cdot \int_\alpha^{\pi/2} \frac{dx (1 - \sin^2\xi \sin^2 x)^{-1/2}}{(2+\varkappa)^{1/2} [\varkappa^{1/2} + (2+\varkappa)^{1/2}]^2}. \tag{18}$$

It is easy to show, in analogy with the preceding case of the nonrelativistic beam, that when $\gamma^2 \ll N$ the maximum amplitude differs from (14) only in the value of the longitudinal electron mass:

$$E_{max} = \frac{m_\parallel c^2}{|e|L} \frac{4}{\pi N} \frac{K^2(k_m)}{[1 + 2\xi_m/\pi N]^2}, \quad m_\parallel \equiv m_0\gamma^3. \tag{19}$$

In the ultrarelativistic case, $\gamma^2 \gg N$, it is easier to determine the amplitude of the phase of the field at the maximum, because the largest contributions to the integrals (17) and (18) are made under these conditions by the values of x close to $\pm\pi/2$. Integrating, we obtain

$$\alpha_{max} = -\pi/2, \quad \xi_{max} = \pi/4 \quad (\varphi_{max} = 0),$$
$$E_{max} = \frac{2}{\pi} \frac{\gamma}{N + {}^1\!/_2}.$$

or, in dimensional units

$$E_{max} = \frac{4 m_0 c^2 \gamma}{|e| L (1 + {}^1\!/_2 N)}. \tag{20}$$

Thus, accurate to a numerical coefficient of the order of unity, the maximum emf induced by the beam in the resonator is equal in this case ($\gamma^2 \gg N$) to the beam energy. The energy lost by the bunch in the decelerating phase of the field during the first quarter of the phase-oscillations period (and acquired then in the accelerating phase) is $2\pi^{-1} m_0 d^2 \gamma$.[1)] Therefore the particle continues to be relativistic even at the minimum of the energy. An estimate of the pulse duration necessary to obtain the maximum field amplitudes can be obtained directly from (17) and (20):

$$\frac{dR^{3/2}}{dn} \approx 12\gamma\mu_*, \quad n_{max} \approx \frac{R_{max}^{3/2}}{12\gamma\mu_*} = \frac{\gamma^{1/2}}{12 N^{3/2} \mu_*}. \tag{21}$$

Knowing the pulse duration necessary to obtain the maximum field amplitude, let us calculate the genera-

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[1)] The amplitude (8) of the traveling wave with which the particle interacts is half as large as the maximum amplitude (20); the second factor 2 in the denominator is due to the fact that the particle traverses half the resonator during slowing-down phase; allowance for the phase drift (sinusoidal form of the field) results in a factor $2/\pi$.

tor efficiency (the ratio of the energy accumulated in the field to the total energy of all the particles passing through the resonator):

$$\eta \equiv \frac{W_f}{W_I} = \frac{E_m^2 SL}{16\pi I_0 \Omega^{-1} m_0 c^2 \gamma n_m}.$$

Substituting here the field from (20), n_m from (21), and μ_* from (5), we obtain

$$\eta \sim (\gamma/N)^{1/2}, \quad \gamma < N, \tag{22}$$

where we have left out a numerical factor of the order of unity, since only the order of magnitude of the duration of the pulse n_m has been determined. Thus, the efficiency of the system increases in proportion to this clear route of the maximum field amplitude.

6. We now proceed to consider the problem of interaction between an unmodulated monochromatic beam and a resonator ($\psi = 1$). We assume here that the dependence of the resonant frequencies on the wave number is nonlinear, so that the synchronism condition $V_{ph} = V_0$ is satisfied for only one spatial harmonic that is excited by the beam.

In the linear approximation, putting $Q^{-1} = 0$ and $\psi = 1$ in (5) and expanding this expression in powers of Δ, we obtain the following integro-differential equation for $\Delta(\tau, \tau_0)$:

$$\frac{\partial^3 \Delta(\tau, \tau_0)}{\partial \tau^3} = \frac{1}{2}\mu \int_0^{2\pi N} d\tau' \Delta(\tau', \tau_0 + \tau - \tau') \sin(\tau - \tau'). \tag{22'}$$

We seek a solution of this equation in the following form:

$$\Delta(\tau, \tau_0) = a(\tau) \exp(i\alpha\tau_0).$$

Substituting this expression in (22′), we obtain an equation for α

$$(\alpha - 1)^3 = \left(\frac{\mu}{2}\pi N\right)'', \quad \alpha = 1 + \frac{1 - i\sqrt{3}}{2}\left(\frac{1}{2}\mu\pi N\right)^{1/3}. \tag{23}$$

Thus, the growth increment ($\epsilon \equiv -\Omega \operatorname{Im} \alpha$) of the instability is proportional to the cubic root of the beam currents. This is a characteristic of the Cerenkov interaction mechanism between a beam and a decelerating medium (the additional factor of 2 in the denominator of the current is due to excitation of a standing wave in the resonator, in contrast to the traveling waves usually considered in an unbounded plasma).

Let us consider now the nonlinear stage of the interaction of an unmodulated nonrelativistic beam ($\beta_0^2 \ll 1$) with the resonator field.

In this case the first integral of the equation of motion (8) depends on the phase of the entrance of the particle τ_0:

$$\dot{\Delta}(R, \varphi, \Delta, \tau_0) = \{R[\sin(\varphi - \tau_0) - \sin(\varphi - \tau_0 + \Delta)]\}^{1/2}\zeta, \tag{24}$$

where $\zeta = +1$ when $\pi/2 < \varphi - \tau_0 < 3\pi/2$ and $\zeta = -1$ when $-\pi/2 < \varphi - \tau_0 < \pi/2$.

The plus sign in (24) corresponds to particles that fall into the accelerating phase of the field at the instant of entry into the resonator, and the minus sign to those that fall into the decelerating phase of the field. Substituting (24) in (7) and introducing a new variable $\vartheta \equiv \varphi - \tau_0$, we obtain the following equations for the amplitude R of the field in the resonator:

$$\frac{dR^{3/2}}{dn} = 6\sqrt{2}\,\mu_* \int_0^{\pi/2} d\vartheta \sin\vartheta [\sin\alpha_-(\vartheta) - \operatorname{sh}\alpha_+(\vartheta)], \tag{25}$$

where the dependence of the phases $\alpha_\pm(\vartheta)$ of the particles accelerated and decelerated by the resonator field on the entry phase ϑ is determined by the following transcendental equations:

$$\begin{aligned} \sqrt{2R}\{\pi N + \vartheta - \arcsin[\sin\vartheta \operatorname{ch}\alpha_+(\vartheta)]\} &= \int_0^{\alpha_+(\vartheta)} \frac{dx}{\sqrt{1 - \sin^2\vartheta \operatorname{ch}^2 x}}, \\ \sqrt{2R}\{\pi N + \vartheta - \arcsin[\sin\vartheta \cos\alpha_-(\vartheta)]\} &= \int_0^{\alpha_-(\vartheta)} \frac{dx}{\sqrt{1 - \sin^2\vartheta \cos^2 x}}. \end{aligned} \tag{26}$$

Expanding (25) and (26) in terms of the small field amplitudes, we obtain

$$\alpha_\pm(\vartheta) = \sqrt{2R}\,\pi N \cos\vartheta \mp (2R)^{3/2} \frac{\pi^2 N^2}{4}\left[\cos\vartheta + \frac{\pi N}{3}\sin\vartheta\right]\sin 2\vartheta. \tag{27}$$

Here, as in (24)–(26), we assume that the beam particles leave the resonator in the same phase (accelerating or decelerating) in which they entered the resonator, so that the sign of the square root in (24)–(26) remains unchanged. Under these conditions, the growth of the field amplitude has an exponential character:

$$dR/dn = \varepsilon_0 R, \quad \varepsilon_0 \equiv {}^1\!/_2 \mu\pi^2 N^2. \tag{28}$$

The upper limit of applicability of (27) and (28) is determined from the condition $\alpha(\vartheta) < 1$. Putting $\alpha = 1$, we obtain an estimate for the maximum field amplitudes in the resonator when the latter is excited by an unmodulated beam:

$$R_{max} = C/2\pi^2 N^2, \quad C \sim 1. \tag{29}$$

Comparison with the corresponding expression (13) for a strongly modulated beam shows that in the absence of modulation the field amplitude is smaller by approximately one order of magnitude, whereas the dependence of the beam energy on the resonator length is the same in both cases.

Physically this decrease of the field amplitude can be explained in the following manner. In the case of strong modulation each bunch falls in the decelerating phase of the field and the growth of the amplitude stops only when this amplitude is large enough to permit the bunch to execute half the phase oscillation during the time of flight. In the case of an unmodulated beam, the growth of the amplitude is determined by the difference effect of the interaction between the resonator and of the particles accelerated and decelerated in the field. In this case the phase-oscillation amplitude at which the increase of the field amplitude stops turns out to be smaller. The maximum field amplitude is accordingly decreased.

The lower limit of the region of applicability of (28) is determined from the condition $\alpha \sim \alpha/N \gg \epsilon_0$. Thus, the limits of applicability of (28) are determined by the inequalities

$$(\mu N^2)^2 \ll R \ll N^{-2}. \tag{30}$$

Since it was assumed above that the inequality $\epsilon T \approx (\mu N^4)^{1/3}$ is satisfied, it is easy to see that the region of field-amplitude values in which the inequality (30) is valid actually exists.

7. We have thus shown that the phase shift introduced by the beam particles into the field generated by them in the resonator limits the growth of the amplitude of this field, both in the case of a modulated beam exciting the resonator by means of the Cerenkov radiation of the bunches, and in the case of an unmodulated beam that experiences automodulation under the influence of the initial perturbation of the field in the resonator and intensifies this field. It should be noted that the method used in this case for integrating the equations of motion of the beam particles in terms of the Lagrange variables remains applicable also under conditions when the trajectories of particles with different initial phases intersect. Assuming the time of flight to be small compared with the reciprocal increment ($q \equiv \epsilon T \ll 1$), we have actually obtained a solution of these equations in the zeroth approximation in the parameter $q \sim e^{2/3}$. In the same approximation, it becomes possible to take into account the thermal scatter of the beam particles and the loss of synchronization between the beam and the slow wave of the resonator ($|V_{ph} - V_0| \neq 0$).

Inasmuch as in fields close to the maximum value the change of the beam particle velocity under the influence of the field becomes comparable with the difference of the phase velocities of the neighboring spatial harmonics of the field in the resonator (in the nonrelativistic case $\Delta V_{ph} \sim V_0/2N$), it follows that the real values of the maximum field amplitudes are apparently somewhat higher than the calculated ones, and the position of the maximum itself is not stable, although generation of neighboring harmonics does occur more slowly than that of the fundamental harmonic (owing to the smallness of the time of interaction between the beam particles and the fields of these harmonics).

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