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STIMULATED EMISSION OF RADIATION BY ELECTRONS MOVING IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

V. G. BAGROV, Yu. I. KLIMENKO, and V. R. KHALILOV

Moscow State University

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The dependence of the stimulated emission of radiation by electrons moving in the field of a plane electromagnetic wave on the polarization, frequency, and spectral width of a second wave perturbing the system is investigated. To within terms linear in Planck's constant $\hbar$, the intensities of the stimulated radiation from a spinor particle (electron) and from a spinless particle (boson) turn out to be identical.

LET us consider an electron moving in a plane electromagnetic wave, which is traveling along the direction of a unit vector **n** (we shall call this wave the first wave). Let a second wave of smaller intensity be incident on this electron at a certain angle θ to the direction **n**. Under the influence of the second perturbing wave the electron will undergo forced transitions, i.e., in principle the phenomenon of stimulated emission and absorption of radiation is possible. The theoretical feasibility of such an effect was established in[1], where the radiation of a scalar particle was considered. In the present article a corresponding calculation is carried out for an electron.

In order to solve the problem about the intensity of the stimulated emission and absorption, as is well known (see, for example,[2,3]) it is necessary to evaluate the matrix elements of the Dirac matrices. In principle such calculations do not differ at all from the corresponding calculations associated with an investigation of the spontaneous emission of radiation by electrons moving in a plane electromagnetic wave, which are well known in the literature,[4,5] and there is no need to cite them here. It is only necessary to indicate the fact that, as a consequence of the nonstationary nature of the external field the electron's energy is not conserved in time, and therefore, as is shown in[1], in order to investigate the intensity of the stimulated radiation it is necessary to define the energy of the radiated photon as the difference between the average energies of the electron's initial and final states. We carry out specific calculations, assuming the first wave to be monochromatic with frequency $\omega_0 = c\kappa_0$ and circular polarization. One can choose the electromagnetic potential of such a wave in the form

$$\mathbf{A} = -\frac{E_0}{\omega_0}\{\mathbf{e}_1 \sin \varkappa_0[ct-(\mathbf{rn})] - g\mathbf{e}_2 \cos \varkappa_0[ct-(\mathbf{rn})]\}, \tag{1}$$

where $\mathbf{e}_i$ (i = 2, 3) is an orthogonal reference frame in the plane orthogonal to **n**, E_0 is the amplitude of the electric field of the first wave, g = 1 corresponds to right-hand circular polarization of the wave, and g = −1 corresponds to left-hand circular polarization. The intensity of the polarized stimulated radiation emitted by an electron whose spin is oriented,[1)] moving in the initial state along the vector **n** with an average velocity $v = c\beta_3$, has the following form:

$$dW = \frac{ce^2}{4\pi^2} \sum_{\varepsilon=\pm1,\, \zeta'=\pm1} R^{-1}R'^{-1}N(\varkappa)(\lambda - \lambda' - \varepsilon\varkappa\cos\theta)\cdot$$
$$\cdot G\,|l_2S_2 + l_3S_3|^2\varkappa\, d\varkappa\, d\Omega,$$
$$S_2 = \{k_0(\lambda'-\lambda)(\boldsymbol{\sigma}\mathbf{s}) + [\varepsilon\lambda\varkappa\sin\theta - k_0\gamma(\lambda'-\lambda)n/a](\boldsymbol{\sigma}\mathbf{n})\}J_n(a)$$
$$-\varepsilon g\gamma k_0(\lambda'+\lambda)J_n'(a)\delta_{\zeta,\zeta'}, \tag{2*}$$
$$S_3 = \{[\varepsilon\lambda\varkappa\cos\theta - k_0^2(1+\gamma^2) + \lambda\lambda']\sin\theta\,\delta_{\zeta,\zeta'} + \gamma k_0(n/a)[(\lambda+\lambda')\cos\theta$$
$$-\varepsilon\varkappa\sin^2\theta]\delta_{\zeta,\zeta'} + ik_0[\varepsilon\varkappa\sin^2\theta - (\lambda'-\lambda)\cos\theta](\boldsymbol{\sigma}[\mathbf{ns}])\}J_n(a)$$
$$\times\, \varepsilon g\gamma k_0(\boldsymbol{\sigma}\mathbf{n})[(\lambda'-\lambda)\cos\theta - \varepsilon\varkappa\sin^2\theta]J_n'(a),$$
$$G = 4c\tau\{4c^2\tau^2[\lambda-\lambda'+\varepsilon\varkappa(1-\cos\theta)]^2+1\}^{-1}.$$

Here the following notation has been introduced:

$$a = \frac{\gamma k_0\varkappa\sin\theta}{\varkappa_0\lambda'},\quad k_0 = \frac{mc}{\hbar},\quad \gamma = \frac{eE_0}{mc\omega_0},\quad \lambda = k_0\left[\frac{(1+\gamma^2)(1-\beta_3)}{1+\beta_3}\right]^{1/2}$$
$$R = k_0^2(1+\gamma^2)+\lambda^2,\qquad R' = k_0^2(1+\gamma^2)+\lambda'^2+\varkappa^2\sin^2\theta, \tag{3}$$
$$\lambda' = \{[\varepsilon(\varkappa\cos\theta - n\varkappa_0) + k_0\beta_3((1+\gamma^2)/(1-\beta_3^2))^{1/2}]^2 + k_0^2(1+\gamma^2)$$
$$+\varkappa^2\sin^2\theta\}^{1/2} - \varepsilon(\varkappa\cos\theta - n\varkappa_0) - k_0\beta_3((1+\gamma^2)/(1-\beta_3^2))^{1/2},$$
$$\boldsymbol{\sigma} = \zeta\mathbf{l}\delta_{\zeta,\zeta'} + \frac{[\mathbf{l}[\mathbf{ln}]] + i\zeta[\mathbf{ln}]}{[1-(\mathbf{ln})^2]^{1/2}}\delta_{\zeta,-\zeta'},\quad \mathbf{s} = \frac{\boldsymbol{\varkappa} - \mathbf{n}(\boldsymbol{\varkappa}\mathbf{n})}{[\varkappa^2-(\boldsymbol{\varkappa}\mathbf{n})^2]^{1/2}}.$$

The quantity $\zeta = +1$ characterizes the two possible orientations of the electron's spin along the direction of the unit vector **l**. The vector $\boldsymbol{\kappa}$, directed at an angle θ to the vector **n**, is the wave vector of the second wave of frequency $\omega = c\kappa$; $N(\boldsymbol{\kappa})$ denotes the number of photons in the second wave with wave vector $\boldsymbol{\kappa}$, J_n and J_n' are the Bessel function of integer order n and its derivative, where n characterizes the number of photons from the first wave entering into the reaction. The parameters l_2 and l_3 characterize the polarization of the second wave.[2)] The quantity τ characterizes the "lifetime" of the electron; one can approximately estimate τ by assuming the second wave to be weak, like the inverse magnitude of the probability of spontaneous emission. From here it follows[5] for $\hbar\omega_0/mc^2 \ll 1$,

$$\tau = \frac{3\hbar c}{2e^2\omega_0}\left(\frac{mc\omega_0}{eE_0}\right)^2,$$

i.e., in the optical range for contemporary lasers $\tau \sim 10^{-9}$ sec, which is much larger than the period of the wave, $T = 2\pi/\omega_0$. Therefore, one can regard the quantity τ as large.

Expressions (2) and (3) completely describe the in-

[1)]For a detailed discussion of the spin states of an electron moving in a plane electromagnetic wave, see [5].

[2)]Concerning the choice of these parameters, see [6].

*$[\mathbf{ns}] \equiv \mathbf{n} \times \mathbf{s}$.

tensity, polarization, and spin effects of the stimulated radiation emitted by an electron moving in a monochromatic circularly polarized plane wave. However, of course expressions (2) and (3) in such form may be utilized, for example, for accurate quantitative calculations upon specific assignment of all of the initial parameters, but these expressions are extremely inconvenient for a physical analysis.

We shall carry out a physical analysis of the derived expressions under the assumption that the energy of the photons in the first and second waves is small in comparison with the electron's rest energy:

$$\frac{\hbar\omega_0}{mc^2} = \frac{\varkappa_0}{k_0} \ll 1, \quad \frac{\hbar\omega}{mc^2} = \frac{\varkappa}{k_0} \ll 1.$$

This is always valid for a choice of the frequencies of both waves in the optical region. Under such an assumption one can expand expressions (2) and (3) in powers of the small parameters (κ/k_0) and (κ_0/k_0), having taken the first nonvanishing approximation. In this approximation *only transitions* without reorientation of the electron's spin occur. Considering the case when the electron's lifetime τ is very large $(\tau \to \infty)$, in the chosen approximation we obtain the following result for the component of linear polarization:

$$W = -\frac{8\pi^2 e^2 (1+\beta_3)[(1-\beta_3^2)(1-q^2)]^{1/2}}{mc^2} \sum_{n=0}^{\infty} \int \frac{1-\cos\theta'}{1+\beta_3\cos\theta'} n\omega_0 \frac{\partial I}{\partial\omega} \times [l_2^2 q^2 J_n'^2(nq\sin\theta') + l_3^2 \operatorname{ctg}^2\theta' J_n^2(nq\sin\theta')]\, d\Omega', \quad q = \gamma(1+\gamma^2)^{-1/2}. \tag{4}$$

In formula (4) we passed from the number of photons $N(\kappa)$ to the spectral intensity

$$I(\varkappa) = \frac{\hbar\omega^3 N(\varkappa)}{8\pi^3 c^2}$$

and changed to a new angle θ' which is related to the angle θ by the Lorentz transformation:

$$\cos\theta = \frac{\cos\theta' + \beta_3}{1+\beta_3\cos\theta'}, \quad \sin\theta = \frac{\sqrt{1-\beta_3^2}\sin\theta'}{1+\beta_3\cos\theta'}. \tag{5}$$

Here the frequency ω of the second wave is related to the frequency ω_0 by

$$\omega = c\varkappa = \frac{1+\beta_3\cos\theta'}{1+\beta_3} n\omega_0. \tag{6}$$

It is known[6] that the expression inside the square brackets in Eq. (4) has a sharp maximum at $n \approx (1+\gamma^2)^{3/2}$, i.e., at $n = 1$ for small values of γ (the amplitude of the first wave is small) and at $n \sim \gamma^3$ for $\gamma \gg 1$ (the first wave has a large amplitude). From here it follows that if $\partial I/\partial\omega < 0$ in the region $n \sim (1+\gamma^2)^{3/2}$, then the total power is positive; therefore the second wave is amplified. For $\partial I/\partial\omega > 0$ the total power is negative, i.e., amplification of the first wave and absorption of the second wave occur. If the angle between the first and second waves is $\theta = \pi/2$, then amplification or absorption of only a certain component of the linear polarization of the second wave occurs (namely, that component in which the electric vector is orthogonal to the vector **n**), whereas the other component of the linear polarization is unchanged. An obvious conclusion follows from formula (4): the power differs from zero under the condition $\theta' \neq 0$ which is equivalent, as follows from Eq. (5), to the condition $\theta \neq 0$.

We perform the subsequent analysis assuming $\beta_3 = 0$ since a transformation to the case $\beta_3 \neq 0$ can always be made by a Lorentz transformation. For $\beta_3 = 0$, in the approximation $\kappa/k_0 \sim \kappa_0/k_0 \ll 1$ the following expressions for the radiated power of the components of linear polarization follow from Eqs. (2) and (3):

$$dW_i = -\frac{2\pi e^2(1-q^2)^{1/2}}{mc^2} I_i(\omega)\left(G_0 F_i^{(1)} - n\omega_0\frac{\partial G_0}{\partial\omega}F_i^{(2)}\right) d\omega\, d\Omega,$$

$$F_2^{(1)} = q^2\left[\left(1 - 6\frac{n\omega_0}{\omega}\cos\theta + 5\frac{n^2\omega_0^2}{\omega^2}\right) J_n'^2(u) + 4\frac{n\omega_0}{\omega}\left(\frac{n\omega_0}{\omega} - \cos\theta\right)\frac{u^2-n^2}{u} J_n(u) J_n'(u)\right],$$

$$F_3^{(1)} = \left[\left(1 - 6\frac{n\omega_0}{\omega}\cos\theta + 5\frac{n^2\omega_0^2}{\omega^2}\right) J_n^2(u) - 4u\frac{n\omega_0}{\omega}\left(\frac{n\omega_0}{\omega} - \cos\theta\right) J_n(u) J_n'(u)\right]\left(\frac{n\omega_0}{\omega}\operatorname{ctg}\theta\right)^2, \tag{7}$$

$$F_2^{(2)} = q^2\left(1 - 2\frac{n\omega_0}{\omega}\cos\theta + \frac{n^2\omega_0^2}{\omega^2}\right) J_n'^2(u),$$

$$F_3^{(2)} = \left(\frac{n\omega_0}{\omega}\operatorname{ctg}\theta\right)^2\left(1 - 2\frac{n\omega_0}{\omega}\cos\theta + \frac{n^2\omega_0^2}{\omega^2}\right) J_n^2(u),$$

$$G_0 = \frac{4c\tau}{1+x^2}, \quad x = 2\tau(\omega - n\omega_0), \quad \frac{\partial G_0}{\partial\omega} = -\frac{4\tau x}{1+x^2}G_0, \quad u = \frac{q\omega\sin\theta}{\omega_0}.$$

In practice the case when $q \ll 1$ is most important (for contemporary lasers $q \sim 10^{-2}$ to 10^{-1}). In this case the maximum of the radiated power is reached for $n = 1$. From Eq. (7) for $q \ll 1$ we obtain

$$W_2 = -\frac{\pi q^2 e^2}{2mc^2}\int G_0 F(\omega)\, d\omega\, d\Omega, \quad W_3 = -\frac{\pi q^2 e^2}{2mc^2}\int G_0\cos^2\theta F(\omega)\, d\omega\, d\Omega,$$

$$F(\omega) = I(\omega)\left(1 - \frac{\omega_0}{\omega}\right)\left(2\frac{\omega_0^2}{\omega^2} + \frac{\omega_0}{\omega} + 1 - 2\frac{\omega_0}{\omega}\cos\theta\right) + \omega_0\frac{\partial I(\omega)}{\partial\omega}\left(1 + \frac{\omega_0^2}{\omega^2} - 2\frac{\omega_0}{\omega}\cos\theta\right). \tag{8}$$

It is obvious that the function G_0 is maximal at the resonance point $\omega = n\omega_0$. In this connection, it follows from (8) that the radiated power is determined by the term $\sim \partial I/\partial\omega$. For regions far away from resonance, the radiated power is small in view of the smallness of the function G_0. It also follows from (7) that for $\theta \sim \pi/2$ (the directions of propagation of the first and second waves are mutually orthogonal) amplification or absorption of only the component W_2 occurs. Hence follow the general conclusions that the radiated power is determined by the behavior of $\partial I/\partial\omega$ for $\omega \approx n\omega_0$ (in the resonance region) and for $\partial I/\partial\omega < 0$ the second wave is amplified, while for $\partial I/\partial\omega > 0$ the second wave is weakened. In the range of angles $\theta \sim \pi/2$ there primarily occurs amplification or absorption of that component of the second wave's linear polarization whose oscillating electric vector is orthogonal to the direction of propagation of the first wave.

Finally we note that in the approximation $\kappa/k_0 \sim \kappa_0/k_0 \ll 1$ the power radiated by an electron and by a spinless particle are identical.[1] This is farily obvious from the fact that an expansion in powers of κ/k_0 is equivalent to an expansion in powers of Planck's constant $\hbar$, but the spin effects appear in the higher approximations in $\hbar$.

Observation of the effect of stimulated emission of radiation by electrons moving in a plane wave should not present serious difficulties since the radiated power is approximately $N(\kappa)$ times greater than the radiated power associated with the Compton effect, which was well investigated experimentally a long time ago. However, $N(\kappa)$ cannot be too large since we assume that the amplitude of the second wave is much smaller than the amplitude of the first.

In conclusion, we note that a smearing-out of the electrons in energy (more precisely, in the quantity λ) cannot substantially change the conditions for observation of amplification or absorption of the wave. In fact, considering for example the case of large τ described by Eq. (4), we see that a different λ corresponds to multiplication of W by a positive factor which depends on β_3. Averaging this factor with respect to a given distribution function can only lead to a change of the numerical coefficient in front of W. The dependence of the radiated power on the parameters of the first wave (on the quantity γ) is essential, but a smearing-out of the electrons in energy does not change this dependence.

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